# Small-$x$ Factorization from Effective Field Theory

**Duff Neill, Aditya Pathak, Iain Stewart**

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# Outline

## Introduction

- The small-$x$ region and the BFKL equation
- LL resummation by Catani and Hautmann

## EFT modes and power counting

## Small-$x$ factorization from Glauber SCET

- Factorization formula
- Collinear function & BFKL evolution
- IR divergences

## BFKL & DGLAP resummation

- Consistency with twist factorization
- BFKL resummation of $F_2$ and $F_L$
- Comparison with previous work

## Backup slides

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BFKL & DGLAP resummation

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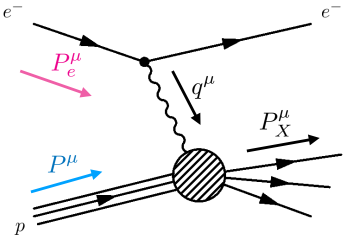
Backup slides

# DIS review: Twist expansion

Consider unpolarized, inclusive DIS:

$$Q^2 = -q^2 > 0 \,, \qquad x_b = \frac{Q^2}{2P\cdot q} \,,$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_b\,\mathrm{d}Q^2}(e^- p \to e^- X) = \frac{2\pi y \alpha^2}{Q^4} L_{\mu\nu}(P_e, q)\, W^{\mu\nu}(P,q) \,.$$




$$W^{\mu\nu} = e_L^{\mu\nu} \frac{1}{x_b} F_L(x_b, Q^2) + e_2^{\mu\nu} \frac{1}{x_b} F_2(x_b, Q^2) \,.$$

Well-known twist-2 factorization:

$$\frac{1}{x_b} F_a(x_b, Q^2) = \sum_\kappa \int_{x_b}^1 \frac{\mathrm{d}\xi}{\xi}\, H_a^{(\kappa)}\Big(\frac{x_b}{\xi}, Q, \mu\Big) f_{\kappa/p}(\xi,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right) .$$

PDF absorbs all the IR divergences.

# DIS review: Twist expansion

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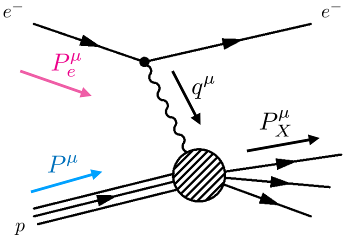



Take Mellin Transform:

$$\bar{F}_p(N,Q^2) = \int_0^1 \frac{dx}{x} x^N \left(\frac{1}{x} F_p(x)\right), \qquad \Rightarrow \qquad \bar{F}_p^{(\kappa)}(N) = \sum_{\kappa'} \bar{H}_p^{(\kappa')}(N) \times \underbrace{\bar{\Gamma}_{\kappa'\kappa}(N,\epsilon)}_{\text{PDF}} \; .$$

IR divergences are exponentiated into PDFs (transition functions),

$$\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2), N, \epsilon) \equiv \mathrm{P}\,\exp\left(\int_0^{\alpha_s(\mu^2)} \frac{d\alpha}{\beta(\epsilon,\alpha)} \gamma^s(\alpha, N)\right)_{\kappa'\kappa} \; .$$

# Mellin Transform

Mellin transform is very useful: (set $N = n+1$)

$$\bar{f}(n) \equiv \int_0^1 \frac{dx}{x} x^{n+1} f(x)$$

Let us note that

| $f(x)$ | $\bar{f}(n)$ | singularity in $x \to 0$ |
|---|---|---|
| $\frac{1}{x} \ln^{\ell-1}(x)$ | $\sim \frac{1}{n^\ell}$ | pole at $n = 0$ |
| $x^{p-1} \ln^{\ell-1}(x)$ | $\sim \frac{1}{(n+p)^\ell}$ | pole at $n = -p$ |

# Small-$x$ limit

Leading terms in $x_b \to 0$ limit:

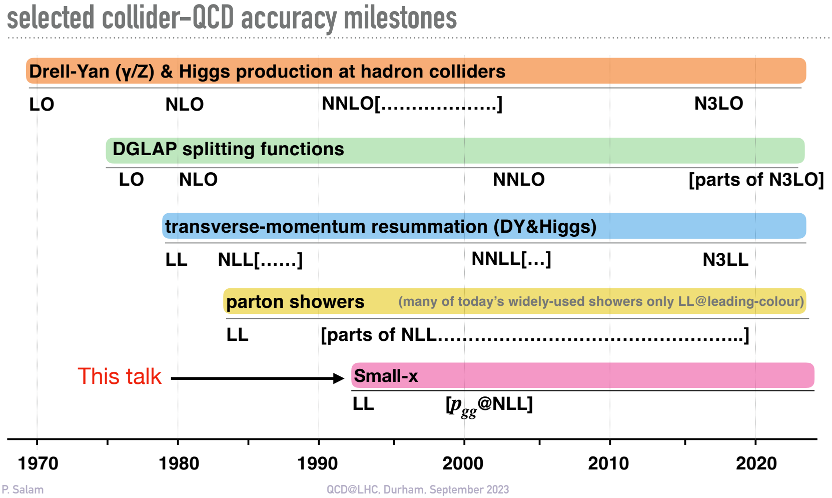
$$\frac{\alpha_s^\ell}{x} \ln^{\ell-1}(x) \quad ,$$

Both coefficient function and the DGLAP anomalous dimension become singular in $x_b \to 0$ limit:

$$\begin{aligned}
\bar{H}_a^{(\kappa)}(n) &\sim \alpha_s \frac{\alpha_s}{n} + \alpha_s \left(\frac{\alpha_s}{n}\right)^2 + \alpha_s \left(\frac{\alpha_s}{n}\right)^3 + \dots\,, \\
\gamma_{gg}(n) &\sim \quad \frac{\alpha_s}{n} + \quad \left(\frac{\alpha_s}{n}\right)^2 + \quad \left(\frac{\alpha_s}{n}\right)^3 + \dots\,, \\
\gamma_{qg}(n) &\sim \alpha_s \frac{\alpha_s}{n} + \alpha_s \left(\frac{\alpha_s}{n}\right)^2 + \alpha_s \left(\frac{\alpha_s}{n}\right)^3 + \dots
\end{aligned}$$

Our goal is to resum these leading logarithmic series.

# Progress (?) in small-$x_b$ resummation



# Phenomenological applications of small-$x$ limit

Small-$x_b$ data at N$^3$LO plays a crucial role in improving PDFs:

## A List of N$^3$LO Ingredients

MSHT20aN3LO: McGowan, Cridge, Harland-Lang, Thorne *Eur.Phys.J.C* 83 (2023) 3, 185

| N$^3$LO Function | No. of Moments | Moments (Even only) | Small-$x$ | Large-$x$ |
|---|---|---|---|---|
| $P_{qq}^{\mathrm{NS}}$ | 8 | $N = 2-16$ [21] | [21] | [21] |
| $P_{qq}^{\mathrm{PS}}$ | 4 | $N = 2-8$ [35, 36] | LL [28] | N/A |
| $P_{qg}$ | 4 | $N = 2-8$ [35, 36] | LL [28] | N/A |
| $P_{gq}$ | 4 | $N = 2-8$ [35, 36] | LL [29–31] | N/A |
| $P_{gg}$ | 4 | $N = 2-8$ [35, 36] | LL & NLL [29–33] | N/A |
| $A_{qq,H}^{\mathrm{NS}}$ | 7 | $N = 2-14$ [50] | N/A | N/A |
| $A_{Hq}^{\mathrm{PS}}$ | 6 | $N = 2-12$ [50] | [53] | [53] |
| $A_{Hg}$ | 5 | $N = 2-10$ [50] | LL [49] | N/A |
| $A_{gq,H}$ | 7 | $N = 2-14$ [50] | [54] | [54] |
| $A_{gg,H}$ | 5 | $N = 2-10$ [50] | N/A | N/A |

**Table A.1:** List of all the N$^3$LO ingredients used to construct the approximate N$^3$LO splitting functions and transition matrix elements. Where only a citation is provided, extensive knowledge i.e. beyond NLL is used. This table is a non-exhaustive list of the current knowledge about these functions, however information beyond that which is provided here is not currently in a usable format for phenomenological studies.

