Power Corrections

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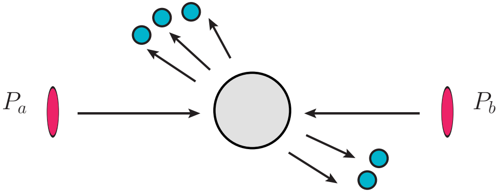
Predictions for the LHC: Factorization

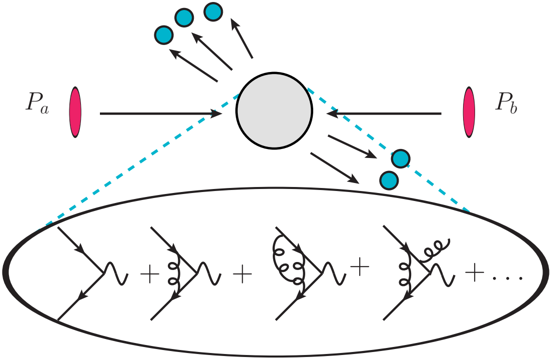
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \; \hat{\sigma}_{ab} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)$$



Factorization of physics at different scales is crucial in making predictions, goes back to [Collins, Soper, Sterman. Nucl. Phys. B250 (1985) 199]

Predictions for the LHC: Fixed-order

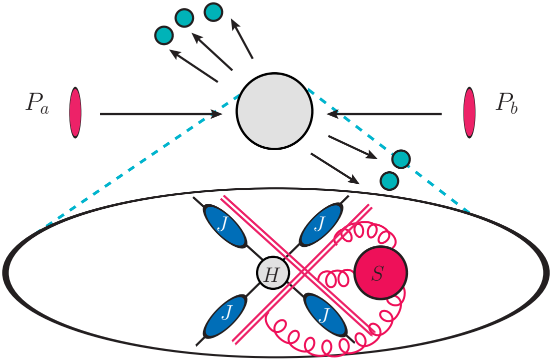
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\, \hat{\sigma}_{ab} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$$



$$d\hat{\sigma}_{ab} = \underbrace{d\hat{\sigma}_{ab}^{(0)}}_{\text{LO}} + \alpha_s \underbrace{d\hat{\sigma}_{ab}^{(1)}}_{\text{NLO}} + \alpha_s^2 \underbrace{d\hat{\sigma}_{ab}^{(2)}}_{\text{NNLO}} + \alpha_s^3 \underbrace{d\hat{\sigma}_{ab}^{(3)}}_{\text{N}^3\text{LO}} + \dots$$

Predictions for the LHC: Singular limits

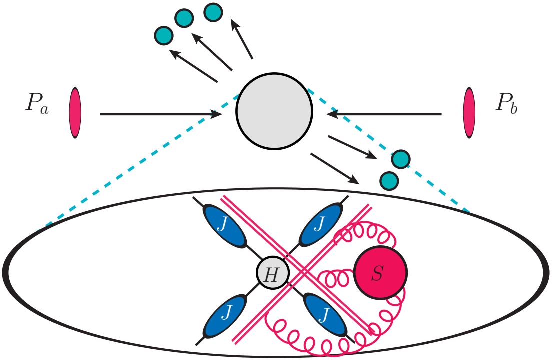
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\mathrm{LP}} + \hat{\sigma}_{ab}^{\mathrm{NLP}} + \ldots \right) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$$



Singular limits of QCD give rise to large logarithms. Resummation of these logarithms is based on factorization of amplitudes in these limits. Expansion in Q_0/Q is necessary to resum large $\ln(Q_0/Q)$.

Predictions for the LHC: Leading power

$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\mathrm{LP}} + \hat{\sigma}_{ab}^{\mathrm{NLP}} + \ldots \right) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$$

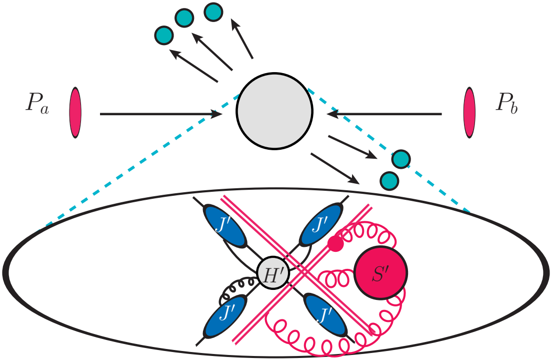


Power expansion in $\tau = (1-z), q_T^2, \ldots$

$$\hat{\sigma}(\tau) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(\tau) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m \tau}{\tau}\right]_+ + d_{nm} \ln^m(\tau) \right) + \ldots \right]$$

Predictions for the LHC: Power corrections

$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab}^{\text{NLP}} + \ldots \right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$



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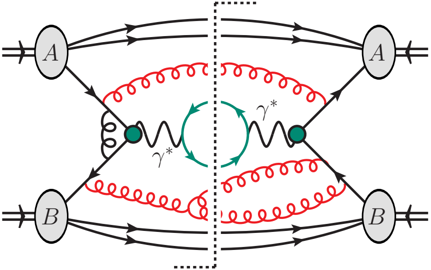
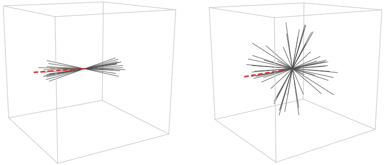
Motivations and focus





[T. Becher, A. Broggio, A. Ferroglia, 1410.1892]

DY threshold $\tau = (1 - z)$

$$\frac{d\sigma}{dQ^2} = f_{a/A} \otimes f_{b/B} \otimes |C^{A0}|^2 \times S_{\rm LP}$$

Event shapes, thrust $\tau = 1 - T$

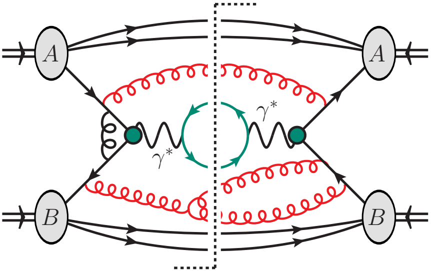
$$\frac{d\sigma}{d\tau} = |C^{A0}|^2 \times J_c^{(q)} \otimes J_{\bar{c}}^{(\bar{q})} \otimes S_{\rm LP}$$

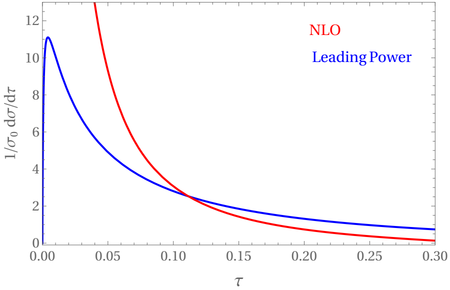
LP logarithms, $\alpha_s^n \left[\frac{\ln^m \tau}{\tau}\right]_+$, $m \leq 2n - 1$ are well studied: LL, NLL, NNLL, N3LL

[T. Becher, M. Neubert, A. Xu, 0710.0680] [T. Becher, M. Schwartz, 0803.0342]

See also Leading jet p_T and numerical codes: Ares, Ceasar. See talks by A. Banfi, J. Gaunt, E. Re.

Motivations and focus





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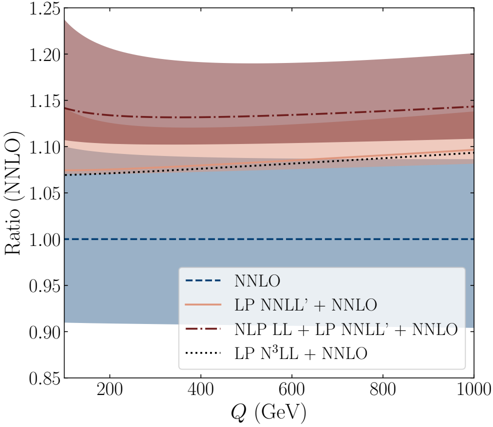
Why are power corrections important?

- Advancing all-order understanding of QFT structure
- Extending applicability of factorization
- Describing power-suppressed processes
- Improving matching and fixed order subtractions
- Getting a handle on theoretical errors
- With the high logarithmic resummation performed at leading power, its natural to ask about leading logarithms at subleading power.
  $\rightarrow$ some phenomenological studies already exist

Phenomenological motivations

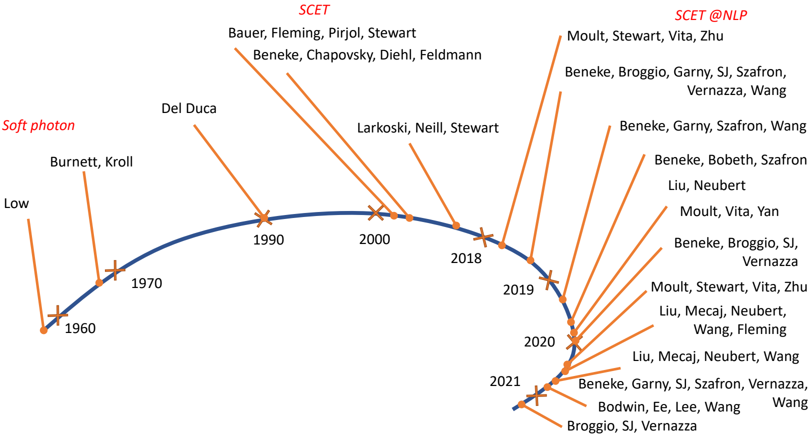
Numerical studies of inclusive Higgs and Drell-Yan production studied in [M. van Beekveld, E. Laenen, J. Sinninghe Damsté, L. Vernazza, 2101.07270].



Studies of Leadling Power NNLL′ matched to NNLO for DY and Higgs concluded that NLP LL has a comparable impact to LP N^3LL.

Towards NLP

Tremendous progress in understanding of the NLP terms has been made in the last few years.



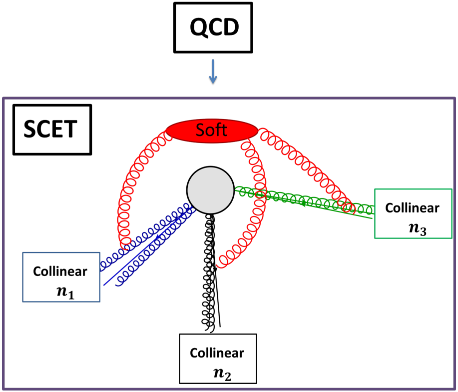
Diagrammatic approach: Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, Melville, Vernazza, White; Anastasiou, Penin; Ajjath, Mukherjee, Ravindran, Sankar, Tiwari

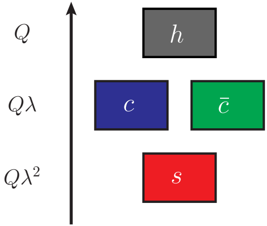
Recent progress

- First NLP resummations
  - [I. Moult, I. Stewart, G. Vita, X. Zhu, 1804.04665]
  - [M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]
  - [N. Bahjat-Abbas, D. Bonocore, J. Sinninghe Damsté, E. Laenen, L. Magnea, L. Vernazza, C. White, 1905.13710]
- End point divergences: thrust and $gg \to h$
  - [M. Beneke, M. Garny, **SJ**, J. Strohm, R. Szafron, L. Vernazza, J. Wang, 2205.04479]
  - [Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]
  - [C. Anastasiou, A. Penin, 2004.03602] [T. Liu, S. Modi, A. Penin, 2111.01820]
- Fixed order subtractions
- TMDs
- Subleading Effects in Soft-Gluon Emission at One-Loop in Massless QCD
  - $\to$ see talk by T. Schellenberger
- Flavour physics and QED corrections: $B \to \mu^+\mu^-$
  - [M. Beneke, C. Bobeth, R. Szafron, 1908.07011]
  - [T. Hurth, R. Szafron, 2301.01739]
  - [C. Cornella, M. König, M. Neubert, 2212.14430]

SCET

Soft collinear effective theory is contained within QCD. It is an EFT which describes energetic particles.





- Process specific description, with collinear sectors formed by energetic particles.
- Interactions between sectors are mediated by the soft degrees of freedom.
- Every interaction is well defined in terms of power counting - this allows for systematic expansion.

