Splitting functions@N3LO

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Collaborators

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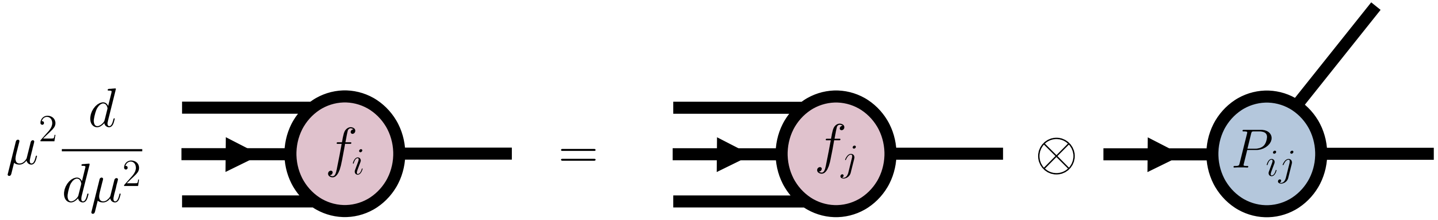
Splitting Basics

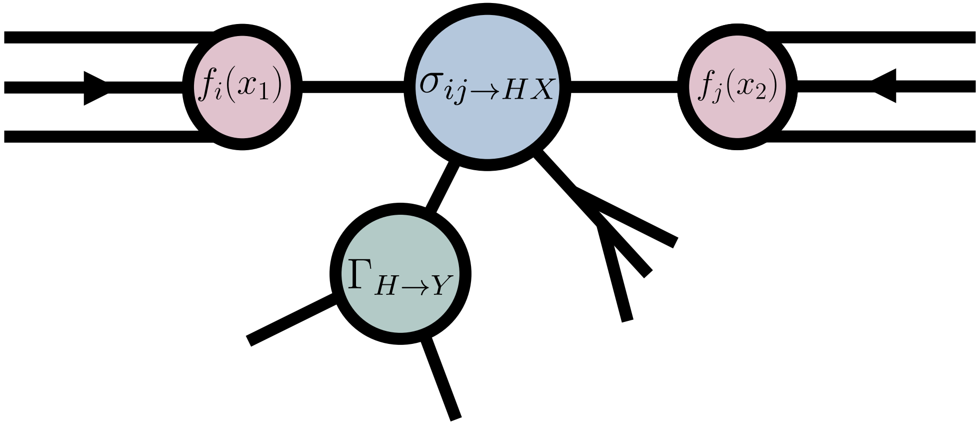
- Splitting functions govern PDF evolution (DGLAB)



$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \int_x^1 \frac{dz}{z} f_j(x/z, \mu^2) P_{ij}(z)$$

Why splitting functions at N3LO?

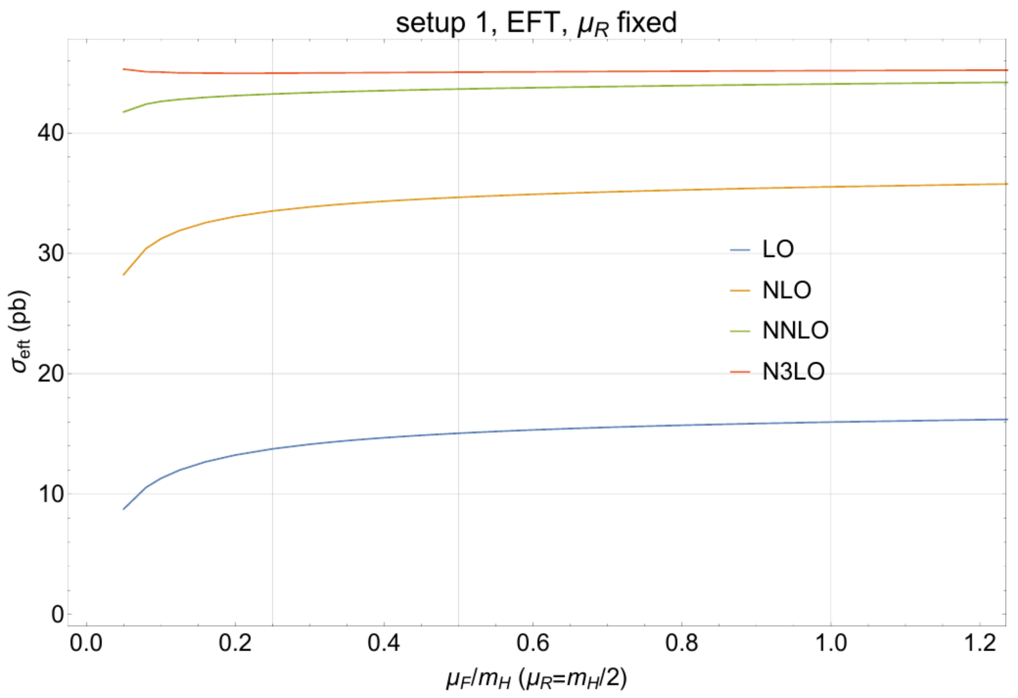
- Missing ingredient at N3LO
  - Needed in N3LO PDF fits
  - Reduce theory error



$$\sigma_{PP\to HX}(\boldsymbol{\mu}) = \sum_{i,j} \int dx_1 dx_2\, f_i(x_1, \boldsymbol{\mu})\, f_j(x_2, \boldsymbol{\mu})\, \sigma_{ij\to HX}(\boldsymbol{\mu})\, \Gamma_{H\to y}$$

Theory uncertainties in Higgs production

- N3LO factorisation scale dependence is artificially low.
- Due to missing *N3LO splitting function* and *N3LO PDF set*.



