Vector boson plus heavy flavour at the LHC

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Durham, United Kingdom

6th of September 2023

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The flavours of the LHC

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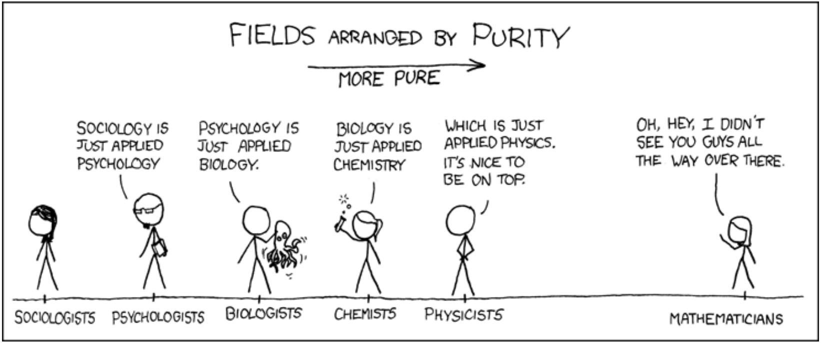
FIELDS ARRANGED BY PURITY
MORE PURE
SOCIOLOGY IS JUST APPLIED PSYCHOLOGY
PSYCHOLOGY IS JUST APPLIED BIOLOGY.
BIOLOGY IS JUST APPLIED CHEMISTRY
WHICH IS JUST APPLIED PHYSICS. IT'S NICE TO BE ON TOP.
OH, HEY, I DIDN'T SEE YOU GUYS ALL THE WAY OVER THERE.
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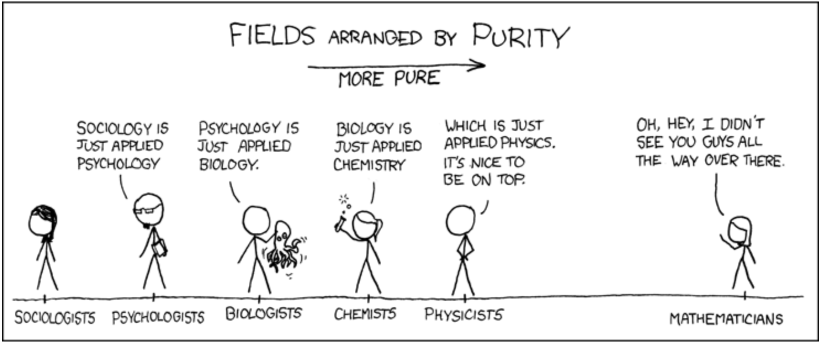
MATHEMATICIANS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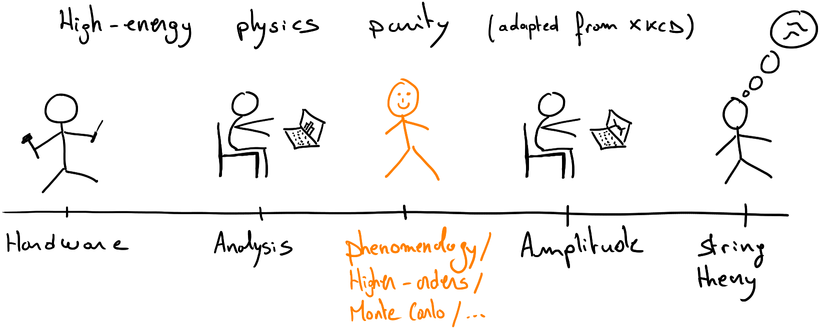

High-energy physics purity (adapted from xkcd)
Hardware
Analysis
Phenomenology/
Higher-orders/
Monte Carlo/...
Amplitude
String theory

What this talk is **NOT**:

→ Topics covered during the conference:

- Quarkonium production
  → See talks by [Flett], [Sridhar], [Lynch]

- $W + b\bar{b}$
  → See talk by [Hartanto]

- top associated production
  → See talks by [Generet], [Stremmer]

Outline:

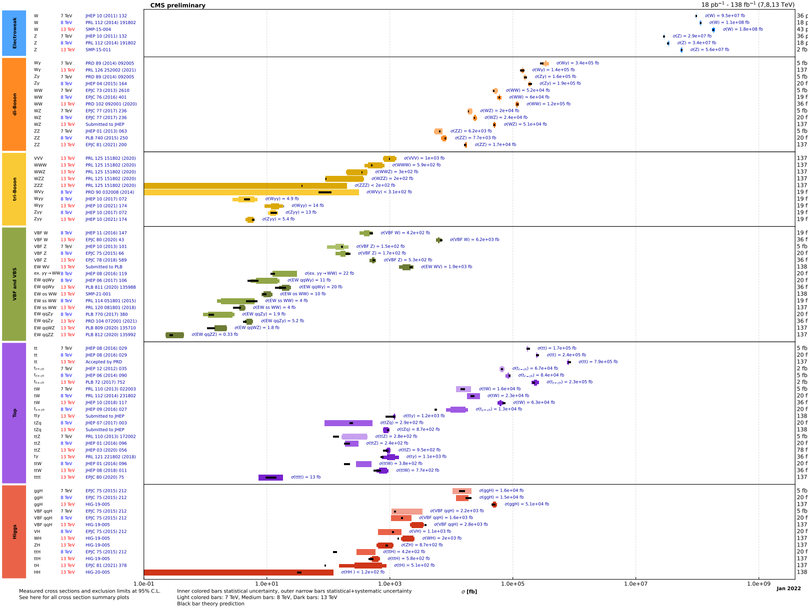
→ Introduction:
Flavours of the LHC

- $pp \to Z + b$

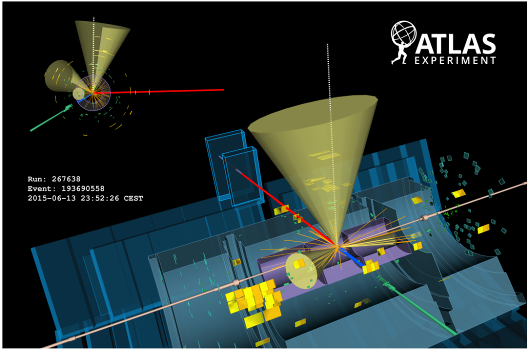
- $pp \to W + c$

- $pp \to Z + c$

Overview of CMS cross section results



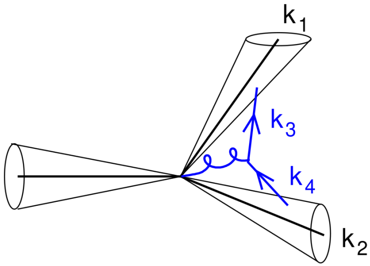
- Triumph of the Standard Model!



- $V+\text{jets}$ among the largest cross sections at LHC
  $\rightarrow$ very well measured / standard candle
  $\rightarrow$ background to many SM processes / BSM searches
- Test of the SM and QCD in particular
- V + **flavoured** jets:
  $\rightarrow$ often related to PDF content of the proton
  $\rightarrow$ interesting processes on their own

Why **flavoured** jets are interesting?

- Beyond NLO in QCD, no IR-safe definition of jets for anti-k_T
  - $\rightarrow$ Flavour k_T algorithm [Banfi, Salam, Zanderighi; hep-ph/0601139]
  - $\rightarrow$ Modified k_T algorithm to account for soft wide-angle $q\bar{q}$



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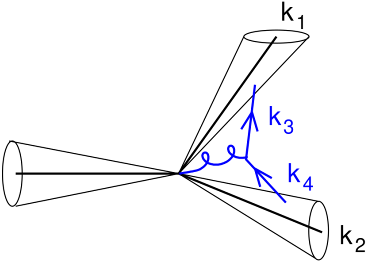


⚠ introduces mismatch with respect to experimental treatment

- Many recent proposals ...

  [Czakon, Mitov, Poncelet; 2205.11879],[Gauld, Huss, Stagnitto; 2208.11138], [Caletti, Larkoski, Marzani, Reichelt; 2205.01109, 2205.01117], [Caola et al.; 2306.07314]

- Vast topic! Flavoured jet algorithms at the LHC [Scyboz]

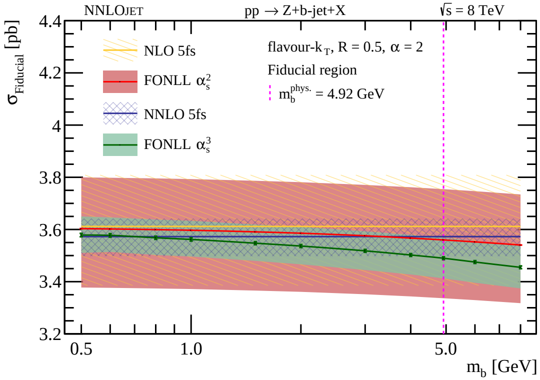
- $\mathrm{pp} \to \mathrm{Z} + \mathrm{b}$

$pp \to Z + b$ [Gauld et al.; 2005.03016]

- NNLO QCD in 5 flavours (5fs) combined with ...
  ... NLO QCD in 4 flavours (4fs)

$$\mathrm{d}\sigma^{\mathrm{FONLL}} = \mathrm{d}\sigma^{5\mathrm{fs}} + \left(\mathrm{d}\sigma^{4\mathrm{fs}}_{m_b} - \mathrm{d}\sigma^{4\mathrm{fs}}_{m_b \to 0}\right)$$

- Allows to incorporate exact b-mass effects



Flavour jet

- Flavour k_{T} algorithm used [Banfi, Salam, Zanderighi; hep-ph/0601139]
- But experimental data [CMS; 1611.06507]
  1. reconstruction of jets with anti-k_{T} algorithm
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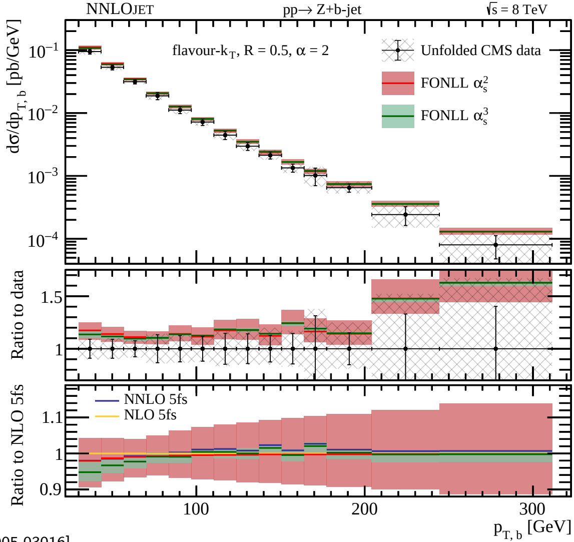
- Applying non perturbative correction to the data gives

$$\sigma^{\mathrm{CMS}}_{\mathrm{Fiducial,f\text{-}k_T}} = 3.134 \pm 0.214^{+0.013}_{-0.025}\,\mathrm{pb}$$