Subleading power effects

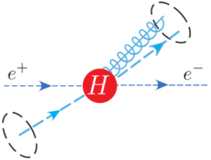
There are three sources of power suppressed corrections in SCET

- 1) Subleading power hard-scattering operators
  - More than one field present in each collinear direction
  - Soft fields present

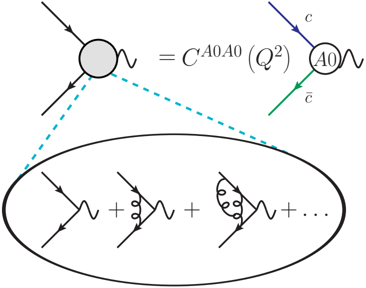
$$\psi(x) \to \underbrace{\psi_1(x) + \cdots + \psi_N(x)}_{N \text{ collinear fermion fields}} + q(x)$$

We keep the *collinear*, *anti-collinear*, and *soft* degrees of freedom. The *hard* modes are integrated out:

$$\bar{\psi}\gamma^\mu\psi = \int \prod_{i=1}^{N}\prod_{k=1}^{n_i} dt_{i_k}\; C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, \dots, t_{i_{n_i}})$$



(from I. Moult)



Subleading power effects

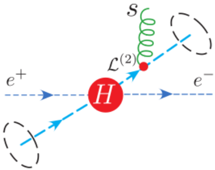
There are three sources of power suppressed corrections in SCET

- 1) Subleading power hard-scattering operators
  - More than one field present in each collinear direction
  - Soft fields present
- 2) Subleading soft-collinear interactions

$$\mathcal{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter $\lambda = \sqrt{1-z}$:

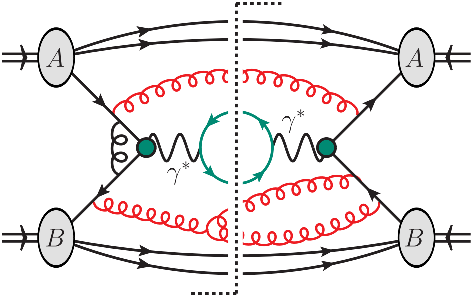
$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots \qquad \text{E.g.} \qquad \mathcal{L}_c^{(1)} = \bar{\chi}_c\, i x_\perp^\mu \left[i n_- \partial \mathcal{B}_\mu^+ \right] \frac{\not{n}_+}{2} \chi_c$$



Subleading power effects

There are three sources of power suppressed corrections in SCET

- 1) Subleading power hard-scattering operators
  - More than one field present in each collinear direction
  - Soft fields present
- 2) Subleading soft-collinear interactions
- 3) Kinematic corrections



The Foundations

The subleading power Lagrangians have been known for a long time [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358] and [D. Pirjol, I. Stewart, hep-ph/0211251]

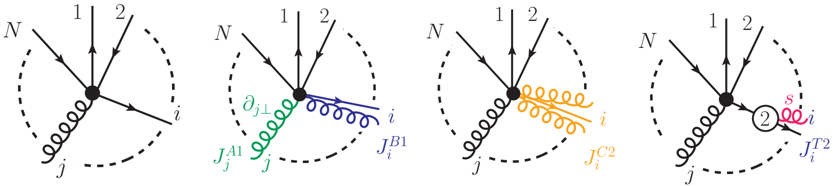
Subleading operator basis for dijets [D. Kolodrubetz, I. Moult, I. Stewart, 1601.02607] [I. Feige, D. Kolodrubetz, I. Moult, I. Stewart, G. Vita, 1703.03411] [I. Moult, I. Stewart, G. Vita, 1703.03408]

and for N-jets [M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

Building blocks (up to $\mathcal{O}(\lambda^2)$):

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

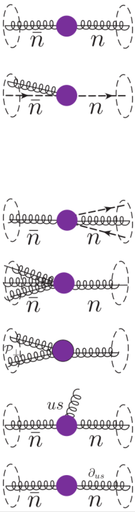
Generic leading power N-jet operator:



Many operators possible

Operator basis at subleading powers for $gg \to H$

Order	Category	Operators	# helicity configs	# of color	$\sigma_{2j}^{\mathcal{O}(\lambda^2)} \neq 0$
$\mathcal{O}(\lambda^0)$	Hgg	$O^{(0)ab}_{\mathcal{B}\lambda_1\lambda_1} = \mathcal{B}^a_{n\lambda_1}\mathcal{B}^a_{\bar{n}\lambda_1} H$	2	1	✓
$\mathcal{O}(\lambda)$	$Hq\bar{q}g$	$O^{(1)a\,\bar{\alpha}\beta}_{\mathcal{B}n,\bar{n}\lambda_1(\lambda_i)} = \mathcal{B}^a_{n,\bar{n}\lambda_1} J^{\bar{\alpha}\beta}_{n\bar{n}\,\lambda_j} H$	4	1	✓
$\mathcal{O}(\lambda^2)$	$Hq\bar{q}Q\bar{Q}$	$O^{(2)\bar{\alpha}\beta\bar{\gamma}\delta}_{qQ1(\lambda_1;\lambda_2)} = J^{\bar{\alpha}\beta}_{(q)n\lambda_1} J^{\bar{\gamma}\delta}_{(Q)\bar{n}\lambda_2} H$	4	2	
		$O^{(2)\bar{\alpha}\beta\bar{\gamma}\delta}_{qQ2(\lambda_1;\lambda_1)} = J^{\bar{\alpha}\beta}_{(q\bar{Q})n\lambda_1} J^{\bar{\gamma}\delta}_{(Q\bar{q})\bar{n}\,\lambda_1} H$	2	2	
		$O^{(2)\bar{\alpha}\beta\bar{\gamma}\delta}_{qQ3(\lambda_1;-\lambda_1)} = J^{\bar{\alpha}\beta}_{(q)n\bar{n}\lambda_1} J^{\bar{\gamma}\delta}_{(Q)n\bar{n}-\lambda_1} H$	2	2	
	$Hq\bar{q}q\bar{q}$	$O^{(2)\bar{\alpha}\beta\bar{\gamma}\delta}_{qq1(\lambda_1;\lambda_2)} = J^{\bar{\alpha}\beta}_{(q)n\lambda_1} J^{\bar{\gamma}\delta}_{(q)\bar{n}\lambda_2} H$	3	2	
		$O^{(2)\bar{\alpha}\beta\bar{\gamma}\delta}_{qq3(\lambda_1;-\lambda_1)} = J^{\bar{\alpha}\beta}_{(q)n\bar{n}\lambda_1} J^{\bar{\gamma}\delta}_{(q)n\bar{n}-\lambda_1} H$	1	2	
	$Hq\bar{q}gg$	$O^{(2)ab\,\bar{\alpha}\beta}_{\mathcal{B}1\lambda_1\lambda_2(\lambda_3)} = \mathcal{B}^a_{n\lambda_1}\mathcal{B}^b_{\bar{n}\lambda_2} J^{\bar{\alpha}\beta}_{n\,\lambda_3} H$	4	3	✓
		$O^{(2)ab\,\bar{\alpha}\beta}_{\mathcal{B}2\lambda_1\lambda_2(\lambda_3)} = \mathcal{B}^a_{\bar{n}\lambda_1}\mathcal{B}^b_{\bar{n}\lambda_2} J^{\bar{\alpha}\beta}_{n\,\lambda_3} H$	2	3	
	$Hgggg$	$O^{(2)abcd}_{4g1\lambda_1\lambda_2\lambda_3\lambda_4} = S\mathcal{B}^a_{n\lambda_1}\mathcal{B}^b_{n\lambda_2}\mathcal{B}^c_{\bar{n}\lambda_3}\mathcal{B}^d_{\bar{n}\lambda_4} H$	3	9	
		$O^{(2)abcd}_{4g2\lambda_1\lambda_2\lambda_3\lambda_4} = S\mathcal{B}^a_{n\lambda_1}\mathcal{B}^b_{\bar{n}\lambda_2}\mathcal{B}^c_{\bar{n}\lambda_3}\mathcal{B}^d_{\bar{n}\lambda_4} H$	2	9	✓
	$\mathcal{P}_\perp$	$O^{(2)a\,\bar{\alpha}\beta}_{\mathcal{P}\chi\lambda_1(\lambda_2)[\lambda_\mathcal{P}]} = \mathcal{B}^a_{n\lambda_1}\{J^{\bar{\alpha}\beta}_{\bar{n}\,\lambda_2}(\mathcal{P}^{\lambda_\mathcal{P}}_\perp)^\dagger\} H$	4	1	✓
		$O^{(2)abc}_{\mathcal{P}\mathcal{B}\lambda_1\lambda_2\lambda_3[\lambda_\mathcal{P}]} = S\,\mathcal{B}^a_{n\lambda_1}\mathcal{B}^b_{\bar{n}\lambda_2}\left[\mathcal{P}^{\lambda_\mathcal{P}}_\perp \mathcal{B}^c_{\bar{n}\lambda_3}\right] H$	4	2	✓
	Ultrasoft	$O^{(2)a\,\bar{\alpha}\beta}_{\chi(us(n))0:(\lambda_1)} = \mathcal{B}^a_{us(n)0} J^{\bar{\alpha}\beta}_{n\bar{n}\,\lambda_1} H$	2	1	
		$O^{(2)a\,\bar{\alpha}\beta}_{\chi(us(\bar{n}))0:(\lambda_1)} = \mathcal{B}^a_{us(\bar{n})0} J^{\bar{\alpha}\beta}_{n\bar{n}\,\lambda_1} H$	2	1	
		$O^{(2)\,\bar{\alpha}\beta}_{\partial\chi(us(i))\lambda_1:(\lambda_2)} = \{\partial_{us(i)\lambda_1} J^{\bar{\alpha}\beta}_{n\bar{n}\,\lambda_2}\} H$	4	1	
		$O^{(2)abc}_{\mathcal{B}(us(n))\lambda_1:\lambda_2\lambda_3} = \mathcal{B}^a_{us(n)\lambda_1}\mathcal{B}^b_{n\,\lambda_2}\mathcal{B}^c_{\bar{n}\,\lambda_3} H$	2	2	✓
		$O^{(2)abc}_{\mathcal{B}(us(\bar{n}))\lambda_1:\lambda_2\lambda_3} = \mathcal{B}^a_{us(\bar{n})\lambda_1}\mathcal{B}^b_{n\,\lambda_2}\mathcal{B}^c_{\bar{n}\,\lambda_3} H$	2	2	✓
		$O^{(2)ab}_{\partial\mathcal{B}(us(i))\lambda_1:\lambda_2\lambda_3} = \left[\partial_{us(i)\lambda_1}\mathcal{B}_{n\,\lambda_2}\right]\mathcal{B}_{\bar{n}\,\lambda_3} H$	4	1	✓



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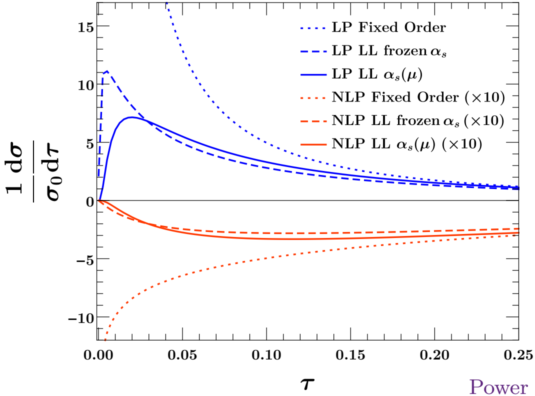
First NLP resummations

After derivation of factorization formulas, and solving the relevant RG equations, the final results for leading NLP logarithms are remarkably simple.
$H \to gg$ event shape:

$$\frac{1}{\sigma_0}\frac{d\sigma_{\rm LL}^{\rm NLP}}{d\tau} = \left(\frac{\alpha_s}{4\pi}\right) 8\, C_A\, \ln(\tau) \exp\left(-\frac{\alpha_s}{4\pi}\Gamma^g_{\rm cusp} \ln^2(\tau)\right)$$

[I. Moult, I. Stewart, G. Vita, X. Zhu, 1804.04665]



First NLP resummations

After derivation of factorization formulas, and solving the relevant RG equations, the final results for leading NLP logarithms are remarkably simple.