$\delta(\text{PDF})$	$\delta(\alpha_s)$
± 0.90 pb	$^{+1.27\text{pb}}_{-1.25\text{pb}}$
$\pm 1.86\%$	$^{+2.61\%}_{-2.58\%}$

$\delta(\text{scale})$	$\delta(\text{trunc})$	$\delta(\text{PDF-TH})$	$\delta(\text{EW})$	$\delta(t,b,c)$	$\delta(1/m_t)$
$^{+0.10\text{ pb}}_{-1.15\text{ pb}}$	± 0.18 pb	± 0.56 pb	± 0.49 pb	± 0.40 pb	± 0.49 pb
$^{+0.21\%}_{-2.37\%}$	$\pm 0.37\%$	$\pm 1.16\%$	$\pm 1\%$	$\pm 0.83\%$	$\pm 1\%$

Arxiv:1602.00695

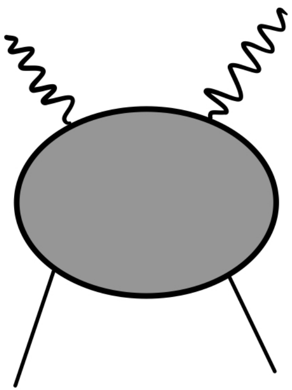
Mellin moments $\gamma_{ij}(N)$

- $\gamma_{ij}(N)$ are important:
  - DGLAB becomes ordinary product in Mellin space
  - are related p→0 Taylor expansion of DIS Forward scattering amplitude
  - are the anomaous dimensions of lightcone operators via the OPE

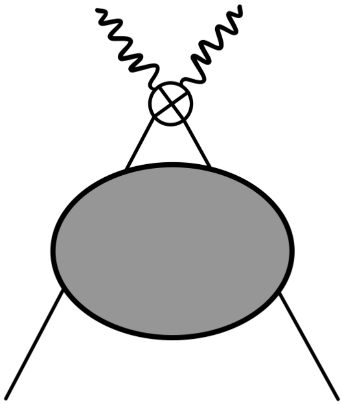
$$\gamma_{ij}(N) = -\int_0^1 dx\, x^{N-1}\, P_{ij}(x)$$

The OPE method

$$\langle 0|T[\bar{\psi}(-p)J(x)J(0)\psi(p)]|0\rangle = \sum_{n,k} F_k^n(x)\langle 0|T[\bar{\psi}(-p)O_k^n(0)\psi(p)]|0\rangle$$

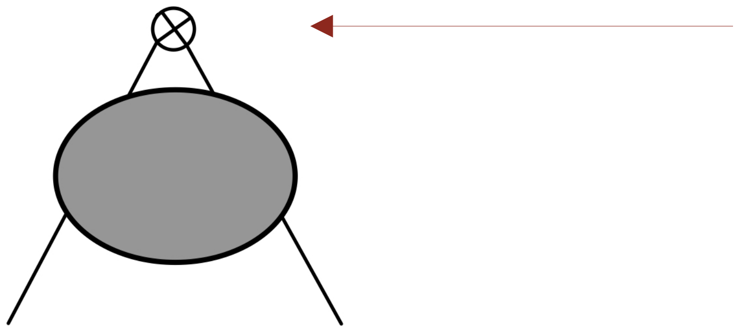


$$= \sum_{n,k} F_k^n(x)$$



→ Only need to compute diagrams with operators inserted!

The OPE method



$$O_g^{\mu_1 \ldots \mu_n} = \mathcal{S} F_\mu^{\mu_1} D^{\mu_2} \ldots D^{\mu_{n-1}} F^{\mu \mu_n}$$

$$O_{q;ns}^{a;\mu_1 \ldots \mu_n} = \mathcal{S} \bar{\psi} \lambda^a \gamma^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} \psi$$

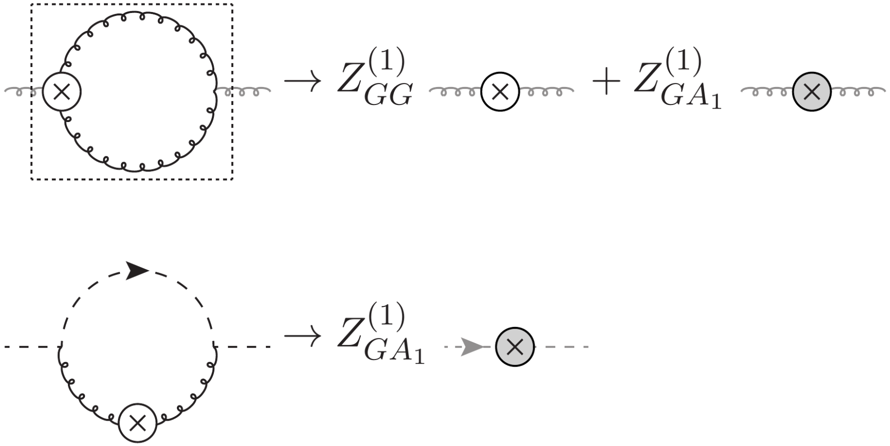
$$O_{q;s}^{\mu_1 \ldots \mu_n} = \mathcal{S} \bar{\psi} \gamma^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} \psi$$

- Need to compute 2-point *off-shell* correlators with operator insertions. (p-integrals)
- This problem is “solved” up to four loops.
  Masters [Baykov, Chetyrkin; Smirnov,Smirnov,Lee] and reductions implemented in FORCER [Ruijl,Ueda, Vermaseren]

Problem: Offshell correlators are ***not*** gauge invariant!