- At $\mathcal{O}(\alpha_{\mathrm{s}}^3)$, the FONLL prediction is

$$\sigma^{\mathrm{FONLL}}_{\mathrm{Fiducial}}(m_b^{\mathrm{phys.}}) = 3.490^{+0.078}_{-0.078}(\mathrm{scales})\ \ \mathrm{pb}$$

NB:
$\delta\sigma\,(\mathrm{PDF}, \alpha_{\mathrm{s}}) = \pm 0.074\,\mathrm{pb}$ estimated at NLO



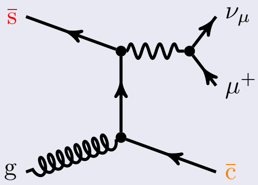
[Gauld et al.; 2005.03016]

Fit to data:

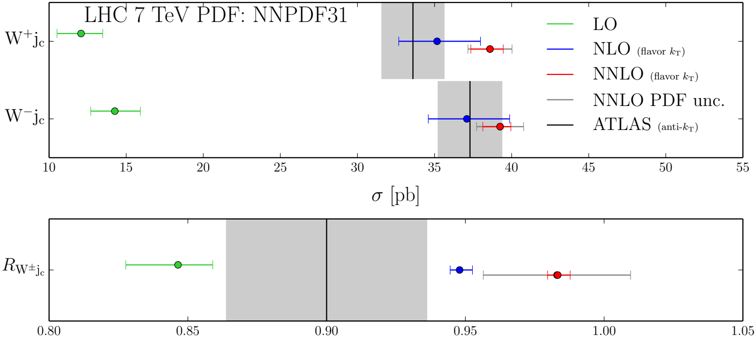
- $\chi^2/\mathrm{N}_{\mathrm{dat}}(\alpha_s^2, p_{\mathrm{T},b}) = 23.4/14$
- $\chi^2/\mathrm{N}_{\mathrm{dat}}(\alpha_s^3, p_{\mathrm{T},b}) = 21.5/14$

- $pp \to W + c$

- pp $\rightarrow$ W + c



- Direct link between W+c measurements and strange PDF
- Test of (perturbative) QCD
  $\rightarrow$ s-$\bar{s}$ asymmetry predicted at 3-loop in QCD
  [Catani, de Florian, Rodrigo, Vogelsang; hep-ph/0404240]



[Czakon, Mitov, MP, Poncelet; 2011.01011] + [ATLAS; 1402.6263]

- $R_{W^\pm j_c} = \frac{\sigma_{W^+ j_c}}{\sigma_{W^- j_c}} \sim \left(|V_{cs}|^2 \bar{s} + |V_{cd}|^2 \bar{d}\right) / \left(|V_{cs}|^2 s + |V_{cd}|^2 d\right)$
- PDF uncertainty dominant over NNLO scale uncertainty

Discussion

$\rightarrow$ Open questions addressed in [Czakon, Mitov, MP, Poncelet; 2212.00467]

- Difference in the jet algorithms (flavoured k_{T} vs. anti-k_{T})
  $\rightarrow$ ~~Estimated to be 12% in Z + b [Gauld et al.; 2005.03016] ...~~
  ... difficult translation to W + c $<$ 3% for OS selection

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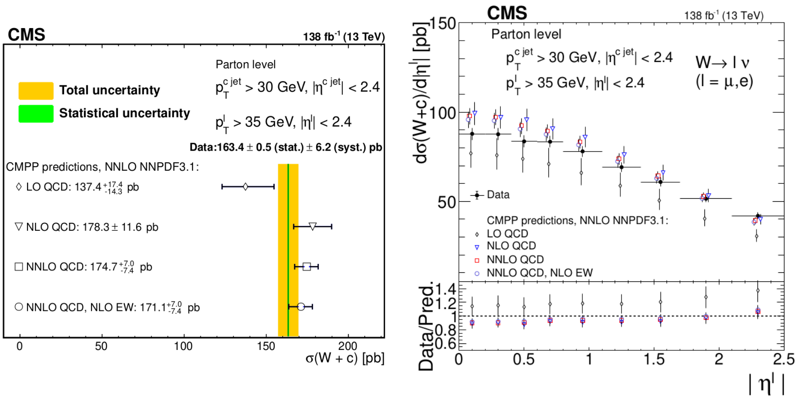
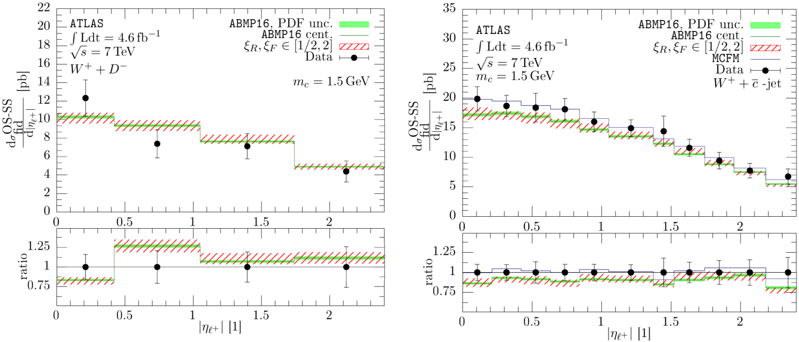
⚠ per-cent precision apart from PDF!