Drell-Yan at threshold:

$$\Delta_{\rm NLP}^{\rm LL}(z) = -\exp\left[4A(\mu_h,\mu) - 4A(\mu_s,\mu)\right] \times 4\frac{\alpha_s C_F}{\pi} \ln\frac{\mu_s}{\mu}\,\theta(1-z)$$

[M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

$$\begin{aligned}\Delta_{\rm NLP}^{\rm LL}(z,\mu) = & -\theta(1-z)\Big\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z) - L_\mu\Big] \\ & + 8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z) - 3L_\mu\ln^2(1-z) + 2L_\mu^2\ln(1-z)\Big] \\ & + 8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z) - 5L_\mu\ln^4(1-z) + 8L_\mu^2\ln^3(1-z) - 4L_\mu^3\ln^2(1-z)\Big] \\ & + \frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z) - 7L_\mu\ln^6(1-z) + 18L_\mu^2\ln^5(1-z) \\ & -20L_\mu^3\ln^4(1-z) + 8L_\mu^4\ln^3(1-z)\Big]\end{aligned}$$

where we define $L_\mu = \ln(\mu/Q)$. Comparison to [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and [D. de Florian, J. Mazzitelli, S. Moch, A. Vogt, 1408.6277]

Also confirmed using diagrammatic techniques [N. Bahjat-Abbas, D. Bonocore, J. Sinninghe Damsté, E. Laenen, L. Magnea, L. Vernazza, C. White, 1905.13710]

Beyond leading logarithms: Problems at the Endpoint

$$\int_0^{\Omega} d\omega \; \underbrace{\left(n_+p\,\omega\right)^{-\epsilon}}_{\text{collinear piece}} \; \underbrace{\frac{1}{\omega^{1+\epsilon}}\frac{1}{(\Omega-\omega)^{\epsilon}}}_{\text{soft piece}}$$

For resummation, we treat the two objects independently, and expand in ϵ prior to performing the final convolution. However, there is a problem! At two loops:

$$J_1^{(1)}(x_a n_+ p_A;\omega) \sim \alpha_s \log(\omega)$$

and

$$S_1\,(\Omega,\omega) \sim \alpha_s\,\delta(\omega) + \mathcal{O}(\alpha^2)$$

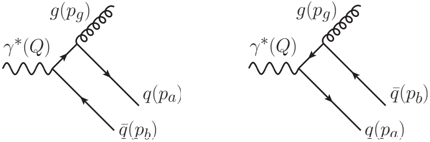
Hence, first expanding in ϵ and performing convolution after yields

$$\Delta^{dyn\,(2)}_{\mathrm{NLP-coll}}(z) = \frac{\alpha_s^2}{(4\pi)^2}\Bigg(C_F^2\left(-\frac{32}{\epsilon^2}-\frac{8}{\epsilon}\left[5-8\ln(1-z)-4\int d\omega\,\delta(\omega)\ln\left(\frac{\omega}{Q}\right)\right]\right) + C_A C_F\,\frac{40}{\epsilon} + \mathcal{O}(\epsilon^0)\Bigg)$$

A couple of issues arise. The convolution $d\omega$ integral is now divergent at the endpoint. This prohibits the application of standard RG methods.

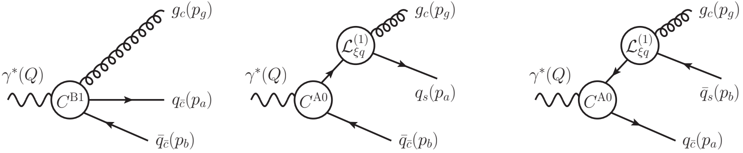
Thrust power corrections: off-diagonal channel

$$e^+e^- \to \gamma^* \to [g]_c + [q\bar{q}]_{\bar{c}}$$



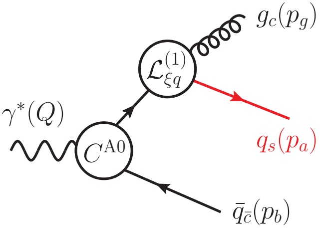
$$\bar{\psi}\gamma_\perp^\mu \psi(0) = \int dt d\bar{t}\, \widetilde{C}^{\rm A0}(t,\bar{t})\, \bar{\chi}_c(tn_+)\gamma_\perp^\mu \chi_{\bar{c}}(\bar{t}n_-) + (c \leftrightarrow \bar{c})$$
$$+ \sum_{i=1,2} \int dt d\bar{t}_1 d\bar{t}_2\, \widetilde{C}_i^{\rm B1}(t,\bar{t}_1,\bar{t}_2)\, \bar{\chi}_{\bar{c}}(\bar{t}_1 n_-)\Gamma_i^{\mu\nu} \mathcal{A}_{c\perp\nu}(tn_+)\chi_{\bar{c}}(\bar{t}_2 n_-) + \dots$$

$$\mathcal{L}_{\xi q}(x) = \bar{q}_s(x_-)\not{\mathcal{A}}_{c\perp}(x)\chi_c(x) + \text{h.c.} \qquad \Gamma_1^{\mu\nu} = \frac{\not{n}_-}{2}\gamma_\perp^\nu\gamma_\perp^\mu \,, \Gamma_2^{\mu\nu} = \frac{\not{n}_-}{2}\gamma_\perp^\mu\gamma_\perp^\nu$$



Integrand level observations

$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R dM_L} = \int d\omega d\omega' \left|C^{A0}\right|^2 \times \mathcal{J}_{\bar{c}}^{(\bar{q})} \times \mathcal{J}_c\left(\omega,\omega'\right) \otimes S_{\mathrm{NLP}}\left(\omega,\omega'\right)$$
$$+ \int dr dr'\, C^{B1}(r) C^{B1}(r')^* \otimes \mathcal{J}_{\bar{c}}^{q\bar{q}}\left(r,r'\right) \times \mathcal{J}_c^{(g)} \times S^{(g)}$$



$$\sim \int_{M_R^2/Q}^{\infty} d\omega \frac{1}{\omega^{1+\epsilon}} + \ldots \qquad\qquad \sim \int_0^1 dr \left[\frac{1}{r^{1+\epsilon}} + \frac{1}{(1-r)^{1+\epsilon}}\right]$$

We see that $S_{\mathrm{NLP}} \to S^{(g)}$ and $\mathcal{J}_{\bar{c}}^{(\bar{q})} \to \mathcal{J}_{\bar{c}}^{q\bar{q}}$. The *integrands* of the two terms should become identical.

Factorization theorem rearrangement

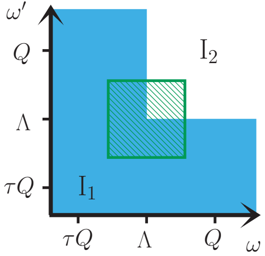
We add the following scaleless integral

$$0 = \frac{2C_F}{Q}\, f(\epsilon)\, |C^{\mathrm{A0}}(Q^2)|^2\, \widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R)\widetilde{\mathcal{J}}_{c}^{(g)}(s_L) \times \int_0^\infty d\omega d\omega'\, \frac{D^{\mathrm{B1}}(\omega Q)}{\omega}\, \frac{D^{\mathrm{B1}*}(\omega' Q)}{\omega'}\, [\![\widetilde{S}_{\mathrm{NLP}}(s_R, s_L, \omega, \omega')]\!]$$

where D^{B1} are universal objects describing the splitting of a collinear quark into a collinear gluon and a soft-collinear quark. The same object appears in the endpoint factorization theorem for $H \to gg$ [Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

We split it by introducing a parameter Λ: $0 = I_1 + I_2$.

Then subtract I_1 from the B-type term and I_2 from the A-type term.



Resummed result for NLP-LL gluon thrust

Putting everything together

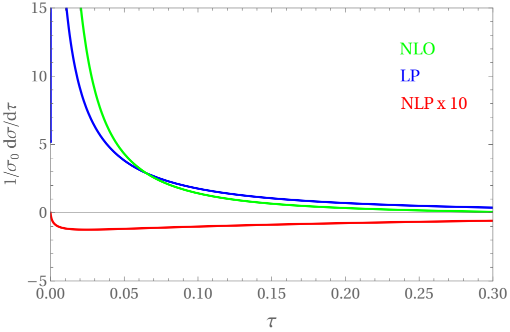
$$\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}|_{\mathrm{LL}} = 2\cdot\frac{2C_F}{Qs_R}\frac{\alpha_s(\mu_c)}{4\pi}\exp\left[4C_F S\left(\mu_h,\mu_{\bar{c}}\right)+4C_A S\left(\mu_s,\mu_c\right)\right]$$
$$\times\left(\frac{Q^2}{\mu_h^2}\right)^{-2C_F A(\mu_h,\mu_{\bar{c}})}\left(\frac{1}{s_L s_R e^{2\gamma_E}\mu_s^2}\right)^{-2C_A A(\mu_s,\mu_c)}$$
$$\times\int_\sigma^Q\frac{d\omega}{\omega}\exp\left[4\left(C_F-C_A\right)S\left(\mu_{s\Lambda},\mu_{h\Lambda}\right)\right]\left(\frac{\omega}{s_R e^{\gamma_E}\mu_{s\Lambda}^2}\right)^{-2(C_F-C_A)A(\mu_{s\Lambda},\mu_{h\Lambda})}$$
$$\times\left(s_R e^{\gamma_E}Q\right)^{2C_F A(\mu_{h\Lambda},\mu_{\bar{c}})+2C_A A(\mu_c,\mu_{h\Lambda})}$$

And in the double logarithmic limit, the result of previous works is recovered

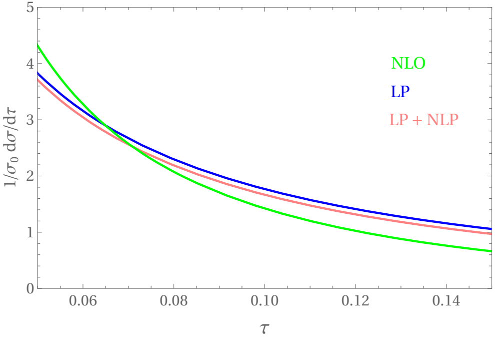
$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}|_{\mathrm{DL}} = \frac{C_F}{C_F-C_A}\frac{1}{\ln(1/\tau)}e^{-\frac{\alpha_s C_A}{\pi}\ln^2\tau}\left\{1-e^{-\frac{\alpha_s}{\pi}(C_F-C_A)\ln^2\tau}\right\}$$

[I. Moult, I. Stewart, G. Vita, H. X. Zhu, 1910.14038] [M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 2008.04943]

Numerics



NLO
LP
NLP x 10
$1/\sigma_0\, d\sigma/d\tau$
τ
15
10
5
0
−5
0.00
0.05
0.10
0.15
0.20
0.25
0.30



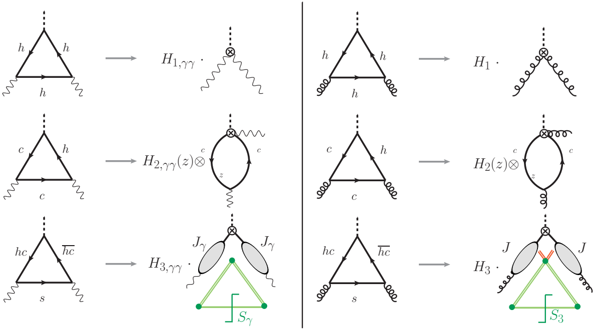
NLO
LP
LP + NLP
$1/\sigma_0\, d\sigma/d\tau$
τ
5
4
3
2
1
0
0.06
0.08
0.10
0.12
0.14

Power corrections in $h \to \gamma\gamma(gg)$

Significant progress in understanding of the structure of large logarthimic corrections in $h \to \gamma\gamma$ and $gg \to h$ mediated by a b quark loop.

[Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2212.10447] [Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018] Z. Liu, B. Mecaj, M. Neubert, X. Wang, 2009.06779] [Z. Liu, B. Mecaj, M. Neubert, X. Wang, 2009.04456] [Z. Liu, B. Mecaj, M. Neubert, X. Wang, S. Fleming, 2005.03013] [Z. Liu, M. Neubert, 1912.08818]

Amplitude level studies, $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$ $\to$ rapidity divergences present.