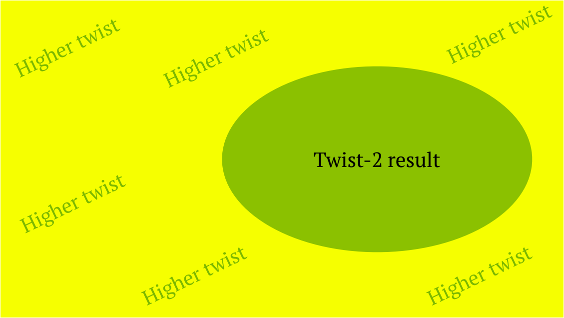
# Two subtleties with small-$x_b$ resummation

The box represents the full expression of the structure function (perturbative as well as nonperturbative):

The complete result for $F_a$

# Two subtleties with small-$x_b$ resummation

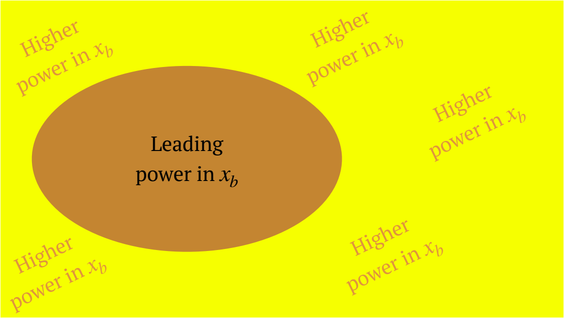
Terms that are retained at leading power in twist expansion $\left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \ll 1\right)$




$$\frac{1}{x_b} F_a(x_b, Q^2) = \sum_\kappa \int_{x_b}^1 \frac{\mathrm{d}\xi}{\xi} H_a^{(\kappa)}\left(\frac{x_b}{\xi}, Q, \mu\right) f_{\kappa/p}(\xi, \mu) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right).$$

# Two subtleties with small-$x_b$ resummation

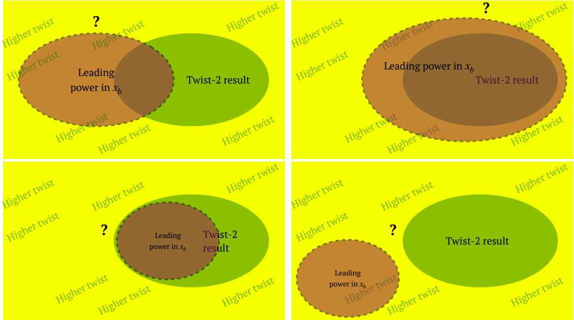
Terms that are leading power in $x_b \ll 1$ expansion ($\sim \frac{1}{x}$ or $n = 0$ pole)




Where is this relative to leading twist terms?

# Two subtleties with small-$x_b$ resummation

Which scenario is correct?