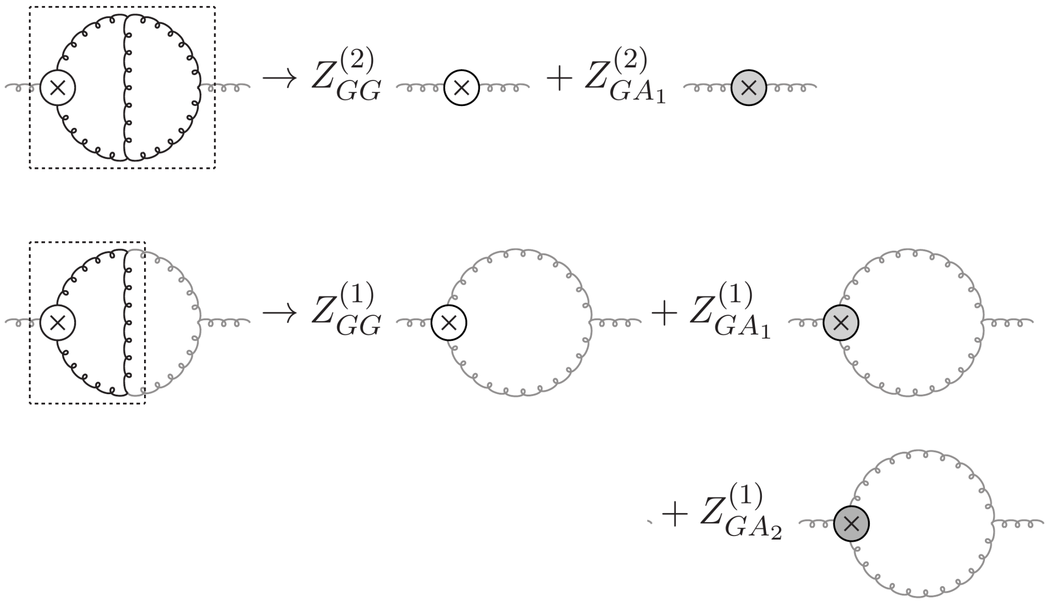
Operator mixing in singlet sector

- Gluonic operator (G) mixes with aliens (A1,A2,...) under renormalisation

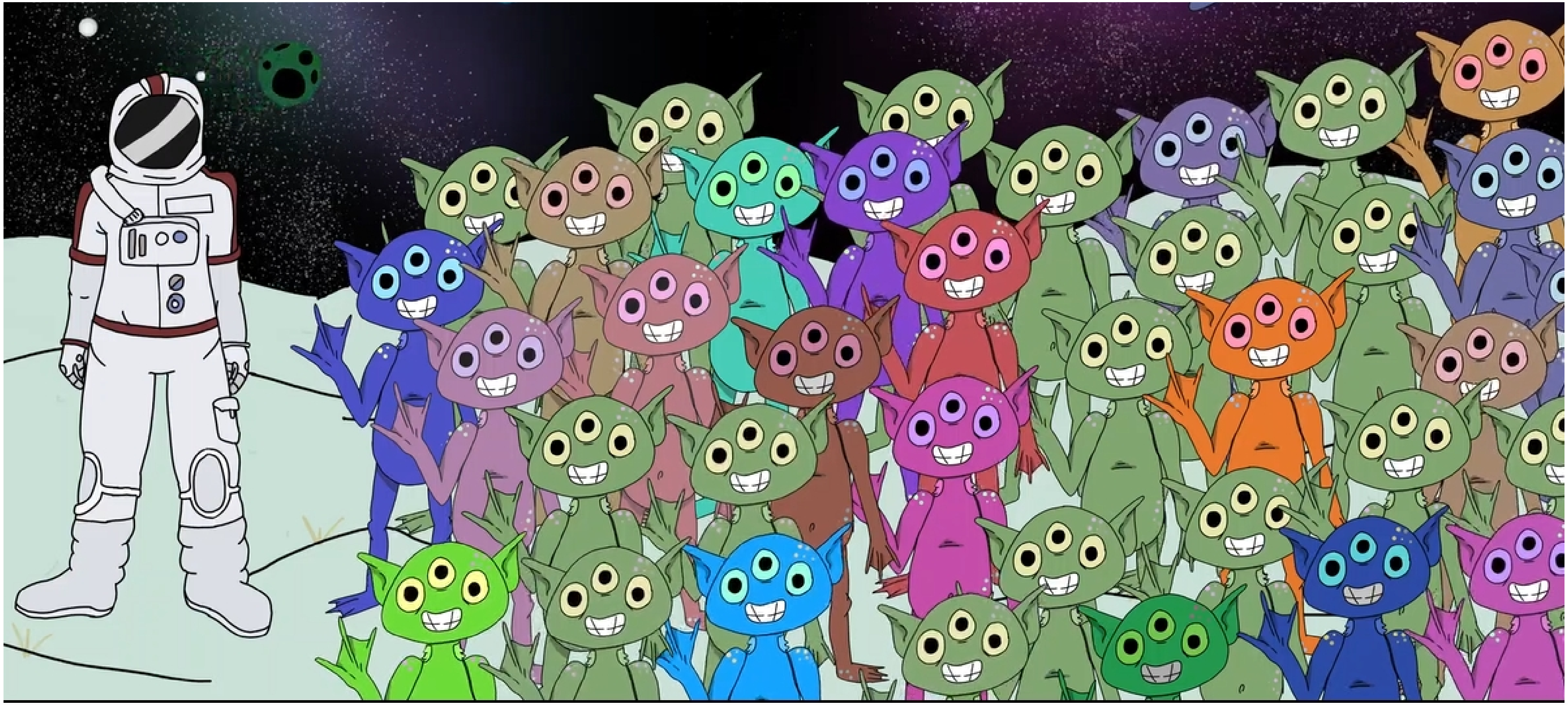


Operator mixing in singlet sector

- Gluonic operator (G) mixes with aliens (A1,A2,...) under renormalisation
- More legs → more Aliens
- More loops→more legs in subgraphs → more Aliens



FORCER allows us to get to 4 loops, but we were were not the first to arrive



Dixon Taylor Alien basis

- Alien and Ghost operators:
  (all Lorentz indices contracted with a massless vector)

$$\overline{O}_A = \overline{F}^{a\alpha}\overline{D}_{\alpha}^{ab}\partial^{m-2}\overline{A}^b - \bar{g}f^{abc}\overline{F}_{\alpha}^{a}\sum_{i=1}^{m-2}\frac{\kappa_i}{\eta}\partial^{\alpha}\left[(\partial^{i-1}\overline{A}^b)(\partial^{m-2-i}\overline{A}^c)\right] + O(\bar{g}^2),$$

$$\overline{O}_\omega = -\xi^a\partial^m\overline{\omega}^a - \bar{g}f^{abc}\bar{\xi}^a\sum_{i=1}^{m-2}\frac{\eta_i}{\eta}\partial\left[(\partial^{m-2-i}\overline{A}^b)(\partial^i\overline{\omega}^c)\right] + O(\bar{g}^2),$$