Refactorization

At the endpoint, the refactorization conditions read

Talk by Philipp

$$[\![H_2(z)]\!] \equiv \lim_{z\to 0} H_2(z) = H_3 \frac{J(zM_h^2)}{z}$$

$$[\![O_2(z)]\!] \equiv \lim_{z\to 0} \langle\gamma\gamma|\, O_2(z)\, |h\rangle = \int_0^\infty \frac{d\ell_+}{\ell_+} J(M_h\, \ell_+)\, S(\ell_+\ell_-)\,,$$

Then $[\![H_2(z)]\!] \otimes [\![O_2(z)]\!]$ has the same integrand with $H_3\, O_3$,
but has different integral region

$$\int_0^1 dz [\![H_2(z)]\!][\![O_2(z)]\!] = H_3 \int_0^{M_h} \frac{d\ell^-}{\ell^-} J(M_h\ell^-) \int_0^\infty \frac{d\ell^+}{\ell^+} J(M_h\ell^+) S(\ell^+\ell^-)$$

useful to do subtraction

$$\int_0^1 dz H_2(z) O_2(z) \to \int_0^1 dz\, (H_2(z)O_2(z) - [\![H_2(z)]\!][\![O_2(z)]\!]) + \int_0^1 dz [\![H_2(z)]\!][\![O_2(z)]\!]$$

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Slide from Z. Liu.

Renormalized Factorization Theorem

T1: Absorb the contributions from infinity-bin and mismatch

T2: Divergences at the endpoints are subtracted

$$\mathcal{M}_b = H_1(\mu)\,\langle O_1(\mu)\rangle$$
$$+\,2\int_0^1 dz\left[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right]$$
$$+\,g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}$$

T3: Cutoff is involve to regulated endpoint divergences

$$\mathcal{M}_b^{\rm NLL} = \frac{N_c\alpha_b}{\pi}\,\frac{y_b(M_h)}{\sqrt{2}}\,m_b\,\varepsilon_\perp^*(k_1)\cdot\varepsilon_\perp^*(k_2)\,\frac{L^2}{2}\left\{\sum_{n=0}^{\infty}\frac{2\Gamma(n+1)}{\Gamma(2n+3)}\left(-\frac{C_F\alpha_s(M_h)}{2\pi}L^2\right)^n\right.$$
$$\left. -\frac{1}{L}\sum_{n=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(2n+2)}\left(-\frac{C_F\alpha_s(M_h)}{2\pi}L^2\right)^n\left[3-\beta_0\,\frac{\alpha_s(M_h)}{2\pi}\,L^2\,\frac{n(n+1)}{(2n+2)(2n+3)}\right]\right\}$$

Slide from Z. Liu.

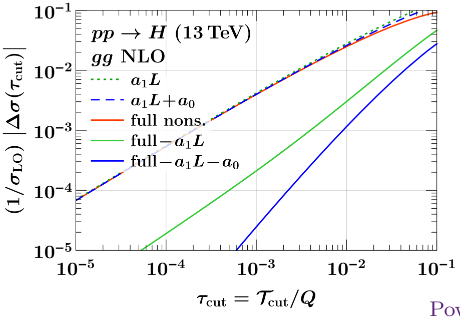
In agreement with [C. Anastasiou, A. Penin, 2004.03602] studies, which were extended to NNLP [T. Liu, S. Modi, A. Penin, 2111.01820].

Fixed order subtractions

Power corrections are useful in improving fixed order subtraction schemes.
At NNLO completely analytic calculations are not always possible → numerical answers → subtraction schemes required: FKS, CS, N-jettiness..

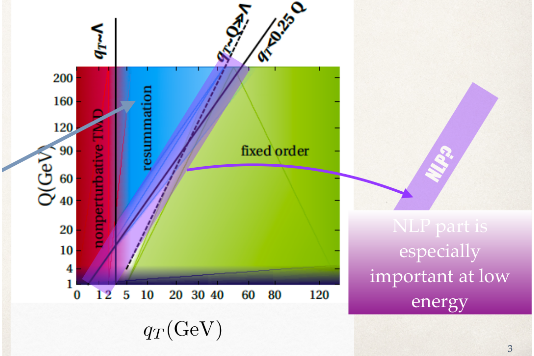
$$\begin{aligned}\sigma(X) &= \int^{\tau_{cut}} d\mathcal{T}_N \frac{d\sigma(X)}{d\mathcal{T}_N} + \int_{\tau_{cut}} d\mathcal{T}_N \frac{d\sigma(X)}{d\mathcal{T}_N} \\ &= \sigma^{\rm sub}(\tau_{cut}) + \int_{\tau_{cut}} d\mathcal{T}_N \frac{d\sigma(X)}{d\mathcal{T}_N} + \underbrace{[\sigma(\tau_{cut}) - \sigma^{\rm sub}(\tau_{cut})]}_{\Delta\sigma(\tau_{cut})}\end{aligned}$$

[J. Gaunt, M. Stahlhofen, F. Tackmann, J. Walsh, 1505.04794] [Ebert, Moult, Stewart, Tackmann, Vita, Zhu, 1807.10764] [R. Boughezal, A. Isgrò, F. Petriello, 1802.00456]



Subleading TMDs

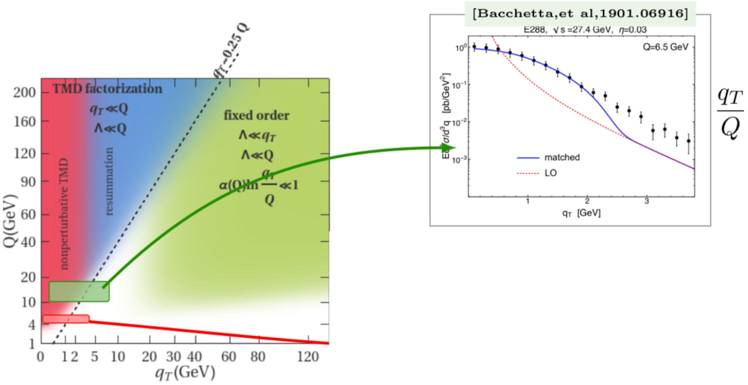
Transverse-momentum-dependent distributions originate from correlation functions with transverse separation (unlike usual PDFs with separation only along light-cone direction)
It is about understanding better the structure of the proton



By now a huge body of work, see the [TMD Handbook, 2304.03302] with almost 500 pages! Also contains subleading chapter. [Picture from I. Scimemi]

TMD power corrections

[J. Collins, L. Gamberg, A. Prokudin, T.C. Rogers, N. Sato, B. Wang, 1605.00671]



[A.Vladimirov, talk at MITP]

Interest in the community! [I. Balitsky, 2012.01588 + 2105.13391] [M. Ebert, A. Gao, I. Stewart, 2112.07680] [S. Rodini, A. Vladimirov 2204.03856] Also considered recently for the case of TMD with jets [R. del Castillo, M. Jaarsma, I. Scimemi, W. Waalewijn, 2307.13025]

Summary

- There has been significant progress in understanding subleading power factorization theorems in the last years.
- The structure of the factorisation formula is more involved than the leading power counter part. New functions appear!
- First NLP LL resummations performed
- We have uncovered issues preventing application of standard RG methods.
- Succeeded in refactorizing and resumming to LL accuracy the off-diagonal “gluon” thrust at cross section level
- NLP NLL results obtained at amplitude level
- Power corrections useful for fixed order calculations
- Away from the LHC, power corrections even more important at future EIC collider where energies are lower.
- Interesting conceptual challenges ahead. Important to understand from the point of view of gauge theories, as well as for delivering precise theoretical predictions.

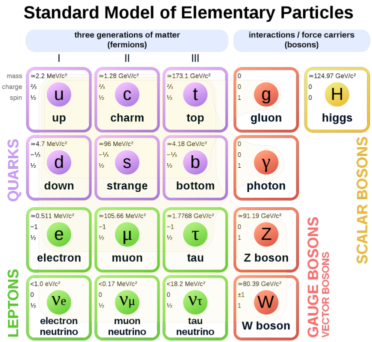
Thank you!

Auxiliary slides

Developing our understanding and striving for precise predictions

Standard Model

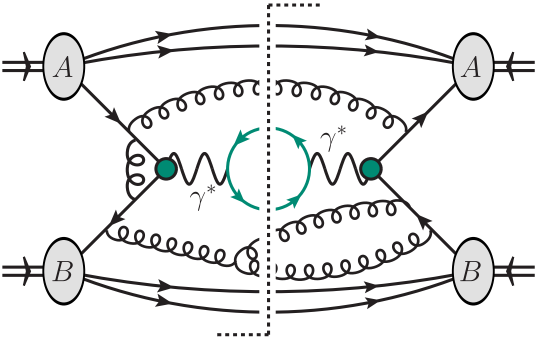
However, we know it is not the complete picture yet



Motivations and focus: High perturbative precision

$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$

Recently calculated to N^{3}LO
[C. Duhr, F.Dulat, B. Mistlberger, 2001.07717]



Motivations and focus: Beyond fixed-order

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

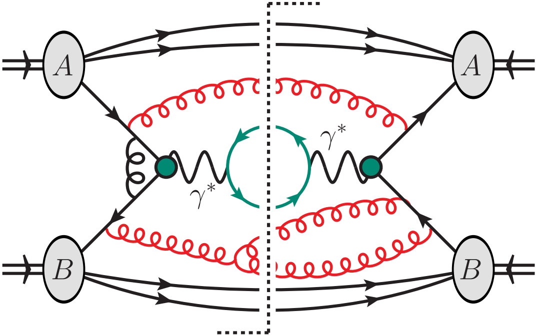
Recently calculated to N^{3}LO
[C. Duhr, F.Dulat, B. Mistlberger, 2001.07717]

Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \to 1$$

Define power counting parameter λ:

$$\lambda = \sqrt{1-z}$$



Schematic form for production cross-sections near threshold, $z \to 1$:

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \right]$$

Motivations and focus: The Drell-Yan process

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

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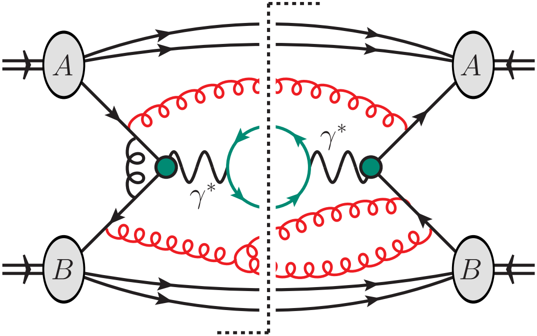

The general idea is to factorise the physics appearing at different scales:

$$\sigma \sim H \otimes J \otimes \ldots \otimes J \otimes S$$

and solve RG equations for each object to sum the large logarithms → SCET

Motivations and focus: The Drell-Yan process

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

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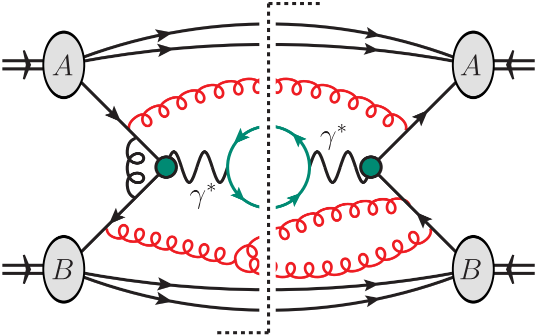

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Motivations and focus: The Drell-Yan process

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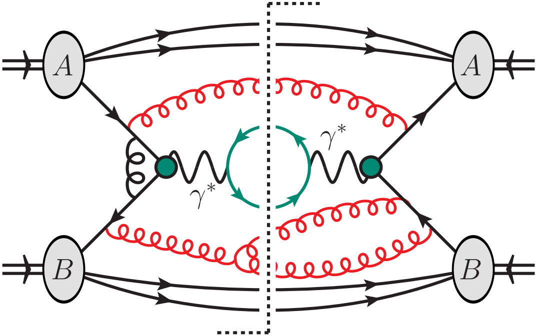

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SCET introduction: N-jet operator basis

Generic N-jet operator has the form:
[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

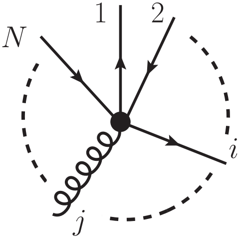
$$J = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k} \; C(\{t_{i_k}\}) \prod_{i=1}^{N} \; J_i(t_{i_1}, t_{i_2}, ..., t_{i_{n_i}})$$

where the Js are constructed by multiplying collinear gauge invariant building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad\qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

by acting on these with derivatives $i\partial^\mu_{\perp i} \sim \lambda$, and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

Generic leading power N-jet operator:



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$$J = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k} \; C(\{t_{i_k}\}) \prod_{i=1}^{N} \; J_i(t_{i_1}, t_{i_2}, ..., t_{i_{n_i}})$$

We adopt the notation: $J_i^{An}, J_i^{Bn}, J_i^{Cn}, J_i^{Tn}$ where:

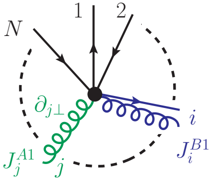
- A, B, C... refers to number of fields in a given collinear direction
- n is the power of λ suppression (relative to $A0$) in a given sector.