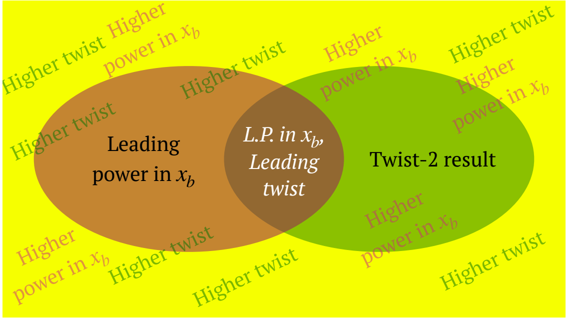



Can partly answer this by inspecting the perturbative series of the coefficient function.

$$H_L^{(g)}(x) \sim \alpha_s\, x(1-x) + \mathcal{O}\left(\frac{\alpha_s^2}{x}\right) \quad \Leftrightarrow \quad \bar{H}_L^{(g)}(x) \sim \alpha_s \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \mathcal{O}\left(\frac{\alpha_s^2}{n}\right)$$

# Subtlety # 1: Non-trivial overlap between twist and small-$x_b$ expansions

Small-$x_b$ expansion is based on rapidity factorization and makes no reference to $\Lambda_{\mathrm{QCD}}$ (more on this coming up), and **hence includes leading as well as higher twist pieces**.



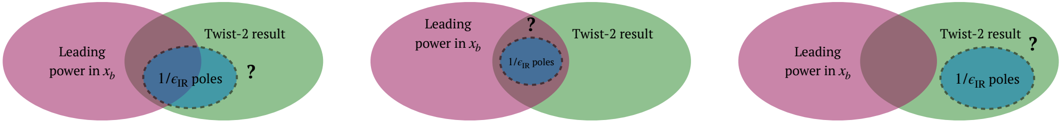

Our first goal is to compute the overlap between the two expansions.

# Factorization of nonperturbative pieces

Using dim-reg $d = 4 - 2\epsilon$ and simple partonic states we can kill higher twist pieces.
IR divergences appear as $1/\epsilon_{\text{IR}}$ poles in the PDF.

$$\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2), n) = \mathsf{P}\exp\left(-\frac{1}{\epsilon}\int_0^{\alpha_s(\mu^2)} \frac{\mathrm{d}\alpha}{\alpha}\gamma^s(\alpha, n)\right)_{\kappa'\kappa}, \qquad \text{(fixed coupling)}$$

Where are exactly all the leading twist-2 $1/\epsilon_{\text{IR}}$ poles?



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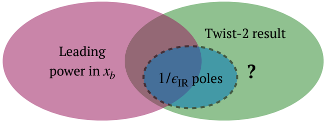
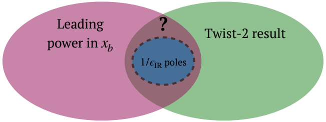
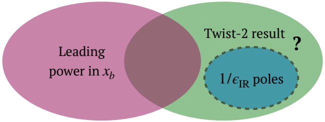


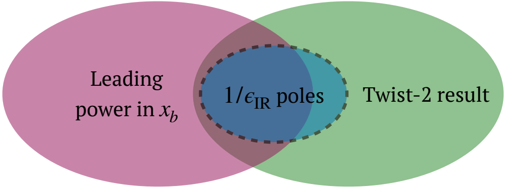

We can answer this by inspecting the perturbative series of the anomalous dimension:

$$\gamma_{gg}(n) = \frac{\alpha_s C_A}{\pi n} + \ldots\,,$$

$$\gamma_{qg}(n) = \frac{\alpha_s T_F}{3\pi} + \mathcal{O}\left(\frac{\alpha_s}{n}\right)$$

# Subtlety # 2: Non-trivial overlap *also of IR poles* between twist and small-$x_b$ expansions

There are IR divergences associated with small-$x_b$ logs as well as those appearing at higher powers in $x_b \ll 1$ expansion:




Our second goal is to consistently factorize IR poles of these two origins into the twist-2 PDF.