Comparison against recent data



[CMS; 2308.02285] based on [Czakon, Mitov, MP, Poncelet; 2212.00467]

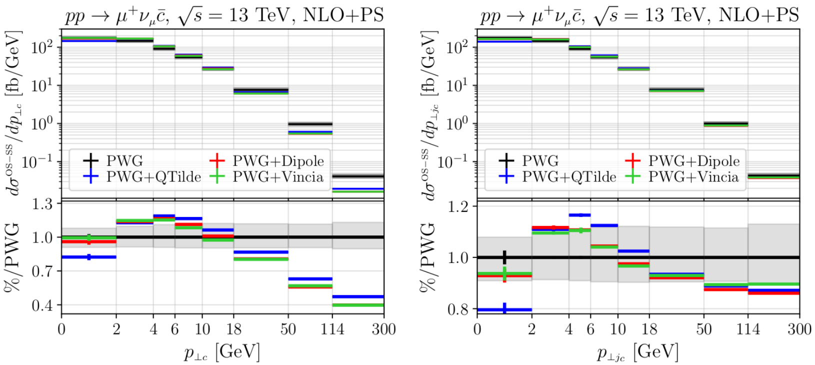
- Perfect agreement between theory and data
  $\rightarrow$ going beyond this precision will be a challenge!
  $\rightarrow$ at 1% accuracy everything is relevant!



- Th. vs. Data [ATLAS; 1402.6263] @ 7 TeV
- Useful comparison between W + D and W + c
  → Estimate of [W + D] → [W + c] effects

NB: Also comparison [CMS; 1811.10021] for W + D @ 13 TeV

NLO QCD + PS in POWHEG [Ferrario Ravasio, Oleari; 2304.13791]



- Comparison between $p_{T,c}$ and p_{T,j_c}
- Non trivial (and different) effect of parton shower
- Up to 10% differences between various parton showers

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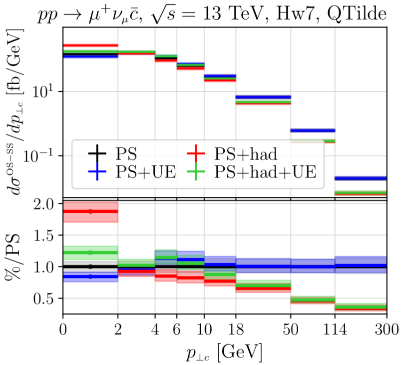

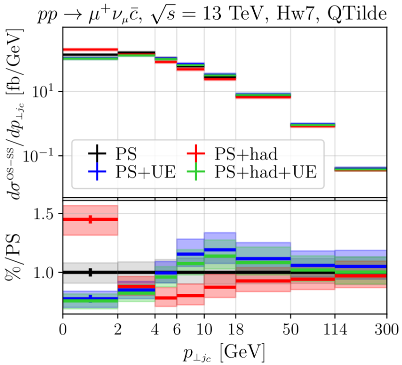

- Underlying Event and hadronisation can have large effects
  $\to$ Different between $p_{\mathrm{T},c}$ and p_{T,j_c}

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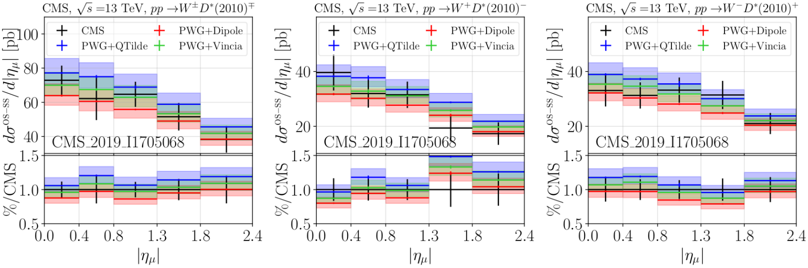


- Th. vs. Data [CMS; 1811.10021] @ 13 TeV
  → both signature separately and their sum
  → Large th. and exp. uncertainties → agreement!

- $pp \to Z + c$

- $pp \to Z + c$

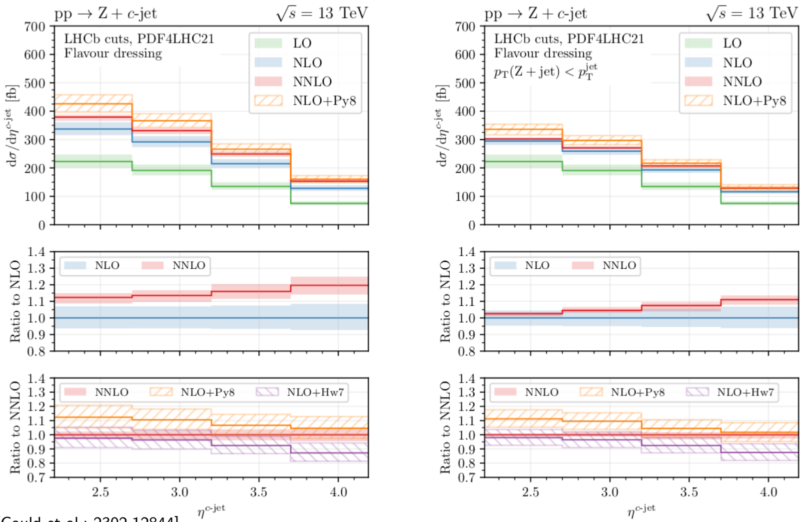
- Sensitive to charm PDF

  [Lipatov, Lykasov, Stepanenko, Bednyakov; 1606.04882], [Bailas, Goncalves; 1512.06007], [Boettcher, Ilten, Williams; 1512.06666]