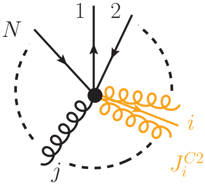
For example, up to $\mathcal{O}(\lambda^2)$ we can construct $J_i^{A1}, J_i^{B1}, J_i^{C2}, J_i^{T2}$ respectively:

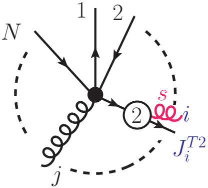
$$i\partial^{\mu}_{\perp i}\chi_i, \quad \chi_i(t_{i_1})\mathcal{A}^{\mu}_{i\perp}(t_{i_2}), \quad \chi_i(t_{i_1})\mathcal{A}^{\nu}_{i\perp}(t_{i_2})\mathcal{A}^{\mu}_{i\perp}(t_{i_3}), \quad i\int d^4z\,\mathbf{T}\left[\chi_i(t_{i_1})\mathcal{L}^{(2)}(z)\right]$$

And $\mathcal{O}(\lambda^2)$ N-jet operators are

$J_1^{A0} \ldots J_i^{B1} J_j^{A1} \ldots J_N^{A0}$, $\quad J_1^{A0} \ldots J_i^{C2} \ldots J_N^{A0}$, $\quad J_1^{A0} \ldots J_i^{T2} \ldots J_N^{A0}, \ldots$







SCET introduction: Decoupling

In a collinear sector

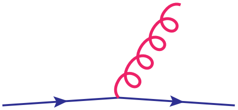
$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots$$

The decoupling transformation, $\xi_c \to Y_+ \xi_c^{(0)}$ and $A_c^\mu \to Y_+ A_c^{(0)\mu} Y_+^\dagger$, separates the soft and collinear sectors at LP [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

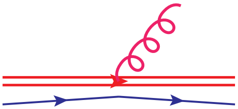
$$\mathcal{L}_c^{(0)} \ni \bar{\xi}_c \left(in_- D_c + g_s n_- A_s\right) \frac{\not{n}_+}{2} \xi_c = \bar{\xi}_c^{(0)}\, in_- D_c^{(0)} \frac{\not{n}_+}{2} \xi_c^{(0)}$$

where

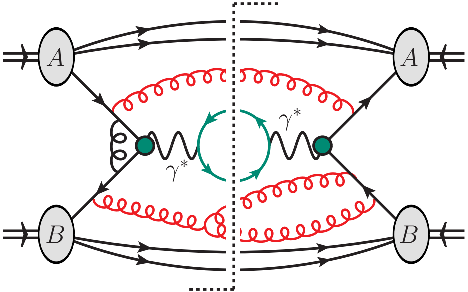
$$Y_\pm(x) = \mathbf{P} \exp\left[ig_s \int_{-\infty}^{0} ds\, n_\mp A_s(x + sn_\mp)\right]$$

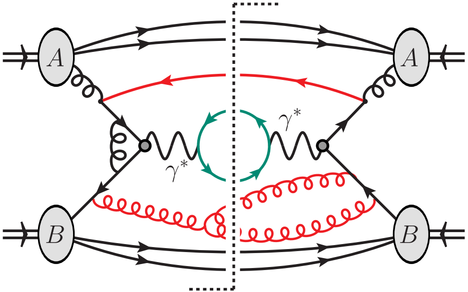


$$\text{LP} \sim \mathbf{T}^A \frac{n_-^\mu}{n_- k}$$



Factorization of the Drell-Yan process

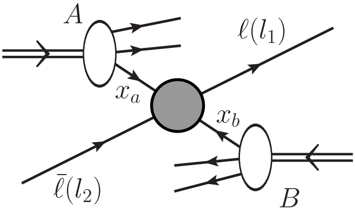




Drell-Yan Process at Threshold: Leading Power

The Drell-Yan Process

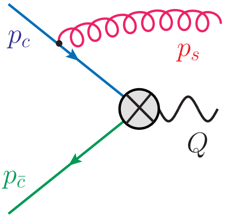
$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)\,] + X(p_X)$$

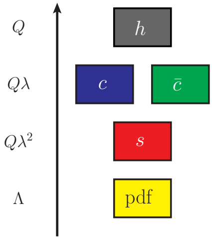


$$z = \frac{Q^2}{\hat{s}} \to 1 \qquad \lambda = \sqrt{(1-z)}$$

$$p_c = (n_+p_c, n_-p_c, p_{c\perp}) \sim Q(1, \lambda^2, \lambda)$$
$$p_{\bar{c}} = (n_+p_{\bar{c}}, n_-p_{\bar{c}}, p_{\bar{c}\perp}) \sim Q(\lambda^2, 1, \lambda)$$
$$p_s = (n_+p_s, n_-p_s, p_{s\perp}) \sim Q(\lambda^2, \lambda^2, \lambda^2)$$

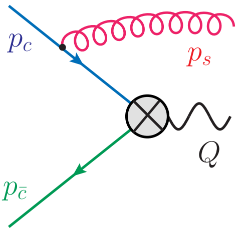
$$Q^2\lambda^2 = Q^2(1-z) \gg \Lambda^2_{\rm QCD} \qquad p_{c-\rm PDF} \sim (Q, \Lambda^2/Q, \Lambda)$$





The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, J_\mu^{A0}(t,\bar{t})$$



$$J_\mu^{A0}(t,\bar{t}) = \bar{\chi}_{\bar{c}}(\bar{t}n_-)\,\gamma_{\perp\mu}\,\chi_c(tn_+)$$

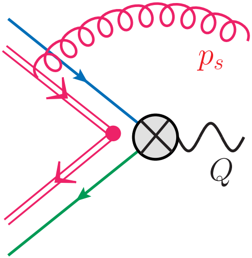
The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t}\,)\, J_\mu^{A0}(t,\bar{t}\,)$$

Use decoupled fields $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$.

Leading power current becomes

$$J_\mu^{A0}(t,\bar{t}\,) = \bar{\chi}_{\bar{c}}^{(0)}(\bar{t}n_-)Y_-^\dagger(0)\gamma_{\perp\mu}Y_+(0)\chi_c^{(0)}(tn_+)$$



The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, J^{A0}_\mu(t,\bar{t})$$

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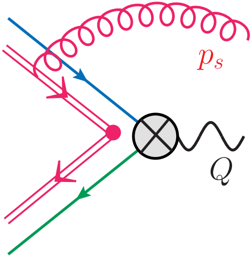


Consider the matrix element:

$$\langle X|\bar{\psi}\gamma^\mu\psi(0)|A(p_A)B(p_B)\rangle = \int \frac{d(n_+p_a)}{2\pi}\frac{d(n_-p_b)}{2\pi} C^{A0}(n_+p_a, -n_-p_b)$$

$$\times\langle X_{\bar{c}}^{\rm PDF}|\hat{\bar{\chi}}_{\bar{c}}^{\rm PDF}(n_-p_b)|B(p_B)\rangle\gamma_\perp^\mu\langle X_c^{\rm PDF}|\hat{\chi}_c^{\rm PDF}(n_+p_a)|A(p_A)\rangle$$

$$\times\langle X_s|\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\right]|0\rangle$$

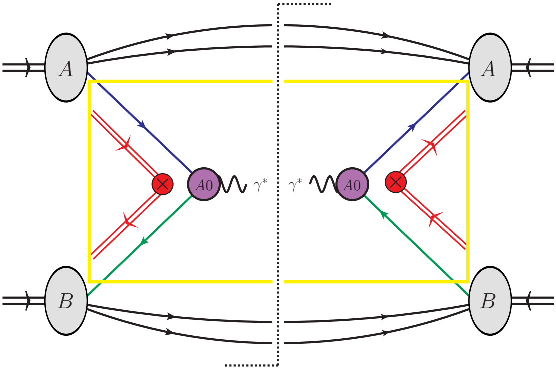
The states factorize: $\langle X| = \langle X_{\bar{c}}^{\rm PDF}|\,\langle X_c^{\rm PDF}|\,\langle X_s|$. The threshold collinear mode does not appear. Only the PDF collinear mode with scaling

$$p_{c-\rm PDF} \sim (Q, \Lambda^2_{\rm QCD}/Q, \Lambda_{\rm QCD})$$

The Drell-Yan Proccess: Leading Power Cross-Section

The leading power factorisation formula has the following form:

$$\frac{d\sigma_{\rm DY}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\rm LP} + \hat{\sigma}_{ab,dyn}^{\rm NLP} + \hat{\sigma}_{ab,kin}^{\rm NLP} + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$



$$\hat{\sigma}^{\rm LP}(z) = Q \left|C^{A0}(Q^2)\right|^2 S_{\rm DY}(\Omega)$$

[G. Sterman 1987] [S. Catani, L. Trentadue 1989] [G. P. Korchemsky G. Marchesini, 1993]
[S. Moch, A. Vogt, hep-ph/0508265] [T. Becher, M. Neubert, G. Xu, 0710.0680]

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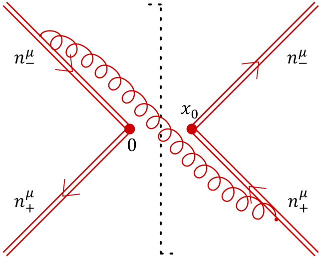
$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{1}{N_c}\, {\rm Tr}\, \langle 0 | \bar{\mathbf{T}}(Y_+^\dagger(x^0) Y_-(x^0))\, \mathbf{T}(Y_-^\dagger(0) Y_+(0)) | 0 \rangle$$



Note, inserting complete set of states, and performing $\int dx_0$

$$\delta(\Omega - 2E_X)$$

Drell-Yan Process at Threshold: Next-to-Leading Power

Factorisation formula at NLP

First let us compare leading power and *next*-to-leading power cross-sections schematically:

$$\frac{d\sigma_{\rm DY}}{dQ^2} \sim \sum_{a,b}\int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\left(\,\hat{\sigma}^{\rm LP}_{ab} + \hat{\sigma}^{\rm NLP}_{ab,dyn} + \hat{\sigma}^{\rm NLP}_{ab,kin} + \ldots\right) + \mathcal{O}\left(\Lambda/Q\right)$$

We have discussed the LP piece

$$\hat{\sigma}^{\rm LP}(z) = Q\,|C^{A0}(Q^2)|^2\; S_{\rm DY}(\Omega)$$

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We have discussed the LP piece

$$\hat{\sigma}^{\rm LP}(z) = Q\,|C^{A0}(Q^2)|^2\; S_{\rm DY}(\Omega)$$

and as will be shown the NLP is given by

$$\hat{\sigma}^{\rm NLP}(z) = \sum_{\rm terms}\,[C \otimes J \otimes \bar{J}\,]^2 \otimes S$$

- C is the hard Wilson matching coefficient
- S is the generalised soft function
- Js are the collinear functions

Let us now motivate the emergence of this structure at next-to-leading power.

Collinear functions at NLP

Beyond LP, the decoupling transformation does not remove soft-collinear interactions in the Lagrangian.