# The BFKL equation

Resummation of small-$x_b$ logs invovles solving the BFKL equation. For a function $f(x, \boldsymbol{q}_\perp)$

$$f(x, \boldsymbol{q}_\perp) \sim x^{p-1}\big(\text{logs of } x\big)\,,$$

that satisfies BFKL equation in $4$ dimensions:

$$x\frac{d}{dx}f(x,\boldsymbol{q}_\perp) = (p-1)f(x,\boldsymbol{q}_\perp) + c\big[K\otimes_\perp f\big](\boldsymbol{q}_\perp)$$

where

$$\big[K\otimes_\perp f\big](\boldsymbol{q}_\perp) \equiv (2\pi)\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^2}\left\{\frac{2f(\boldsymbol{k}_\perp)}{(\boldsymbol{q}_\perp-\boldsymbol{k}_\perp)^2} - \frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp-\boldsymbol{k}_\perp)^2}f(\boldsymbol{q}_\perp)\right\},$$

In the $n$-space we have an iterative equation

$$\bar{f}(n,\boldsymbol{q}_\perp) = \frac{1}{n+p}\times\underbrace{f(x=1,\boldsymbol{q}_\perp)}_{\text{Boundary condition}} - \frac{c}{n+p}\big[K\otimes_\perp\bar{f}(n)\big](\boldsymbol{q}_\perp)$$

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Eigenfunctions of BFKL Kernel:

$$\left[K \otimes_\perp \left(\frac{1}{\boldsymbol{k}_\perp^{2(1-\gamma)}}e^{\mathrm{i}n\phi}\right)\right](\boldsymbol{q}_\perp) = \chi(n,\gamma)\frac{1}{\boldsymbol{q}_\perp^{2(1-\gamma)}}e^{\mathrm{i}n\phi} \, , \qquad 0 < \mathrm{Re}\,\gamma < 1 \, .$$

# Bad boundary condition = IR divergence!

**What happens for $\gamma = 0$?**

$$\gamma = 0: \qquad \left[K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^2}\right](\boldsymbol{q}_\perp) = \frac{1}{\boldsymbol{q}_\perp^2}(2\pi)\int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2}$$

This Integral is divergent! But we can make sense of it in dimensional regularization:

$$(2\pi) I_\epsilon\left[\boldsymbol{q}_\perp^2\right] \equiv (2\pi)\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{\mathrm{d}^{2-2\epsilon} k_\perp}{(2\pi)^{2-2\epsilon}}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2}$$

$$= \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\Gamma(-\epsilon) e^{\epsilon\gamma_E}\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

$$= -\frac{1}{\epsilon} + \log\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\,.$$

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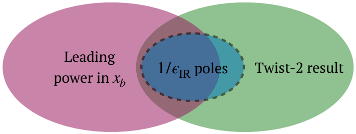


***This is relevant:*** Nature produces bad boundary conditions for the BFKL equation and these IR divergences go into the PDF, but *not every* IR divergence is generated this way.

# LL small-$x$ resummation by Catani and Hautmann

LL small-$x_b$ resummation consistent with twist factorization by Catani and Hautmann [CH94]:

$$\bar{F}_L^{(g)}(n) = h_L(\gamma_{gg}) \times R(n) \times \left(\frac{Q^2}{\mu^2}\right)^{\gamma_{gg}} \times \bar{\Gamma}_{gg} \,,$$

- At LL, the IR divergences in $F_L^g$ appear in $\bar{\Gamma}_{gg}$.
- $h_L$: describes coupling with photon, IR finite, defined via an *off-shell* cross section.
- $R$: scheme chosen to factorize the IR divergences.

$$h_L(\gamma) = \gamma \int_0^\infty \frac{\mathrm{d}\boldsymbol{k}_\perp^2}{\boldsymbol{k}_\perp^2} \left(\frac{\boldsymbol{k}_\perp^2}{Q^2}\right)^\gamma \hat{\sigma}_L^g\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}, \alpha_s, \epsilon = 0\right).$$

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- $R$: scheme chosen to factorize the IR divergences.