- Was used by Van-Neerven&Hamberg to *correctify* OPE computations at 2-loop

2022 Beyond Dixon Taylor basis

- General Method for Alien basis only available since *last year*
  - **Based on Generalised gauge+BRST invariance** [Falcioni, FH]

A1 $$\mathcal{O}^{(N),1}_{\mathrm{EOM}} = \eta\,(D.F)^a\,\partial^{N-2}A^a \qquad \mathcal{O}^{(N),1}_{G} = -\eta\,(\partial\bar{c}^a)\,(\partial^{N-1}c^a)\,,$$

A2 $$\mathcal{O}^{(N),2}_{\mathrm{EOM}} = g(D.F)^a \sum_{\substack{i+j=\\N-3}} C^{abc}_{ij}(\partial^i A^b)(\partial^j A^c) \qquad \mathcal{O}^{(N),2}_{G} = -g \sum_{\substack{i_1+i_2\\=N-3}} \tilde{C}^{a;a_1\,a_2}_{i_1\,i_2}\,(\partial\bar{c}^a)\,(\partial^{i_1}A^{a_1})\,(\partial^{i_2+1}c^{a_2})\,,$$

A3 $$\mathcal{O}^{(N),3}_{\mathrm{EOM}} = g^2(D.F)^a \sum_{\substack{i+j+k\\=N-4}} C^{abcd}_{ijk}\,(\partial^i A^b)(\partial^j A^c)(\partial^k A^d) \qquad \mathcal{O}^{(N),3}_{G} = -g^2 \sum_{\substack{i_1+i_2+i_3\\=N-4}} \tilde{C}^{a;a_1\,a_2\,a_3}_{i_1\,i_2\,i_3}\,(\partial\bar{c}^a)\,(\partial^{i_1}A^{a_1})\,(\partial^{i_2}A^{a_2})\,(\partial^{i_3+1}c^{a_3})\,,$$

A4 $$\mathcal{O}^{(N),4}_{\mathrm{EOM}} = g^3(D.F)^a \sum_{\substack{i+j+k+l\\=N-5}} C^{abcde}_{ijkl}\,(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e) \qquad \mathcal{O}^{(N),4}_{G} = -g^3 \sum_{\substack{i_1+..+i_4\\=N-5}} \tilde{C}^{a;a_1\,a_2\,a_3\,a_4}_{i_1\,i_2\,i_3\,i_4}\,(\partial\bar{c}^a)\,(\partial^{i_1}A^{a_1})\,(\partial^{i_2}A^{a_2})\,(\partial^{i_3}A^{a_3})\,(\partial^{i_4+1}c^{a_4})\,.$$

  - **Alternative approach not based on operators** [Gehrmann, von Manteuffel, Yang]

History of splitting functions

Please accept my apologies: this is not complete

- LO (one loop)
  - *OPE in physical gauge:*
    1973: Georgi & Politzer; Gross & Wilczek
  - *OPE in Feynman gauge:*
    1974: Dixon and Taylor
- NLO (two loops)
  - *OPE in Feynman gauge:*
    1977-1979 Floratos, Ross & Sachrajda **with ERROR in singlet**
    1980 Gonzalez-Arroyo&Lopez **with ERROR in singlet**
    1993 Van Neerven & Hamberg **CORRECT!**
  - *Factorisation in axial gauge:*
    1980 Curci, Furmanski, Petronzio **CORRECT!**
- NNLO (three loops)
  - *Extracted from DIS:*
    2000 (nonsinglet), 2004 (singlet) Moch, Vermaseren, Vogt
  - N3LO calculations:
    2016 [Anastasiou, Duhr, Dulat, FH, Mistlberger]; 2020 [Duhr, Dulat, Mistlberger]
  - Massive onshell OPE:
    2017 (singlet NF) [Ablinger, Behring, Blümlein, De Freitas, von Manteuffel];..
  - OPE:
    2021 (non-singlet) [Blümlein, Marquard, Schneider, Schönwald];
  - Complete OPE:
    2022 [Gehrmann, von Manteuffel, Yang]

N3LO/4-loops and beyond

- OPE method
  - 1996 **Leading Nf all orders** [Gracey]
  - Non-singlet:
    - 2017 **Subleading Nf; Leading Nc; Approx. from N<22** [Davies, Vogt, Ruijl, Ueda, Vermaseren]
    - 2018 **5-loop N=2,3** [Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt]
  - Singlet:
    - 2023 **Approx N<22 for P_{qq} and P_{qg}** [Falcioni, FH Moch, Vogt]
    - 2023 **Subleading Nf for P_{qq}** [Gehrmann, von Manteuffel,Yang]

- DIS Taylor expansion
  - Singlet:
    - 2021-23 **N<14** [Moch, Ruijl, Ueda, Vermaseren, Vogt]

Results: 4-loop pure singlet

- First 20 moments known:

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N=2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=6) &= -46.03061374\, n_f + 4.744075766\, n_f^2 + 0.042548957\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=8) &= -24.01455020\, n_f + 3.235193483\, n_f^2 - 0.007889256\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=10) &= -13.73039387\, n_f + 2.375018759\, n_f^2 - 0.021029241\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=12) &= -8.152592251\, n_f + 1.819958178\, n_f^2 - 0.024330231\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=14) &= -4.840447180\, n_f + 1.438327380\, n_f^2 - 0.024479943\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=16) &= -2.751136330\, n_f + 1.164299642\, n_f^2 - 0.023546009\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=18) &= -1.375969240\, n_f + 0.960873318\, n_f^2 - 0.022264393\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N=20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3 \,.
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N=20) = \; & n_f C_F^3 \Bigg(\frac{2128032487727689123396891103081423002879945894061}{2362988589594296007960167341127989233623040000000} \\
& - \frac{7463032385600125416449}{780446422304229681000 0}\zeta_3 + \frac{9834028074797}{14900178793500}\zeta_4 - \frac{178084}{153615}\zeta_5 \Bigg) \\
& + n_f C_A C_F^2 \Bigg(-\frac{844228173134903089150028231588375751525961 5913}{1413270687556397133947468505459323704320000000} \\
& - \frac{1351234593414493006 4021}{104059522973897290800 00}\zeta_3 - \frac{7936779238702}{3725044698375}\zeta_4 + \frac{89042}{460845}\zeta_5 \Bigg) \\
& + n_f C_A^2 C_F \Bigg(\frac{2504501090182155536693337518632638071230 28219}{11901226842580186391136576888078515404800 0000} \\
& + \frac{406254244378961149952 30699}{18397723661785041013440000}\zeta_3 + \frac{164760066767}{112031419500}\zeta_4 - \frac{4694036}{1382535}\zeta_5 \Bigg) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \Bigg(\frac{1240469880166297818093184994697 46921}{184011824338334566011505267 2000000} + \frac{3466020543326488599 4007}{1100342324269440252000}\zeta_3 \\
& - \frac{48237328}{460845}\zeta_5 \Bigg) \\
& + n_f^2 C_F^2 \Bigg(-\frac{2055309173013029770227661860695365 5791}{717720537479570533869346306432512 00000} - \frac{2842660003013}{7450089396750}\zeta_3 + \frac{89042}{460845}\zeta_4 \Bigg) \\
& + n_f^2 C_A C_F \Bigg(\frac{6885600202313786463969272150511308 32957}{160272932053708607939245002263500 8000000} + \frac{1316792611}{7223745375}\zeta_3 - \frac{89042}{460845}\zeta_4 \Bigg) \\
& + n_f^3 C_F \Bigg(-\frac{462358173460692018715852418 41}{990993385042051234188945600000} + \frac{178084}{6912675}\zeta_3 \Bigg) \,. \qquad \text{(A.12)}
\end{aligned}
$$

Strategy for approximations

- Fit 80 *trial* functions which include
  - known small-x (BFKL) amd large-x (soft) terms
  - 10 polynomials different for each trial function

e.g.

$$P^{(3)}_{\rm ps,\,A}(n_f = 3, x) \;=\; p^{(n_f=3)}_{\rm ps,0}(x) + 67731\, x_1 L_0/x + 274100\, x_1/x + 40006\, L_0^3 + 10620\, L_0^2 + 353656\, x_1 L_0 - 2365.1\, x_1 L_1^2 - 7412.1\, x_1 L_1 + 1533.0\, x_1^2 L_1^2 - 104493\, x_1 (1+2x) + 34403\, x_1 x^2\,,$$

Known (rounded) endpoint behaviour:

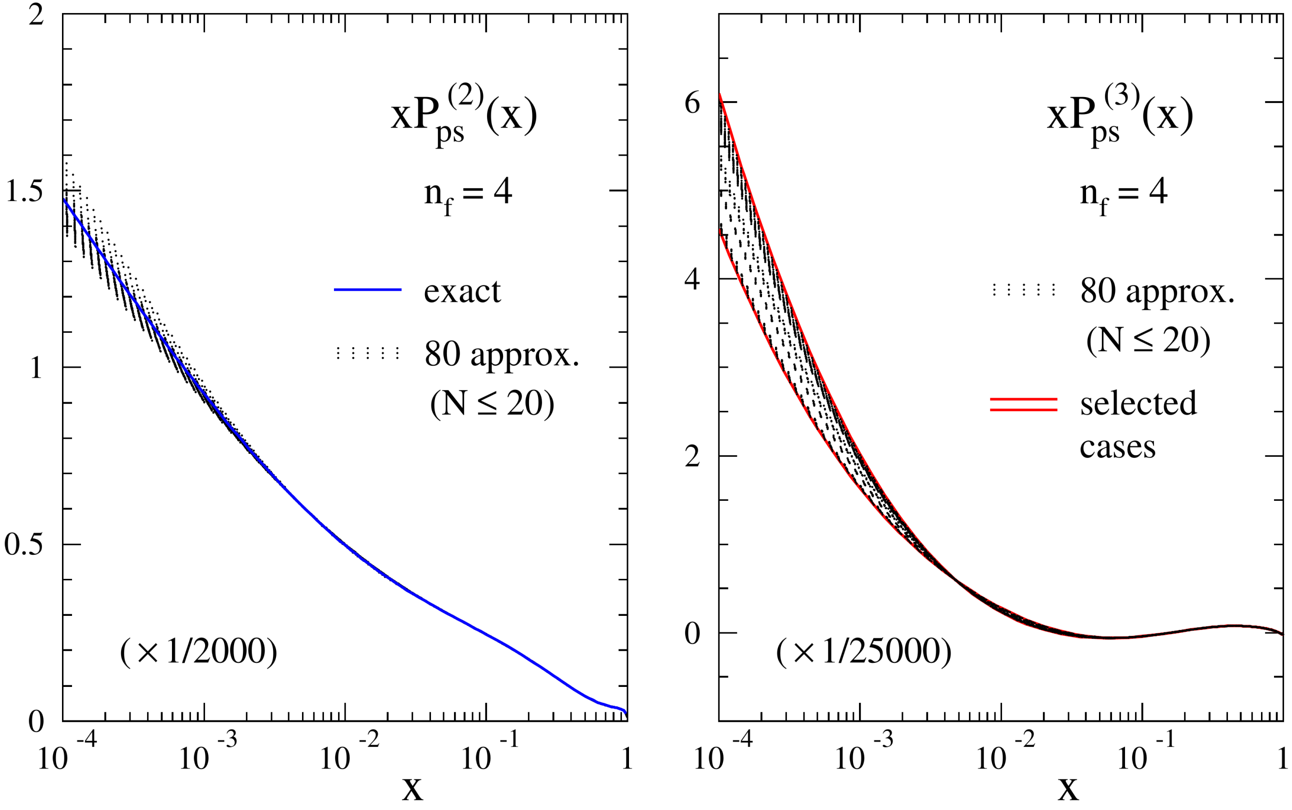
$$p^{(n_f)}_{\rm ps,0}(x) \;=\; n_f \Big\{ 1749.227\, L_0^2/x - (7.506173 - 0.7901235\, n_f) L_0^6 + (28.54979 + 3.792593\, n_f) L_0^5 - (854.8001 - 77.36626\, n_f + 0.1975309\, n_f^2) L_0^4 - (199.1111 - 13.69547\, n_f) x_1^2 L_1^3 - 13.16872\, x_1^2 L_1^4 - (247.5505 - 40.55967\, n_f + 1.580247\, n_f^2) x_1 L_1^3 - (56.46091 - 3.621399\, n_f) x_1 L_1^4 \Big\}\,.$$