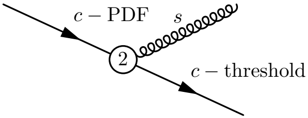
Consider an example of subleading SCET Lagrangian:

[M. Beneke and Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}^{(2)}_{2\xi} = \frac{1}{2}\bar{\chi}^{(0)}_c z^\mu_\perp z^\rho_\perp \left[i\partial_\rho i n_- \partial \mathcal{B}^+_\mu(z_-) \right] \frac{\not{n}_+}{2} \chi^{(0)}_c \qquad \text{where} \qquad \mathcal{B}^\mu_\pm = Y^\dagger_\pm \left[i D^\mu_s Y_\pm \right]$$

The subleading power Lagrangian terms enter the basis through time-ordered product operators

$$\left(J^{T2}_{A0,2\xi}(t) \right)^\mu = i \int d^4 z \, \mathbf{T} \left[J^\mu_{A0}(t) \, \mathcal{L}^{(2)}_{2\xi}(z) \right]$$



Collinear functions at NLP

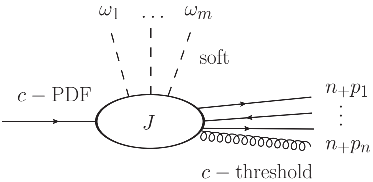
PDF collinear modes can be radiated into the final state:
$p_c \sim Q(1, \lambda^2, \lambda)$ and $p_{c-\text{PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$

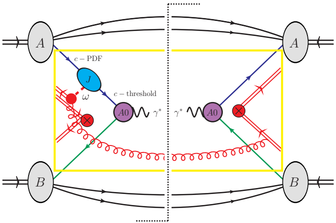
$$i \int d^4z \, \mathbf{T}\Big[\chi_c\,(tn_+)\;\mathcal{L}^{(2)}(z)\Big] = 2\pi \sum_i \int du \int dz_- \,\tilde{J}_i\,(t, u; z_-)\,\chi_c^{\text{PDF}}(un_+)\,\mathfrak{s}_i(z_-)$$

$$\mathfrak{s}_i(z_-) \in \left\{ \frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\,,\; \frac{i\partial_{[\mu_\perp}}{in_-\partial}\mathcal{B}^+_{\nu_\perp]}(z_-)\,,\; \frac{1}{(in_-\partial)}\left[\mathcal{B}^+_{\mu_\perp}(z_-), \mathcal{B}^+_{\nu_\perp}(z_-)\right], \ldots \right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]





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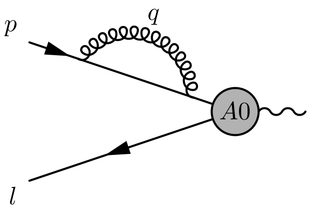
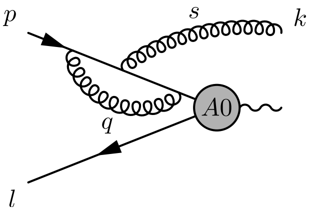
[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

In the diagrammatic approach, soft emissions from jets are captured by the *radiative jet function* [V. Del Duca, 1990], [D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza, C.D. White, 1503.05156], [D. Bonocore, E. Laenen, L. Magnea, L. Vernazza, C.D. White, 1610.06842]

Absence of collinear functions at LP





The purely threshold-collinear loops are scaleless and vanish.

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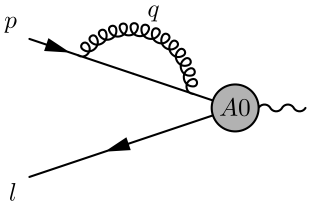
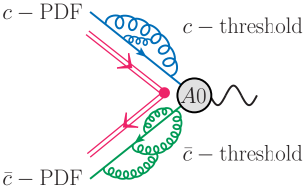




The purely threshold-collinear loops are scaleless and vanish.

At leading power, we can apply the decoupling transformation $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$ which removed the soft-collinear interactions from the Lagrangian.

[C. Bauer, D. Pirjol, and I. Stewart, hep-ph/0109045]

The threshold-collinear modes can be identified with c-PDF modes, $\chi_c \to \chi_c^{\mathrm{PDF}}$

$$\chi_c(tn_+) = \int du\, \tilde{J}(t,u)\, \chi_c^{\mathrm{PDF}}(un_+) \qquad \tilde{J}(t,u) = \delta(t-u)$$

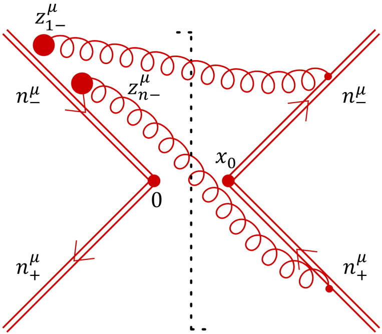
Generalised soft functions

The necessary presence of the time-ordered products at NLP gives rise to the concept of generalized soft functions with explicit gauge fields. Schematically we have

$$S(\Omega,\{\omega\}) = \int \frac{dx^0}{4\pi} e^{ix^0\Omega/2} \left(\prod_{j=1}^{n} \int \frac{d(n_+z_j)}{4\pi}\, e^{-i\omega_j(n_+z_j)/2} \right)$$
$$\times \,\mathrm{Tr}\, \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \times \mathcal{L}_s^{(n)}(z_{1-}) \times \ldots \times \mathcal{L}_s^{(n)}(z_{n-}) \right] |0\rangle$$

$\mathcal{L}_s^{(n)}(z_{j-})$ contains $\mathcal{B}^+_{\perp\nu}(z_{j-})$ fields, not only Wilson lines.

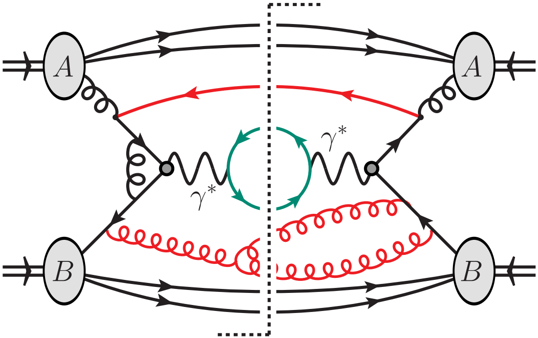
[M. Beneke , F. Campanario, T. Mannel, B.D. Pecjak, hep/0411395]



Factorisation formula at NLP

The off-diagonal DY process

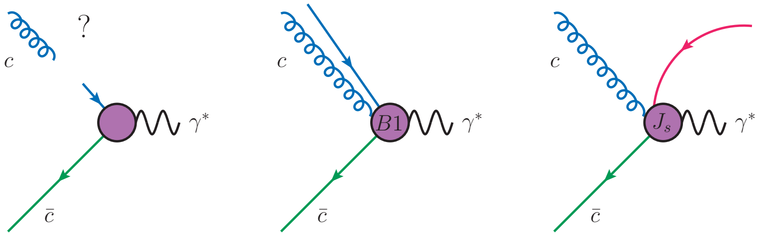
Now we consider initial state gluon from a proton A, which must be converted into a collinear quark via an emission of a soft antiquark.



In the EFT picture

This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\rho\psi(0) = \sum_{m_1,m_2} \int \{dt_k\}\,\{d\bar{t}_{\bar{k}}\}\,\widetilde{C}^{\,m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)\,J_s(0)\,J_\rho^{\,m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)$$



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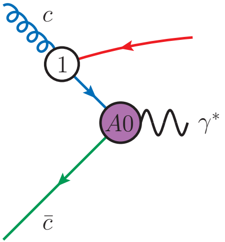
$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t}\,)\ \underbrace{\bar{\chi}_{\bar{c}}(\bar{t}n_-)\,\gamma_{\perp\mu}\,\chi_c(tn_+)}_{J^{A0}_\mu(t,\bar{t}\,)}$$



Use an insertion of a subleading power Lagrangian term with a soft quark, starting at $\mathcal{O}(\lambda)$

$$\left(J^{T2}_{A0,\xi q}(t)\right)^\mu = i\int d^dz\,\mathbf{T}\left[\chi_c\,(tn_+)\ \mathcal{L}^{(1)}_{\xi q}(z)\right] \qquad \mathcal{L}^{(1)}_{\xi q} = \bar{q}_+ \not{\mathcal{A}}_{c\perp}\chi_c + \text{h.c.}$$

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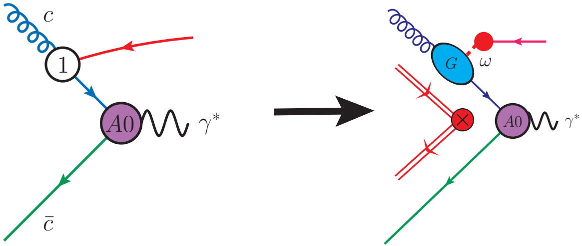

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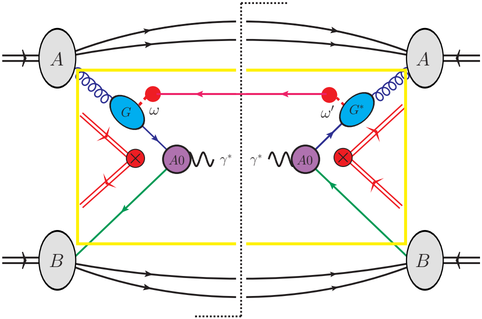
NLP factorization formula for Drell-Yan: $g\bar{q}$

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \qquad \Delta_{g\bar{q},\mathrm{NLP}}(z) = \Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z)$$

where [**SJ**, PhD thesis][A. Broggio, **SJ**, L. Vernazza, 2306.06037]

$$\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z) = 8H(Q^2) \int d\omega\, d\omega'\, G^*_{\xi q}(x_a n_+ p_A;\omega')\, G_{\xi q}(x_a n_+ p_A;\omega)\, S(\Omega,\omega,\omega')$$



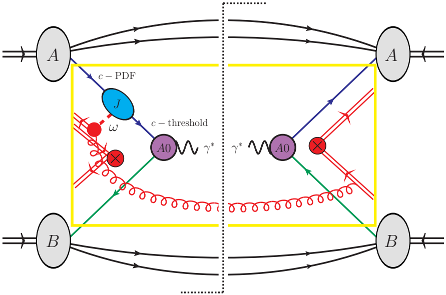
Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\Delta^{dyn}_{\rm NLP}(z) = -2\,Q\left[\left(\frac{\not{n}_-}{4}\right)\gamma_{\perp\rho}\left(\frac{\not{n}_+}{4}\right)\gamma^{\rho}_{\perp}\right]_{\beta\gamma}\int d(n_+p)\; C^{\,A0}\,(n_+p, x_b n_-p_B)$$

$$\times\, C^{*A0}\,(x_a\, n_+p_A,\, x_b n_-p_B)\sum_{i=1}^{4}\int\{d\omega_j\}\; J_i\,(n_+p, x_a\, n_+p_A; \{\omega_j\})\; S_i(\Omega; \{\omega_j\}) + \text{h.c.}$$



For comparison, LP result is:

$$\Delta_{\rm LP}(z) = |C(Q^2)|^2\; Q\; S_{\rm DY}(Q(1-z))$$

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$$\times\, C^{*A0}\,(x_a\, n_+p_A,\, x_b n_- p_B) \sum_{i=1}^{4} \int \{d\omega_j\}\; J_i\,(n_+p, x_a\, n_+p_A; \{\omega_j\})\; S_i(\Omega; \{\omega_j\}) + \text{h.c.}$$

where the *generalised* soft functions have the structure:

$$\widetilde{S}_i\,(x; \{\omega_j\}) = \int \left\{ \frac{dz_{j-}}{2\pi} \right\}\; e^{-i\omega_j z_{j-}} \times \frac{1}{N_c} \text{Tr}\langle 0|\bar{\mathbf{T}}\left(\left[Y_+^\dagger\, Y_-\right](x)\right) \mathbf{T}\left(\left[Y_-^\dagger\, Y_+\right](0)\, \mathfrak{s}_i\,(\{z_{j-}\})\right)|0\rangle$$

with

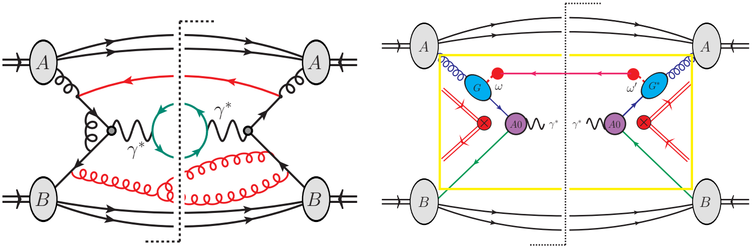
$$\mathfrak{s}_i(\{z_{j-}\}) \;\in\; \Bigg\{ \frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_{1-}),\, \frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_{1-}),\left[in_-\partial\mathcal{B}^+_{\mu_\perp}(z_{1-})\right]\right],$$

$$\frac{1}{(in_-\partial)}\mathcal{B}^+_{\mu_\perp}(z_{1-})\mathcal{B}^+_{\nu_\perp}(z_{2-}),\, \frac{1}{(in_-\partial)^2} q_{+\sigma}(z_{1-})\bar{q}_{+\lambda}(z_{2-}) \Bigg\}$$

Validation

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991]

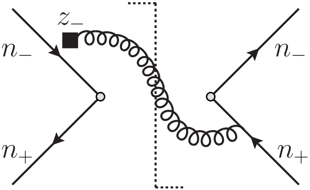
$$\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z) \; = 8H(Q^2) \int d\omega \, d\omega' \, G^*_{\xi q}(x_a n_+ p_A; \omega') \, G_{\xi q}(x_a n_+ p_A; \omega) \, S(\Omega, \omega, \omega')$$



Soft function at NLO

The only generalised soft function with non-vanishing contributions at $\mathcal{O}(\alpha_s)$ is

$$S_1(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi}\, e^{ix^0\Omega/2 - i\omega(n_+z)/2} \times \frac{1}{N_c} \,\mathrm{Tr}\, \langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right] \mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\, \frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right]|0\rangle$$



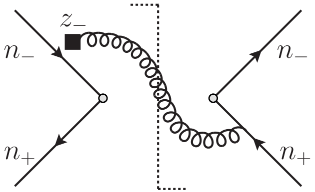
$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\,\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\; e^{iz_-k}$$

$\rightarrow \delta(n_-k - \omega)$

Soft function at NLO

The only generalised soft function with non-vanishing contributions at $\mathcal{O}(\alpha_s)$ is

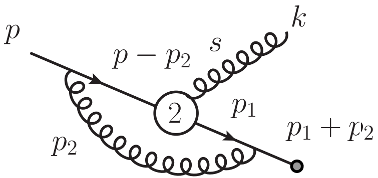
$$S_1(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi}\, e^{ix^0\Omega/2 - i\omega(n_+z)/2}$$
$$\times \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right]|0\rangle$$



$$S_1^{(1)}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\,\frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]}\,\frac{1}{\omega^{1+\epsilon}}\,\frac{1}{(\Omega-\omega)^\epsilon}\,\theta(\omega)\theta(\Omega-\omega)\ .$$

After $\int d\omega$ the NLO cross section $\Delta_{\mathrm{NLP}}^{dyn(1)}$ is correctly reproduced.