  $\to$ especially at forward kinematic (like LHCb [LHCb; 2109.08084])

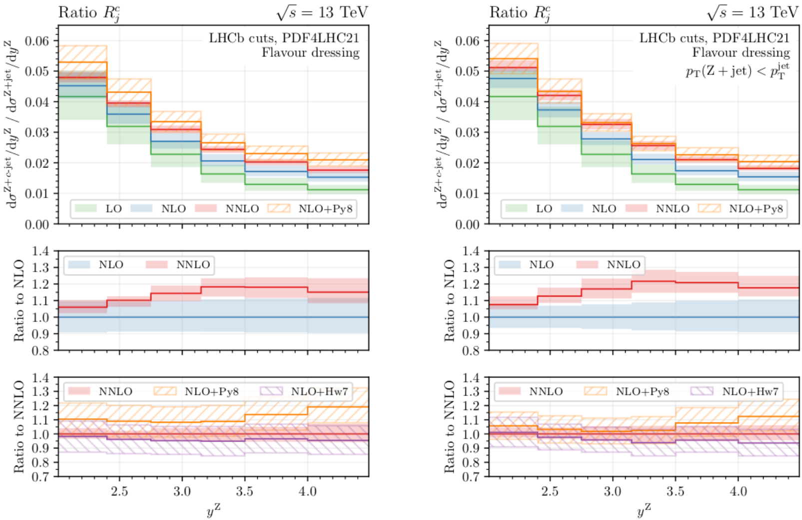
- Probe of instrinsic charm

  [Brodsky, Hoyer, Peterson, Sakai; Phys. Lett. B 93 (1980) 451–455], [Brodsky et al.; 1504.06287], [Ball et al.; 2208.08372], [Hou et al.; 1707.00657], [Guzzi et al.; 2211.01387]



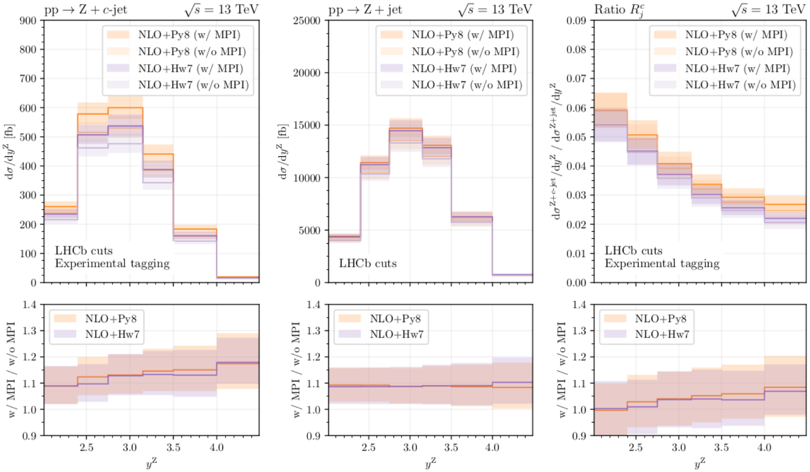
[Gauld et al.; 2302.12844]

- $(10 - 20)\%$ corrections @ NNLO QCD
  → perturbative convergence improved with $p_{\mathrm{T},\mathrm{Z}+\mathrm{j}} < p_{\mathrm{T},j}$
- Coherent picture between NNLO QCD and NLO QCD + PS



[Gauld et al.; 2302.12844]

- $R_j^c = \sigma(Z + c - \text{jet})/\sigma(Z + \text{jet})$
  $\rightarrow$ Same (flavour) algorithm for both computations
  [Gauld, Huss, Stagnitto; 2208.11138]



[Gauld et al.; 2302.12844]

- Large MPI corrections for $\mathrm{Z} + \mathrm{c}$
  $\rightarrow$ no comparison to data [LHCb; 2109.08084]
  ⚠ different flavour tagging in exp. vs. th.

Final word

a IRC-safe definition of jet flavour. Only a joint effort of both communities, theory and experimental, will enable to exploit in the best way the huge amount of data that LHC will provide us in the next decades, better enabling the use flavour signatures as a powerful window into short-distance interactions from GeV to TeV energy scales.