[CH94] resummed $(\frac{\alpha_s}{n})^\ell$ terms in $\gamma_{gg}(\alpha_s, n)$ by inventing a *gluon Green's function* $\bar{\mathcal{F}}_g^{(0)}$:

$$\bar{\mathcal{F}}_g^{(0)}(n, \boldsymbol{q}_\perp) = \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) + \frac{\bar{\alpha}_s}{n}\big[K \otimes_\perp \bar{\mathcal{F}}_g^{(0)}(n)\big](\boldsymbol{q}_\perp) \,, \qquad \bar{\alpha}_s \equiv \frac{\alpha_s C_A}{\pi}$$

$\bar{\mathcal{F}}_g^{(0)}$ is determined completely by the $\delta^{(2-2\epsilon)}(q_\perp)$ boundary condition and iterations of the BFKL kernel.

# LL small-$x$ resummation by Catani and Hautmann

Notice how BFKL kernel acts on $\delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp)$:

$$K \otimes_\perp \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) \sim \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon}$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon} \sim \frac{1}{\epsilon} \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-2\epsilon}$$

$$\vdots$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\ell\epsilon} \sim \frac{1}{\ell\epsilon} \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-(\ell+1)\epsilon}$$

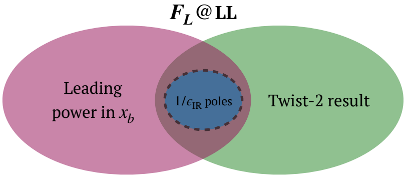
This generates an IR divergent series solution for $\bar{\mathcal{F}}_g^{(0)}$:

$$\bar{\mathcal{F}}_g^{(0)} \sim \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) + \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \sum_{\ell=1}^{\infty} c_\ell(\epsilon) \left(\frac{\bar{\alpha}_s}{n} \left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon}\right)^{\ell}, \qquad c_\ell(\epsilon) = \frac{1}{\ell!} \left(-\frac{1}{\epsilon}\right)^{\ell} \left(1 + \mathcal{O}(\epsilon^2)\right)$$

# A special property of the LL series and $F_L$ channel

$$\bar{\mathcal{F}}_g^{(0)} \sim \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) + \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}} \sum_{\ell=1}^{\infty} c_\ell(\epsilon) \left( \frac{\bar{\alpha}_s}{n} \left( \frac{\boldsymbol{k}_\perp^2}{\mu^2} \right)^{-\epsilon} \right)^{\ell} , \qquad c_\ell(\epsilon) = \frac{1}{\ell!} \left( -\frac{1}{\epsilon} \right)^{\ell} \left( 1 + \mathcal{O}(\epsilon^2) \right)$$

**A special property of the LL series and $F_L$ channel:** All the IR divergences at LL for $F_L$ are generated by BFKL equation

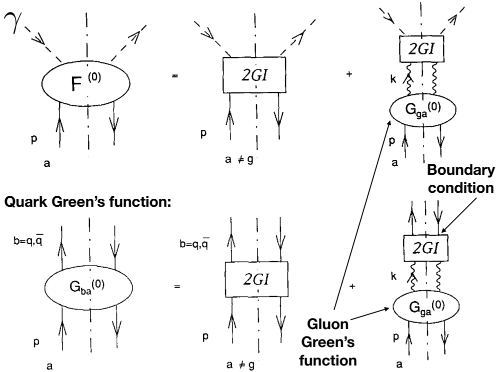


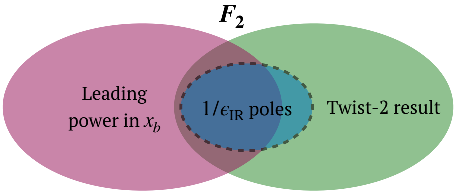

$\bar{\Gamma}_{gg}$ absorbs the IR divergences in $\bar{\mathcal{F}}_g^{(0)}$:

$$\bar{\mathcal{F}}_g^{(0)}(n, \boldsymbol{q}_\perp) = \frac{1}{\pi \boldsymbol{k}_\perp^2} \times \gamma_{gg} \times \tilde{R}(n, \boldsymbol{k}_\perp, \epsilon) \times \bar{\Gamma}_{gg} \, .$$

# LL small-$x$ resummation by Catani and Hautmann

- Resummation of $F_2$ and $\gamma_{qg}$ is not straightforward in this framework, because $F_2$ involves IR divergences NOT generated by BFKL evolution alone!.
- They introduced a new *quark's Green's function* to capture this non-BFKL divergence.