$$x_1 = 1 - x$$

$$L_1 = \log 1 - x$$

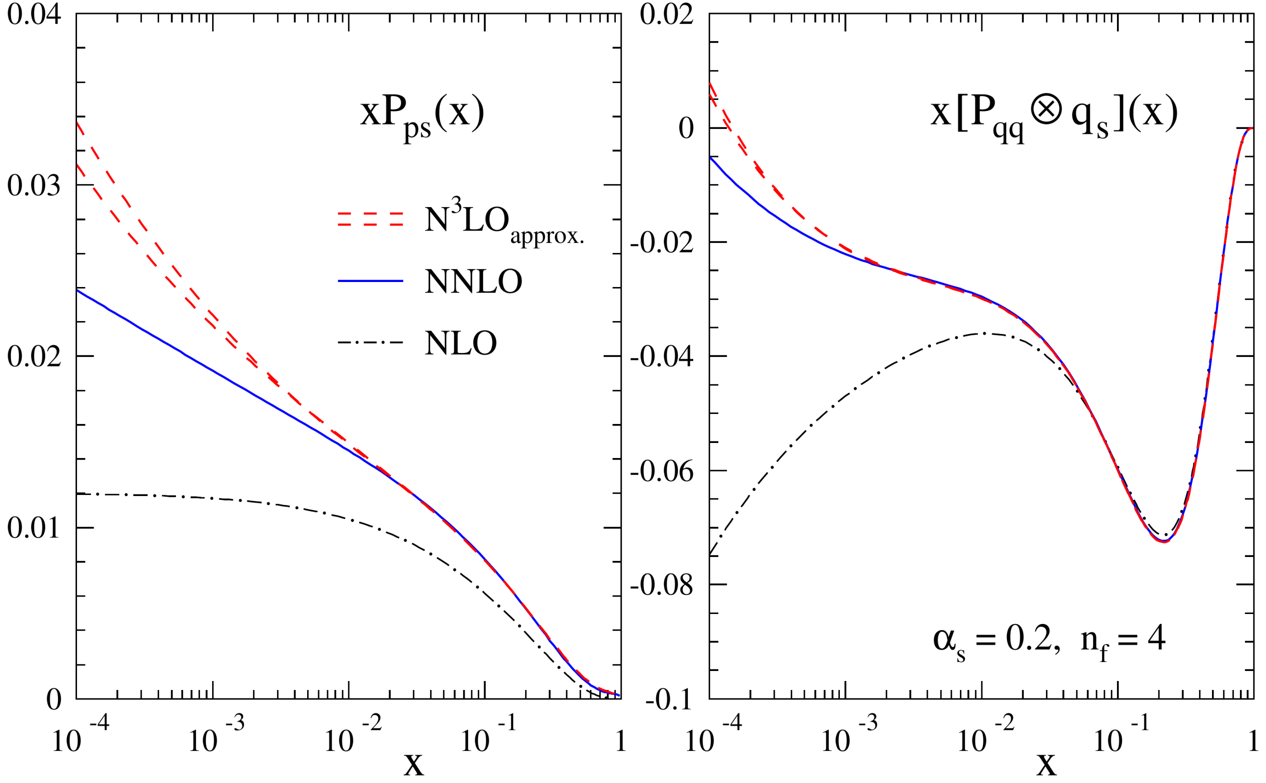
$$L_0 = \log x$$

Approximations P_{ps}



- Strategy works well at NNLO
- Small x uncertainties increase for $x<10^{-3}$

Approximations P_{ps}

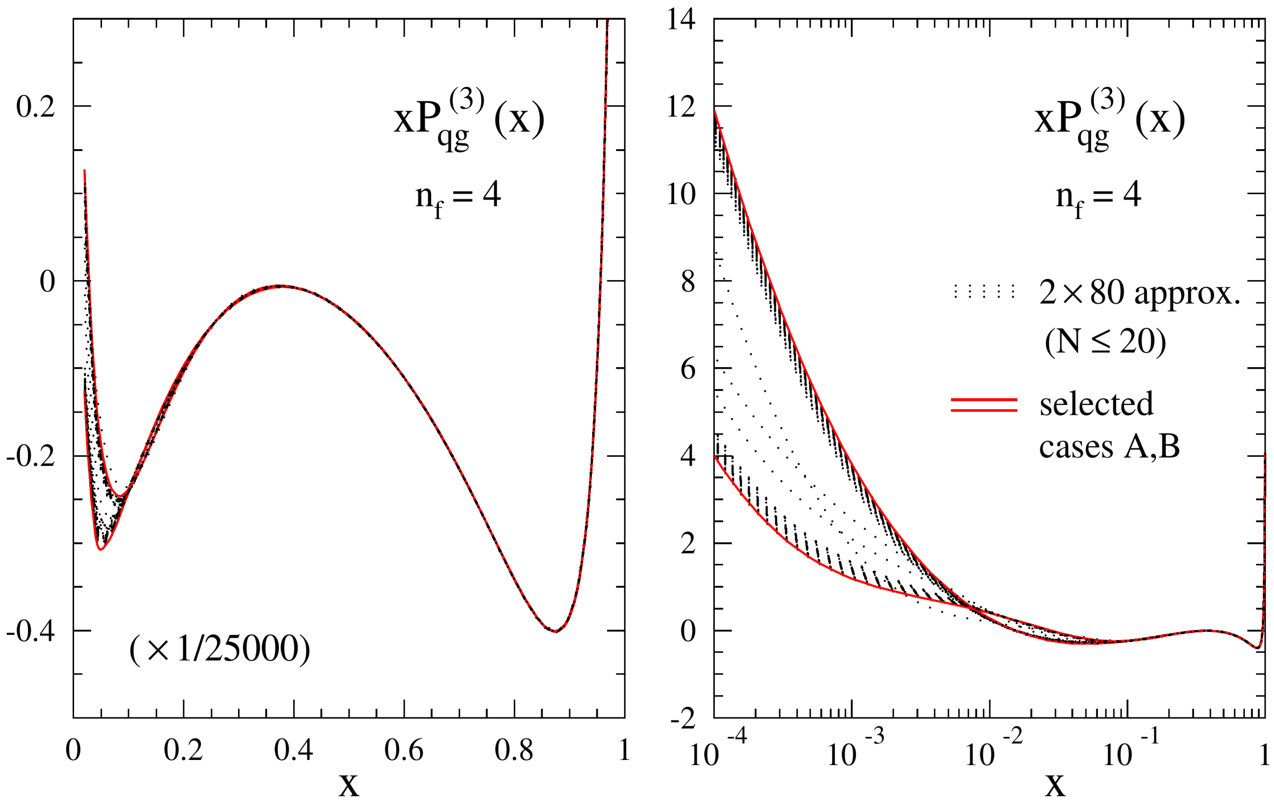


- Perturbative corrections large at small x