[Gauld et al.; 2302.12844]

Summary

New computations available:

- Z+b [Gauld et al.; 2005.03016]
- W+c [Czakon, Mitov, MP, Poncelet; 2011.01011, 2212.00467], [Bevilacqua, Garzelli, Kardos, Toth; 2106.11261], [Ferrario Ravasio, Oleari; 2304.13791]
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- Decisive information for SM measurements/BSM searches
  → Precision programme at the LHC
- Crucial interplay between theory and experiment
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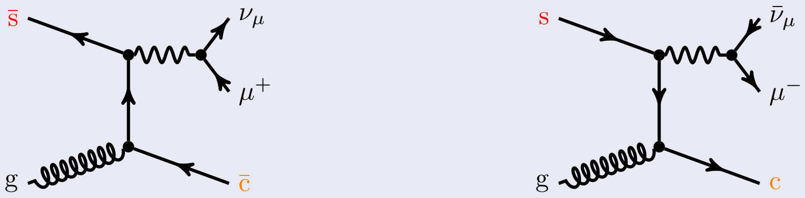
Thank you

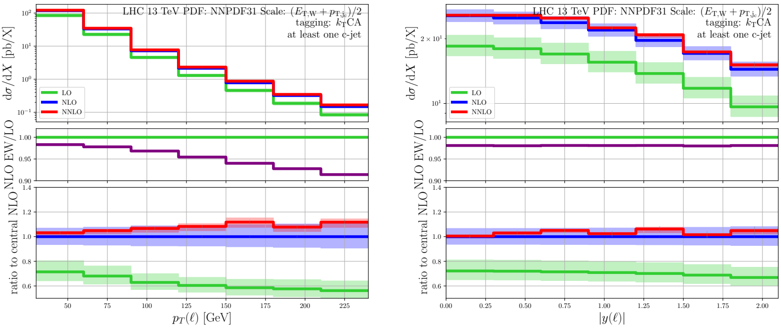
BACK-UP

Feature of the (new) computation [Czakon, Mitov, MP, Poncelet; 2212.00467]



- Full CKM dependence up to NNLO QCD
- NLO EW
- Study of flavour-jet algorithm
- Study of experimental definition
- 13 TeV setup

Best predictions @ 13 TeV - Differential distributions



- Good perturbative behaviour for QCD corrections
- Sudakov logarithm for EW corrections

Best predictions @ 13 TeV - cross sections

Order	$\sigma_{W^+j_c}$ [pb]	$\sigma_{W^-j_c}$ [pb]	$R_{W^\pm j_c} = \sigma_{W^+j_c}/\sigma_{W^-j_c}$
LO	$113.817(2)^{+12.4\%}_{-9.87\%}$	$119.711(2)^{+12.4\%}_{-9.88\%}$	$0.95076(2)^{+0.013\%}_{-0.021\%}$
NLO	$162.4(1)^{+7.2\%}_{-6.6\%}$	$168.1(1)^{+6.9\%}_{-6.4\%}$	$0.9659(9)^{+0.29\%}_{-0.21\%}$
NNLO	$168.6(8)^{+0.7\%\ +3.8\%(\mathrm{PDF})}_{-2.1\%\ -3.8\%(\mathrm{PDF})}$	$173.9(1.9)^{+0.6\%\ +3.7\%(\mathrm{PDF})}_{-1.8\%\ -3.7\%(\mathrm{PDF})}$	$0.96(1)^{+0.2\%\ +2.1\%(\mathrm{PDF})}_{-0.3\%\ -2.1\%(\mathrm{PDF})}$

- PDF uncertainty dominant at NNLO QCD

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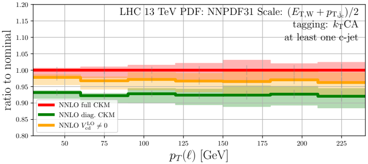
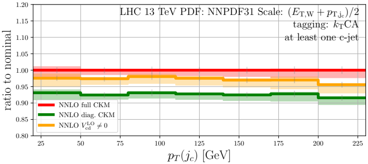
NLO EW	117.399(2)	111.627(2)	0.95084(2)
$\delta_{\mathrm{NLO\ EW}}$[%]	-1.93	-1.92	-0.01

- EW corrections null in the ratio

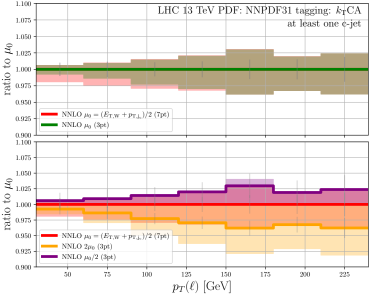
Effect of non-diagonal CKM @ NNLO QCD

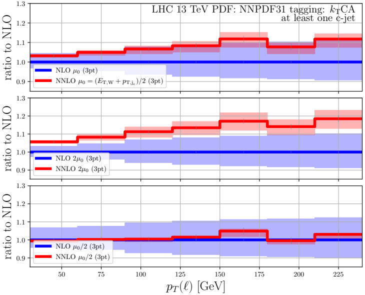




- full CKM / no CKM $\sim 7.5-11\%$
- full CKM / $V_{\text{cd}}^{\text{LO}} \neq 0 \sim 3\%$
  $\rightarrow$ Original approximation rather good
  $\rightarrow$ Full CKM dependence up to NNLO QCD for precise predictions