Collinear functions at one-loop



Use NLP Feynman rules to calculate these new objects up to $\mathcal{O}(\alpha_s)$. [M. Beneke, M. Garny, R. Szafron, J. Wang, 1808.04742]

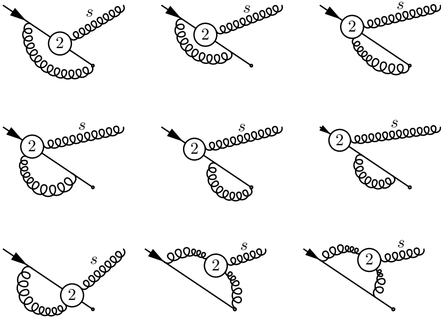
Some non-standard features present!

$$X^\mu = \partial^\mu \left[(2\pi)^d \delta^{(d)} \left(\sum p_{\text{in}} - \sum p_{\text{out}}\right)\right]$$

$$\mathcal{O}(\lambda^2): \qquad i g_s \mathbf{T}^A A^{\rho\nu}(k,p,p') \frac{\not{n}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu})$$

$$A^{\rho\nu}(k,p,p') \equiv \frac{1}{2}\left[(n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left(\frac{\not{p}'_\perp}{n_+ p'}\gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p}\right)\right]$$

Collinear functions at one-loop



$$J_1\left(n_+q, n_+p;\, \omega\right) = -\frac{1}{n_+p}\delta(n_+q - n_+p) + 2\,\frac{\partial}{\partial n_+q}\delta(n_+q - n_+p)$$

$$+\frac{\alpha_s}{4\pi}\,\frac{1}{(n_+p)}\left(\frac{n_+p\,\omega}{\mu^2}\right)^{-\epsilon}\frac{e^{\epsilon\gamma_E}\,\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]}$$

$$\times\left(C_F\left(-\frac{4}{\epsilon}+3+8\epsilon+\epsilon^2\right) - C_A\left(-5+8\epsilon+\epsilon^2\right)\right)\delta(n_+q - n_+p)$$

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

Validation at Fixed Order: $\Delta^{dyn\,(2)}_{\rm NLP-coll}$ piece

Focus on one piece of the factorization formula

$$\Delta^{dyn\,(2)}_{\rm NLP-coll}(z) \quad = \quad 4Q \int d\omega\, J^{(1)}_{1,1}(x_a\, n_+ p_A;\omega)\, S^{(1)}_1(\Omega;\omega)\,.$$

The factorization formula is valid for unrenormalized objects. Performing the convolution in d - dimensions reproduces fixed NNLO result:

$$\Delta^{(2)}_{\rm NLP-coll} = \frac{\alpha_s^2}{(4\pi)^2}\Bigg(C_A C_F\left(\frac{20}{\epsilon} - 60\log(1-z) + 8 + \mathcal{O}(\epsilon^1)\right)$$
$$+C_F^2\left(\frac{-16}{\epsilon^2} - \frac{20}{\epsilon} + \frac{48}{\epsilon}\log(1-z) + 60\log(1-z) - 72\log^2(1-z) + \mathcal{O}(\epsilon^1)\right)\Bigg)$$

after we set the scale to hard. In agreement with equation (4.22) of [D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza, C. White, 1503.05156]

We have also computed and checked the remaining contributions

- Hard: 1-loop hard and NLO soft functions, $\Delta^{dyn(2)}_{\rm NLP-hard}$.
  [M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]
- Soft: NNLO soft functions, $\Delta^{dyn(2)}_{\rm NLP-soft}$ → focus on next.
  [A. Broggio, **SJ**, L. Vernazza, 2107.07353]

NLP factorization formula for Drell-Yan

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \qquad \Delta_{\rm NLP}(z) = \Delta^{dyn}_{\rm NLP}(z) + \Delta^{kin}_{\rm NLP}(z)$$

where [M. Beneke, A.Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\begin{aligned}
\Delta^{dyn}_{\rm NLP}(z) = & -\frac{2}{(1-\epsilon)} Q \left[\left(\frac{\not{n}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{n}_+}{4} \right) \gamma^{\rho}_{\perp} \right]_{\beta\gamma} \\
& \times \int d(n_+p)\, C^{A0,A0}(n_+p, x_b n_- p_B)\, C^{*A0A0}(x_a n_+ p_A, x_b n_- p_B) \\
& \times \sum_{i=1}^{4} \int \{d\omega_j\}\; J_{i,\gamma\beta}(n_+p, x_a n_+ p_A; \{\omega_j\})\; S_i(\Omega; \{\omega_j\}) + \text{h.c.}
\end{aligned}$$

We now focus on $\mathcal{O}(\alpha_s^2)$ calculation of:

$$S_i(\Omega; \{\omega_j\}) = \int \frac{dx^0}{4\pi} e^{i\Omega\, x^0/2} \int \left\{ \frac{dz_{j-}}{2\pi} \right\} e^{-i\omega_j z_{j-}} S_i(x_0; \{z_{j-}\})$$

Generalised Soft Functions

We make use of the soft building blocks

$$\mathcal{B}^{\mu}_{\pm} = Y^{\dagger}_{\pm}\,[i\,D^{\mu}_{s}\,Y_{\pm}]\;, \quad q^{\pm} = Y^{\dagger}_{\pm}\,q_s$$

The relevant soft functions are

$$S_1(x^0; z_-) = \frac{1}{N_c}\,\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{i\partial^{\nu}_{\perp}}{in_-\partial}\mathcal{B}^{+}_{\nu_\perp}(z_-)\right)|0\rangle$$

$$S_3(x^0; z_-) = \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right] \times \mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_-),\left[in_-\partial\mathcal{B}^{+}_{\mu_\perp}(z_-)\right]\right]\right)|0\rangle$$

$$S^{AB}_{4;\mu\nu,bf}(x^0; z_{1-}, z_{2-}) = \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba} \times \mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\,\mathcal{B}^{+A}_{\mu_\perp}(z_{1-})\,\mathcal{B}^{+B}_{\nu_\perp}(z_{2-})\right)|0\rangle$$

$$S_{5;bfgh,\sigma\lambda}(x^0; z_{1-}, z_{2-}) = \frac{1}{N_c}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba} \times \mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\frac{g_s^2}{(in_-\partial_{z_1})(in_-\partial_{z_2})}\,q_{+\sigma g}(z_{1-})\,\bar{q}_{+\lambda h}(z_{2-})\right)|0\rangle$$

Not functions of Wilson lines only!

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\,\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\;e^{iz_-k}$$

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\ e^{iz_-k}$$

Double emission:

$$\langle g^{K_1}(k_1)g^{K_2}(k_2)|\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\right]|0\rangle =$$

$$g_s^2\mathbf{T}^{K_2}\,\mathbf{T}^{K_1}\frac{1}{(n_-k_1)}\frac{n_-^{\eta_2}}{(n_-k_2)}\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-k_1}$$

$$+\dots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\Bigg(-\frac{\left(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2}\right)n_-^{\eta_1}}{(n_-k_1)} + \frac{\left(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1}\right)\ n_-^{\eta_2}}{(n_-k_2)}$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\ \left(k_{1\perp}^2 + k_{1\perp}\cdot k_{2\perp}\right)$$

$$-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp}+k_{2\perp}^2\right)\Big]\Bigg)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-(k_1+k_2)}$$

$$+\dots$$

→ $\delta(\omega - n_-k_1)$ constraint in the integrand.

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\;\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\;e^{iz_-k}$$

Double emission:

$$\langle g^{K_1}(k_1)g^{K_2}(k_2)|\mathbf{T}\left[\;Y_-^\dagger(0)Y_+(0)\;\frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\right]|0\rangle =$$

$$g_s^2\mathbf{T}^{K_2}\,\mathbf{T}^{K_1}\frac{1}{(n_-k_1)}\frac{n_-^{\eta_2}}{(n_-k_2)}\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\;e^{iz_-k_1}$$

$$+\dots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\left(-\frac{(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2})\,n_-^{\eta_1}}{(n_-k_1)} + \frac{(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1})\;n_-^{\eta_2}}{(n_-k_2)}\right.$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\;\left(k_{1\perp}^2 + k_{1\perp}\cdot k_{2\perp}\right)$$

$$\left.-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp} + k_{2\perp}^2\right)\Big]\right)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\;e^{iz_-(k_1+k_2)}$$

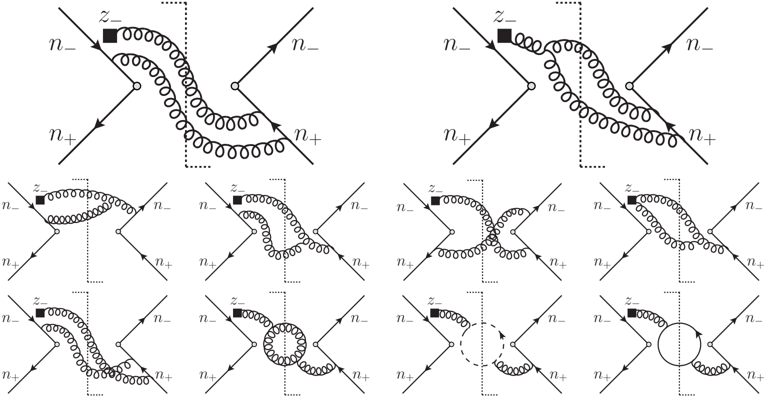
$$+\dots$$

$\rightarrow$ $\delta(\omega - n_-k_1 - n_-k_2)$ constraint in the integrand.

Some sample diagrams

For the S_1 soft function:

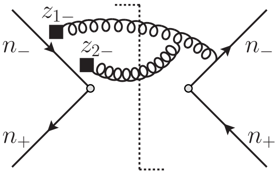
$$S_1(x^0; z_-) \;=\; \frac{1}{N_c}\,\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle$$

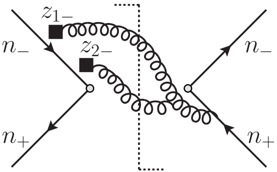


. . .