- Convolution with PDF surpresses small-x uncertainty

$$xq_s(x,\mu_0^2) \;=\; 0.6x^{-0.3}(1-x)^{3.5}\left(1+5.0x^{0.8}\right)$$

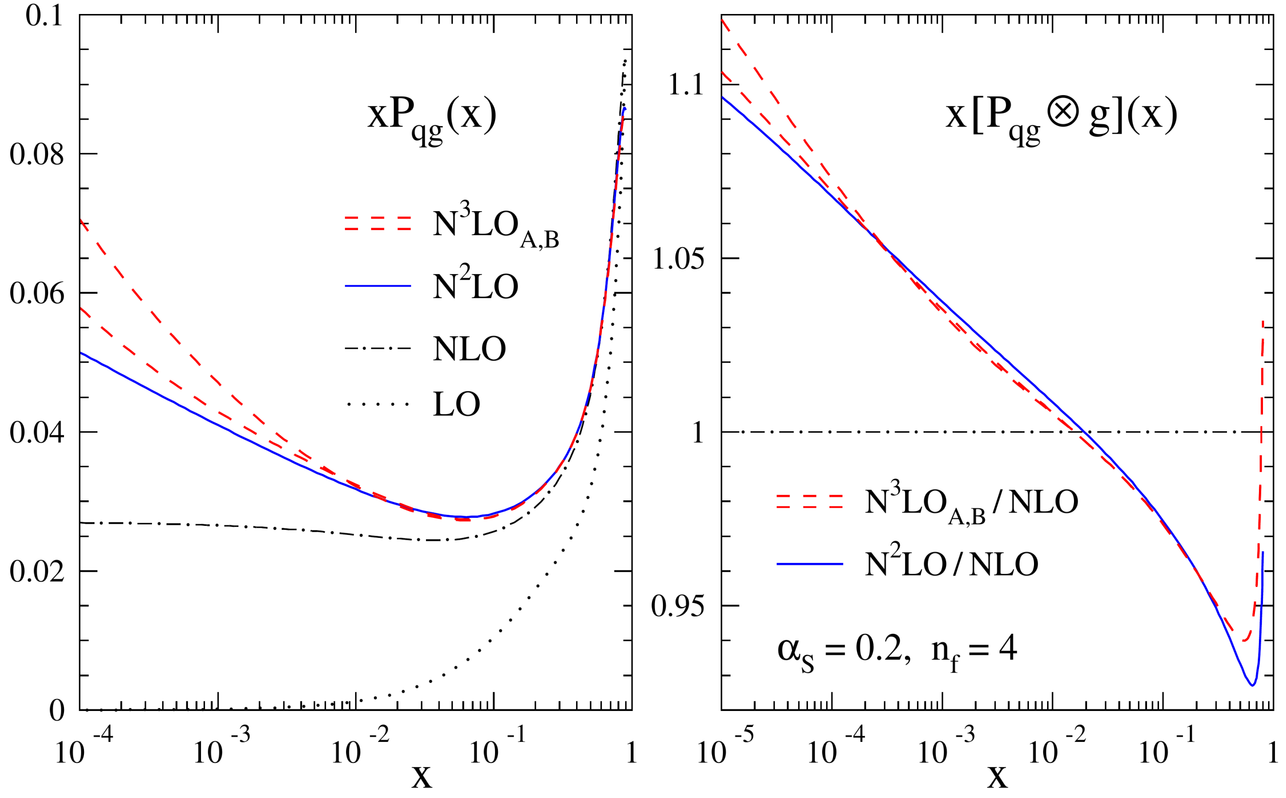
Approximations P_{qg}

- Approximations for P_{qg} are worse than for P_{ps}
- Large errors already at x~0.005