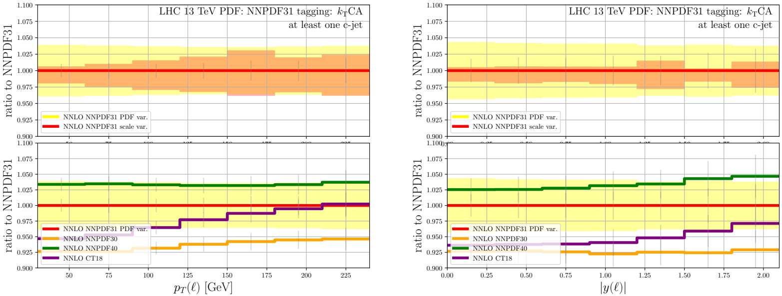
Scale setting





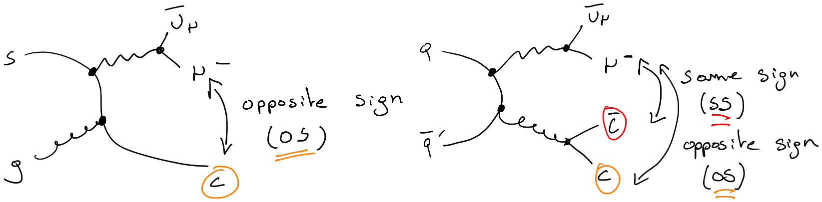
- $\mu_0 = \frac{1}{2}\left(E_{\mathrm{T,W}} + p_{\mathrm{T,j_c}}\right)$
- For $p_{\mathrm{T},\ell}$, $\mu_0/2$ best / For $p_{\mathrm{T,j_c}}$, μ_0 best / For cross section, $2\mu_0$ best
  $\rightarrow$ μ_0 good choice with good perturbative convergence

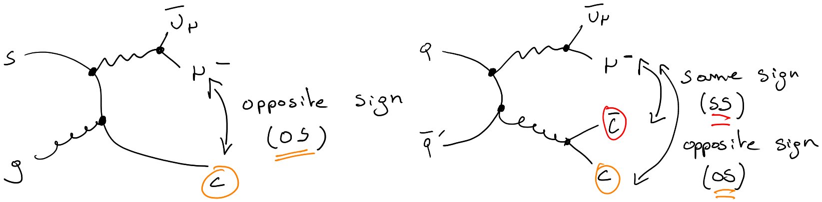
PDF uncertainty



- NNPDF3.1 variation $\sim \pm 4\%$
- Spread of various PDF sets $\sim 10\%$
  $\rightarrow$ PDF error is the largest theoretical uncertainty

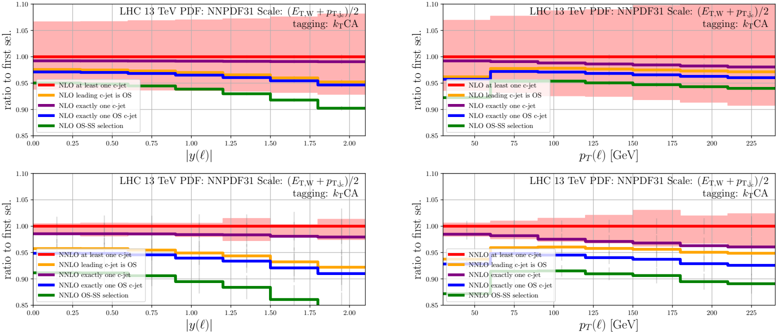
Event selection(s)





- Experiments measure OS-SS
  → More sensitivity to strange PDF
- Many possibilities...
  → most inclusive: at least one c-jet

Event selection(s)



- At NLO QCD, differences covered by scale uncertainty
- At NNLO QCD, differences $> 10-15\%$

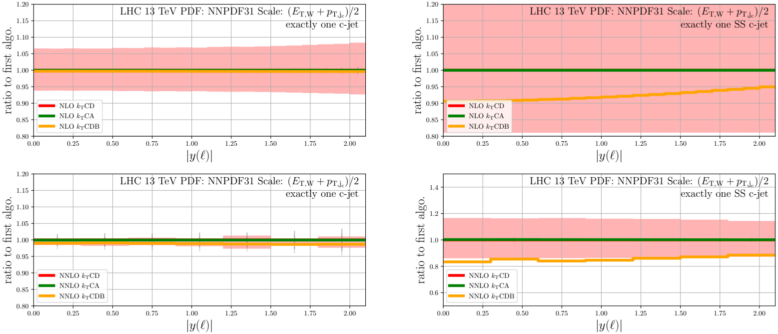
Jet algorithms - definitions (1)

→ Freedom in choosing whether cc and $\bar{c}\bar{c}$ are flavoured

Variation of flavour k_{T} algorithm [Banfi, Salam, Zanderighi; hep-ph/0601139]

- flavoured k_{T} algorithm, charge dependent (k_{T}CD)
- flavoured k_{T} algorithm, charge agnostic (k_{T}CA)
- flavoured k_{T} algorithm, charge dependent, with beam definition including W momenta (k_{T}CDB)

Jet algorithm (1)



- No difference at NLO and NNLO for exactly one-jet
- Large differences for exactly one SS c-jet

Jet algorithms - definitions (2)