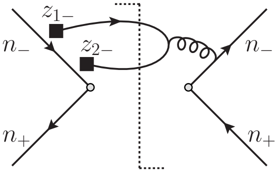
Some sample diagrams

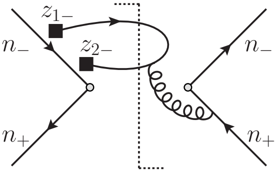
For the S_4 soft function:





For the S_5 soft function:



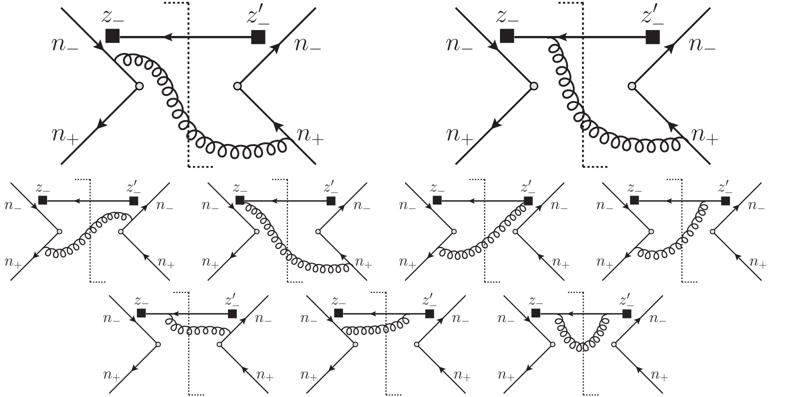


Some sample two-loop diagrams: double real

$$\mathcal{S}(\Omega,\omega,\omega') = \sum_{s,\lambda} \int \frac{d^d k_1}{(2\pi)^{d-1}} \delta^+(k_1^2) \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_2^2) \delta(\omega - n_- k_1 - n_- k_2) \delta(\omega' - n_- k_1)$$

$$\times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left(\frac{g_s}{i n_- \partial_{z'}} \, \bar{q}_{+\sigma k}(z'_-) \; \mathbf{T}^D \left\{ Y_+^\dagger(0) Y_-(0) \right\} \right) | X_s \rangle$$

$$\times \frac{\not{n}_{-\sigma\beta}}{4} \langle X_s | \mathbf{T} \left(\left\{ Y_-^\dagger(0) Y_+(0) \right\} \mathbf{T}^D \, \frac{g_s}{i n_- \partial_z} \, q_{+\beta}(z_-) \right) | 0 \rangle \delta(\Omega - 2k_1^0 - 2k_2^0)$$



The calculation

Methods developed for calculations of two-loop soft functions at *leading power.*

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglia, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = n_-(k_1 + k_2),$$

$$P_4 = k_1^2 \,, \quad P_5 = k_2^2 \,, \quad P_6 = \big(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\big) \,, \quad P_7 = \big(\omega - n_- k_1\big)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E} \mu^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i}\left[\frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+}\right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

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- The reduction is implemented in `LiteRed`
- 9 topologies are needed to reduce the soft functions
- We have 8 Master Integrals (MIs)
  - 5 MIs implementing the $\delta(\omega - n_- k_1)$ constraint: $\hat{I}_1$ – $\hat{I}_5$
  - 2 MIs with $\delta(\omega - n_- k_1 - n_- k_2)$: $\hat{I}_6$ and $\hat{I}_7$
  - 1 MI with $\delta(\omega_1 - n_- k_1)\,\delta(\omega_2 - n_- k_2)$: $\hat{I}_8$

The calculation

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[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

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First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = k_1^2\,, \quad P_4 = k_2^2\,, \quad P_5 = \left(\omega - n_- k_1\right)$$

$$P_6 = \left(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\right)\,, \quad P_7 = \left(\omega' - n_- k_1 - n_- k_2\right)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

- The reduction is implemented in `LiteRed`
- 4 + 1 topologies are needed to reduce the soft functions
- We have 6 + 2 Master Integrals (MIs)
  - 4 MIs implementing the $\delta(\omega - n_- k_1)\delta(\omega' - n_- k_1 - n_- k_2)$ and $\omega \leftrightarrow \omega'$ constraint: $\hat{I}_1 - \hat{I}_4$
  - 1 MI with $\delta(\omega - n_- k_1)\delta(\omega - \omega')$: $\hat{I}_5$
  - 1 MI with $\delta(\omega - n_- k_1 - n_- k_2)\,\delta(\omega - \omega')$: $\hat{I}_6$

Reduced expressions

Double real

$$S_{g\bar{q}}^{(2)2r0v}(\Omega,\omega,\omega')$$

$$= \frac{\alpha_s^2}{(4\pi)^2} T_F \Bigg[C_F \left(\frac{(4-\epsilon)(1-\epsilon)(1-2\epsilon)}{\epsilon^2\omega^2(\Omega-\omega)} \hat{I}_6 + \frac{4(2-3\epsilon)(1-3\epsilon)}{\epsilon^2\omega(\Omega-\omega)^2} \hat{I}_5 \right) \delta(\omega-\omega')$$

$$+ (C_A - 2C_F) \left(\frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\,\omega\omega'(\Omega-\omega)(\omega-\omega')} \hat{I}_3 - \frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\,\omega\omega'(\Omega-\omega')(\omega-\omega')} \hat{I}_1 \right.$$

$$\left. + \frac{(\Omega-\omega)(\omega+\omega')}{2\omega\omega'} \hat{I}_4 + \frac{(\Omega-\omega')(\omega+\omega')}{2\omega\omega'} \hat{I}_2 \right) \Bigg]$$

where for example

$$\hat{I}_1(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(0,0,1,1,1,1,1), \quad \hat{I}_2(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(1,1,1,1,1,1,1)$$

and family $\mathcal{A}$ propagators are

$$P_1 = (k_1+k_2)^2, \quad P_2 = n_+k_2, \quad P_3 = k_1^2, \quad P_4 = k_2^2,$$
$$P_5 = \big(\Omega - n_-k_1 - n_-k_2 - n_+k_1 - n_+k_2\big), \quad P_6 = \big(\omega - n_-k_1\big),$$
$$P_7 = \big(\omega' - n_-k_1 - n_-k_2\big)$$

The integrals are calculated using the differential equations method.

DE method for MIs

Convenient to change to dimensionless variable $\omega \to r\,\Omega$

$$I_1'(r) = \frac{1}{\Omega^2}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_1(\Omega, r), \qquad I_2'(r) = \frac{1}{\Omega}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_2(\Omega, r),$$

$$I_3'(r) = \left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_3(\Omega, r), \qquad I_4'(r) = \Omega\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_4(\Omega, r),$$

$$I_5'(r) = \Omega^2\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_5(\Omega, r)$$

System of DEs can be put into canonical form [J. Henn, 1304.1806]

$$\frac{d\vec{I}(r)}{dr} = \epsilon\, A(r)\cdot\vec{I}(r)$$

$$I_1'(r) = \frac{2(1-r)^2}{2-9\epsilon+9\epsilon^2} I_1(r),$$

$$I_3'(r) = \frac{1}{\epsilon^2} I_3(r),$$

$$I_4'(r) = -\frac{1}{\epsilon^2(1-r)} I_4(r),$$

$$I_5'(r) = -\frac{1+r}{2\epsilon^2(1-r)r} I_4(r) + \frac{1}{\epsilon^2 r} I_5(r)$$

$$A(r) = \begin{bmatrix} -\frac{1}{r}+\frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

Virtual real result

This contribution is given by [A. Broggio, **SJ**, L. Vernazza, 2306.06037]

$$S_{g\bar{q}}^{(2)1r1v}(\Omega,\omega,\omega') = \frac{\alpha_s^2\,T_F}{(4\pi)^2}(2C_F - C_A)\frac{e^{2\epsilon\gamma_E}\,\Gamma[1+\epsilon]}{\epsilon\,\Gamma[1-\epsilon]}$$

$$\cdot\,\mathrm{Re}\Bigg\{\left[\frac{\omega'-\omega}{\omega(\omega')^2}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,-\frac{\omega}{\omega'}\right) - \frac{1}{\omega\omega'}\right]\left(\frac{\mu^4}{\omega\omega'(\Omega-\omega')^2}\right)^{\epsilon}\theta(\omega)$$

$$+\frac{2(\omega+\omega')}{\omega\omega'(\omega'-\omega)}\left(\frac{\mu^4}{(\omega'-\omega)^2(\Omega-\omega')^2}\right)^{\epsilon}\frac{\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]}\theta(\omega'-\omega)\Bigg\}\theta(\omega')\theta(\Omega-\omega')$$

Integrating over ω,ω'

$$\Delta_{g\bar{q}}^{(2)}(z)|_{\mathrm{NLP,s,1r1v}} = -\left(\frac{\alpha_s}{4\pi}\right)^2 T_F(C_A - 2C_F)\left(\frac{\mu^2}{\Omega^2}\right)^{2\epsilon}$$

$$\times\,\frac{2\mathrm{Re}[e^{-i\epsilon\pi}]e^{2\epsilon\gamma_E}\Gamma[1-2\epsilon]\Gamma[1-\epsilon]^2\Gamma[1+\epsilon]^2}{\epsilon^3\Gamma[1-4\epsilon]}$$

Results at NNLO

The S_1 soft function is the only one with C_F^2 contributions. Combine our results for two-loop soft with tree-level collinear according to factorisation theorem

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) = 4\,Q\,\Omega\,H^{(0)}(Q^2)\int dr\,J^{(0)}_{1,1}\left(x_a(n_+p_A);\omega\right)S^{(2)2r0v}_{1,C_F^2}(\Omega,r)$$

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) \quad = \quad & \frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\Bigg(\frac{32}{\epsilon^3}-\frac{128}{\epsilon^2}\ln(1-z)+\frac{256}{\epsilon}\ln^2(1-z)-\frac{112\pi^2}{3\epsilon} \\ & +\frac{32}{3}\left(-32\ln^3(1-z)+14\pi^2\ln(1-z)-62\zeta(3)\right)+\mathcal{O}(\epsilon)\Bigg)\end{aligned}$$

The S_1 1r1v gives leading logarithms proportional to C_FC_A, which are cancelled exactly by the leading poles of 2r0v contribution to S_1

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_FC_A}}(z) \quad = \quad & \frac{\alpha_s^2}{(4\pi)^2}\,C_FC_A\Bigg(\frac{8}{\epsilon^3}-\frac{4}{3\epsilon^2}(24\ln(1-z)-11) \\ & -\frac{16}{9\epsilon}\left(-36\ln^2(1-z)+33\ln(1-z)+6\pi^2-16\right) \\ & -\frac{256}{3}\ln^3(1-z)+\frac{352}{3}\ln^2(1-z)+\frac{128}{3}\pi^2\ln(1-z) \\ & -\frac{1024}{9}\ln(1-z)-\frac{616\zeta(3)}{3}-\frac{154\pi^2}{9}+\frac{1484}{27}+\mathcal{O}(\epsilon)\Bigg)\end{aligned}$$

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

Results at N^{3}LO

Interestingly, we can partially validate our results at N^{3}LO by comparing to [N. Bahjat-Abbas, J. Sinninghe Damsté, L. Vernazza, C. White, 1807.09246]

$$\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad 4Q\int d\omega\, J^{(1)}_{1,1}(x_a\, n_+p_A;\omega)\, S^{(2)}_{1,C_F^2}(\Omega;\omega)$$

Using one-loop collinear function from [M.Beneke, A.Broggio, **SJ**, L.Vernazza, 1912.01585]

$$S^{(2)2r0v}_{1,C_F^2}(\Omega,\omega) = 8\,\frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\left(\frac{\omega\,(\Omega-\omega)^3}{\mu^4}\right)^{-\epsilon}\frac{1}{\omega}\,\frac{1}{\epsilon^2}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]}{\Gamma[1-3\epsilon]}\theta(\Omega-\omega)\theta(\omega)$$

We find a closed d-dimensional result, expanding

$$\begin{aligned}\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad & \frac{\alpha_s^3}{(4\pi)^3}C_F^3\Bigg(-\frac{64}{\epsilon^4}+\frac{80(4\ln(1-z)-1)}{\epsilon^3}+\frac{16}{\epsilon^2}\Big(-50\ln^2(1-z)\\ & +25\ln(1-z)+7\pi^2-6\Big)+\frac{1}{\epsilon}\Big(\frac{4000}{3}\ln^3(1-z)-1000\ln^2(1-z)\\ & -560\pi^2\ln(1-z)+480\ln(1-z)+2624\zeta(3)+140\pi^2-128\Big)\\ & -\frac{5000}{3}\ln^4(1-z)+\frac{5000}{3}\ln^3(1-z)+1400\pi^2\ln^2(1-z)\\ & -1200\ln^2(1-z)-700\pi^2\ln(1-z)+640\ln(1-z)\\ & +\zeta(3)(3280-13120\ln(1-z))+\frac{62\pi^4}{5}+168\pi^2-192\Bigg)\end{aligned}$$