# Importance of higher order small-$x_b$ resummation

- The approach of Catani and Hautmann [CH94] has not been extended beyond LL.
- Higher order resummation is crucial: Large corrections from next-to-leading log small-$x_b$ resummation.

***Goal of this work:*** provide a new framework for higher order resummation using a factorization derived in SCET with Glauber operators of Rothstein and Stewart [RS16].

*See also Ciafaloni et al. [Cia+04], Altarelli, Ball, and Forte [ABF06], and Thorne [Tho01] and references therein.*

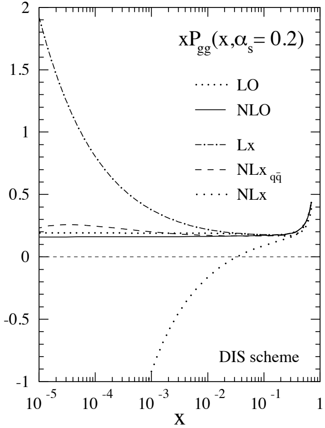



Figure from Blumlein et al. [Blu+98].

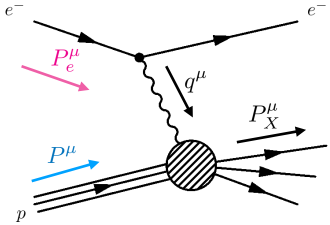
# Outline

# EFT modes and power counting

Center of mass light cone coordinates:

$$P^\mu = \frac{\sqrt{s}}{2} n^\mu \,, \quad P_e^\mu = \frac{\sqrt{s}}{2} \bar{n}_\mu \quad n^2 = \bar{n}^2 = 0\,, \quad n\cdot\bar{n} = 2\,.$$

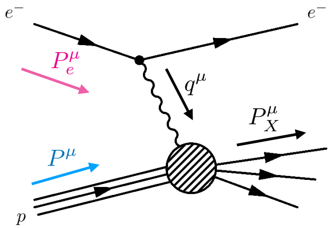
$$p^\mu = p^+ \frac{\bar{n}^\mu}{2} + p^- \frac{n^\mu}{2} + p_\perp^\mu \,, \qquad p^2 = p^+ p^- - \boldsymbol{p}_\perp^2$$




Power counting parameters: $\lambda' \sim \frac{\Lambda_{\mathrm{QCD}}}{Q}$ and $\lambda \sim x_b$ .

# EFT modes and power counting

Center of mass light cone coordinates:




$$P^\mu = \frac{\sqrt{s}}{2} n^\mu \,, \quad P_e^\mu = \frac{\sqrt{s}}{2} \bar{n}_\mu \quad n^2 = \bar{n}^2 = 0 \,, \quad n \cdot \bar{n} = 2 \,.$$

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Power counting parameters: $\lambda' \sim \frac{\Lambda_{\rm QCD}}{Q}$ and $\lambda \sim x_b$ .

Two possible scenarios based on the scaling of the invariant mass of hadronic state:

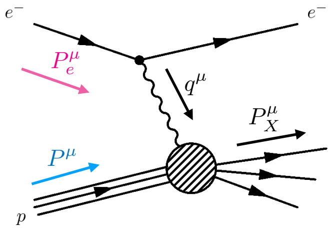
| | ***Hard scattering*** | | ***Forward scattering*** |
|---|---|---|---|
| $\frac{\boldsymbol{P_X^2}}{\boldsymbol{s}} = \frac{(q+P)^2}{s} = \frac{Q^2}{s}\frac{(1-x_b)}{x_b}$ | $\sim \lambda^0$ | or | $\sim \lambda$ |
| $\boldsymbol{q^\mu} = -\frac{Q^2}{\sqrt{s}}\frac{n^\mu}{2} + \frac{Q^2}{x_b\sqrt{s}}\frac{\bar{n}^\mu}{2} + q_\perp^\mu$ | $\sim \sqrt{s}(1, \lambda, \sqrt{\lambda})$ (collinear to $e^-$) | or | $\sim \sqrt{s}(\lambda, \lambda^2, \lambda)$ |