→ Flavoured anti-k_{T} algorithm

$$d_{ij}^{(flavored)} = d_{ij}^{(standard)} \times \begin{cases} \mathcal{S}_{ij}\,, & \text{if both } i \text{ and } j \text{ have non-zero flavor of OS,} \\ 1\,, & \text{otherwise.} \end{cases}$$

where

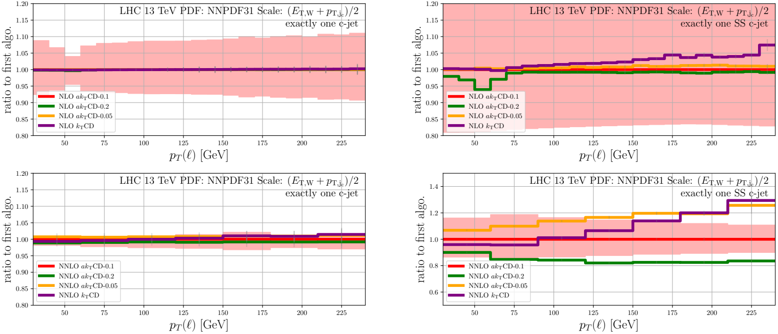
$$\mathcal{S}_{ij} = 1 - \theta\left(1 - \kappa_{ij}\right) \cos\left(\frac{\pi}{2}\kappa_{ij}\right) \quad \text{with} \quad \kappa_{ij} \equiv \frac{1}{a}\,\frac{k_{T,i}^2 + k_{T,j}^2}{2k_{T,\max}^2}\,.$$

[Czakon, Poncelet, Mitov; 2205.11879]

Variation of anti-k_{T} algorithm

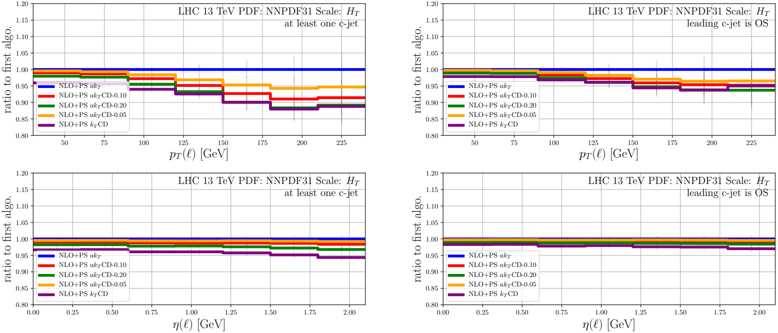
- flavoured anti-k_{T} algorithm, charge dependent, with $a = 0.2, 0.1, 0.05$ (ak_{T}CD-0.2, ak_{T}CD-0.1, ak_{T}CD-0.05)
- flavoured anti-k_{T} algorithm, charge agnostic, with $a = 0.1$ (ak_{T}CA-0.1).

<u>NB:</u> Alternatives [Caletti, Larkoski, Marzani, Reichelt; 2205.01117, 2205.01109], [Gauld, Huss, Stagnitto; 2208.11138]



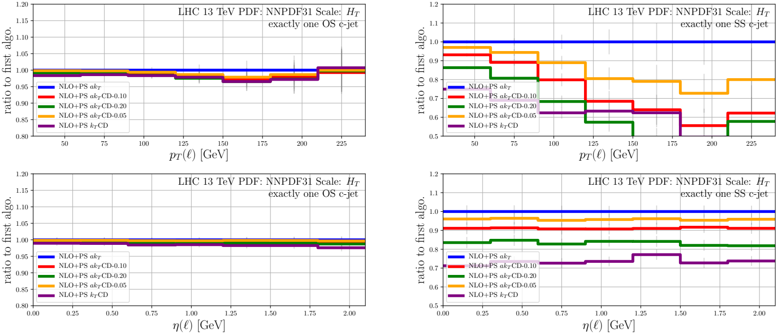
- No(small) difference at NLO and NNLO for exactly one-jet
- Large differences for exactly one SS c-jet

Jet algorithm (3) at **NLO+PS**



- 5−10% differences for at least one c-jet (inclusive)
- Below 5% differences for leading c-jet is OS

Jet algorithm (4) at NLO+PS



- $<3\%$ differences for exactly one OS c-jet
- Huge differences for exactly one SS c-jet
  $\rightarrow$ exactly one OS c-jet is preferred in this respect

Cuts @ 13 TeV

- Charged lepton

$$p_{\mathrm{T},\ell} > 30\,\mathrm{GeV}, \qquad |\eta_\ell| < 2.5.$$

- At least one c-tagged jet

$$p_{\mathrm{T},\mathrm{j}_c} > 20\,\mathrm{GeV}, \qquad |\eta_{\mathrm{j}_c}| < 2.5.$$

CKM effect

σ_{NNLO} [pb]	full CKM	$V_{\mathrm{cd}}^{\mathrm{LO}} \neq 0$	no CKM
+	$168.6(8)^{+0.7\%\ +3.8\%(\mathrm{PDF})}_{-2.1\%\ -3.8\%(\mathrm{PDF})}$	$164.4(8)^{+1.0\%\ +3.9\%(\mathrm{PDF})}_{-2.4\%\ -3.9\%(\mathrm{PDF})}$	$156.7(8)^{+0.7\%\ +4.2\%(\mathrm{PDF})}_{-2.1\%\ -4.2\%(\mathrm{PDF})}$
-	$173.9(1.9)^{+0.6\%\ +3.7\%(\mathrm{PDF})}_{-1.8\%\ -3.7\%(\mathrm{PDF})}$	$168.5(1.9)^{+1.0\%\ +3.8\%(\mathrm{PDF})}_{-2.2\%\ -3.8\%(\mathrm{PDF})}$	$156.7(1.9)^{+0.5\%\ +4.2\%(\mathrm{PDF})}_{-1.6\%\ -4.2\%(\mathrm{PDF})}$