Approximations P_{qg}

- As for P_{ps}, PDF convolution compresses small-x uncertainty

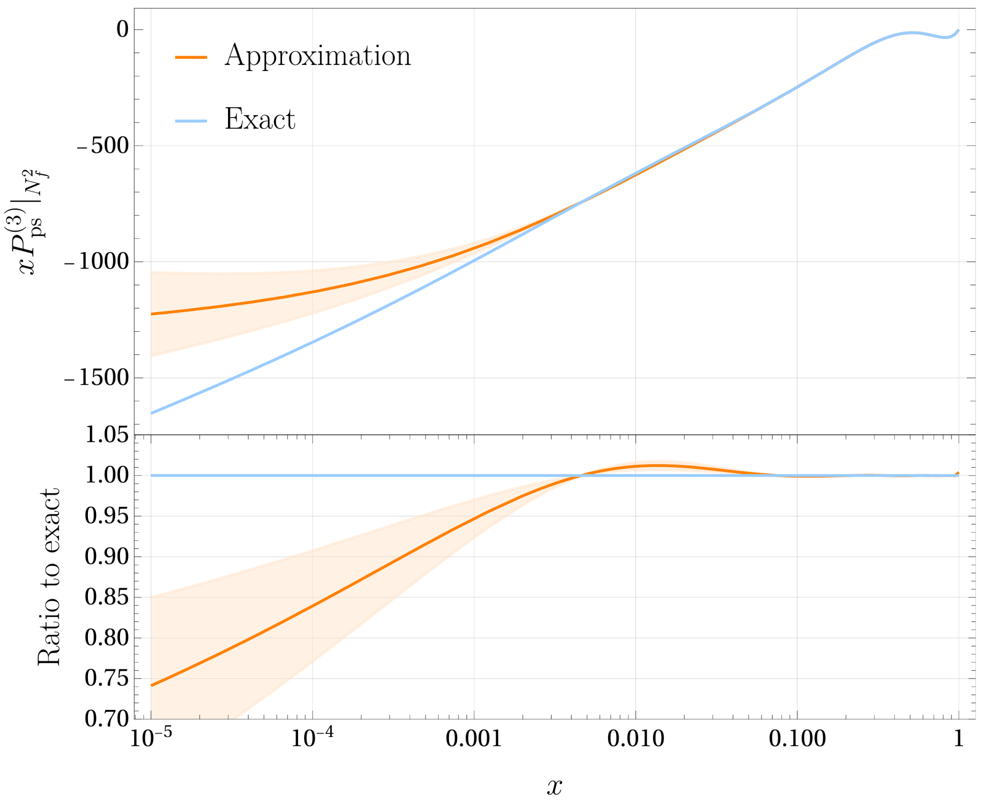


Summary/Outlook

- Reviewed status of N3LO OPE calculations for splitting functions
  - Alien's operator basis finally well understood
- Presented approximations for P_{ps} and P_{qg} from first 10 Mellin moments + known endpoint behaviour
  - Small x-uncertainties still fairly large, (more knowledge about logs could help!)
  - Pdf convolution has small uncertainties even at small x
  - Likely sufficient for most pheno-applications
- Calculations are P_{gq} and P_{gg} are almost finished and we expect to publish first 20 moments & approximations soon!

Backup

Plot in 2308.07958



General Structure of Aliens

$$F^{\mu;a} = \Delta_\nu \, F^{\mu\nu;a}, \qquad A^a = \Delta_\mu A^{\mu;a}, \qquad D = \Delta_\mu \, D^\mu, \qquad \partial = \Delta_\mu \partial^\mu \, .$$

$$\Delta^2 = 0$$

$$\mathcal{O}_G^{(N)} + \mathcal{O}_{\mathrm{EOM}}^{(N)} = s(\partial \bar{c}^a \mathcal{G}^a) + \frac{\delta S}{\delta A_a} \mathcal{G}^a$$

$$\mathcal{G}^a = \sum_{\substack{i_1 + \dots + i_k \\ = N - k - 1}} C^{a;a_1 \dots a_k}_{i_1 \dots i_k} \left(\partial^{i_1} A^{a_1}\right) \dots \left(\partial^{i_k} A^{a_k}\right)$$

Alien examples N=2,4

- N=2:

$$\mathcal{O}_2^{(2)} = (D.F)^a\, A^a + \bar{c}^a \partial^2 c^a$$

- N=4

$$\mathcal{O}_2^{(4)} = (D.F)^a \Big[\partial^2 A^a + g f^{a a_1 a_2} A^{a_1} \partial A^{a_2}\Big] - \partial \bar{c}^a\, \partial^3 c^a - g f^{a a_1 a_2}\, \partial \bar{c}^a \Big[2 A^{a_1} \partial^2 c^{a_2} + \partial A^{a_1}\, \partial c^{a_2}\Big]$$
$$- g^2 (f f)^{a a_1 a_2 a_3}\, \partial \bar{c}^a\, A^{a_1} A^{a_2}\, \partial c^{a_3},$$

$$\mathcal{O}_3^{(4)} = d^{a a_1 a_2 a_3} \Big[(D.F)^a A^{a_1} A^{a_2} A^{a_3} - 3\, \partial \bar{c}^a\, A^{a_1} A^{a_2}\, \partial c^{a_3}\Big].$$

Including limits in pure singlet

terms in the limits $x \to 0, 1$. At small x, the coefficient of the leading logarithm $(\ln^2 x)/x$ is known since long [47], as well as those of the highest three sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$, see ref. [48]. At large x, the leading terms are of the form $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$. The coefficients for $k = 4, 3$ are known [49] for all j. With the 10 Mellin moments $N \leq 20$ in

[47] S. Catani and F. Hautmann, *High-energy factorization and small x deep inelastic scattering beyond leading order*, *Nucl. Phys. B* **427** (1994) 475–524, hep-ph/9405388.

[48] J. Davies, C. H. Kom, S. Moch and A. Vogt, *Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering*, *JHEP* **08** (2022) 135, arXiv:2202.10362.

[49] G. Soar, S. Moch, J. A. M. Vermaseren and A. Vogt, *On Higgs-exchange DIS, physical evolution kernels and fourth-order splitting functions at large x*, *Nucl. Phys. B* **832** (2010) 152–227, arXiv:0912.0369.