# EFT modes and power counting

Center of mass light cone coordinates:

$$P^\mu = \frac{\sqrt{s}}{2} n^\mu\,, \quad P_e^\mu = \frac{\sqrt{s}}{2}\bar{n}_\mu \quad n^2 = \bar{n}^2 = 0\,, \quad n\cdot\bar{n} = 2\,.$$

$$p^\mu = p^+ \frac{\bar{n}^\mu}{2} + p^- \frac{n^\mu}{2} + p_\perp^\mu\,, \qquad p^2 = p^+ p^- - \boldsymbol{p}_\perp^2$$




Power counting parameters: $\lambda' \sim \frac{\Lambda_{\rm QCD}}{Q}$ and $\lambda \sim x_b$ .

| | |
|---|---|
| photon momentum in forward scattering: | $q^\mu \sim \sqrt{s}(\lambda, \lambda^2, \lambda) \quad \Leftrightarrow \quad \frac{Q^2}{s} \sim \lambda^2$ |
| Collinear modes in the proton: | $p_c^\mu \sim \sqrt{s}\Big(\frac{\Lambda^2_{\rm QCD}}{s}, 1, \frac{\Lambda_{\rm QCD}}{\sqrt{s}}\Big) \sim \sqrt{s}\Big((\lambda\lambda')^2, 1, \lambda\lambda'\Big)$ |
| Small-$x_b$ resummation requires collinear modes with higher virtuality $p_n^2 \sim Q^2$ : | $p_n^\mu \sim \sqrt{s}(\lambda^2, 1, \lambda)$ |

We do not enforce $\lambda' \ll 1$ until later.

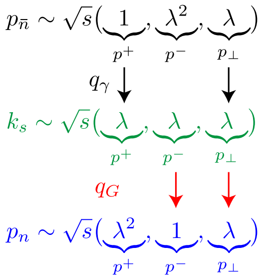
# EFT modes and power counting

***Forward scattering***

$$P_X^2/s \sim \lambda$$

$$q^\mu \sim \sqrt{s}(\lambda, \lambda^2, \lambda)$$

$$p_n^\mu \sim \sqrt{s}(\lambda^2, 1, \lambda)$$




The photon cannot interact directly with collinear mode without knocking it offshell. The leading terms start at $\mathcal{O}(\alpha_s^2)$ due to intermediate soft sector:

$$p_s = (p_s^+, p_s^-, p_{s\perp}) \sim \sqrt{s}(\lambda, \lambda, \lambda)\,.$$

Need additional Glauber modes for soft-collinear interaction:

$$q_G^\mu = q'^\mu \sim \sqrt{s}(\lambda^2, \lambda, \lambda)\,.$$

Having only soft and collinear particles in the final state is consistent with $P_X^2/s \sim \lambda$:

$$P_X^2 \sim (p_n + p_s)^2 \sim p_n^- p_s^+ \sim s\lambda\,.$$

# Outline

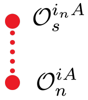
# SCET with Glauber operators

SCET Lagrangian:

$$\mathcal{L}_{\text{SCET}} = \sum_{n_i} \mathcal{L}_{n_i} + \mathcal{L}_s + \mathcal{L}_G \,.$$

Glauber operators derived in Rothstein and Stewart [RS16] account for forward scattering phenomena.

$$S_G^{(n_i s)} = 8\pi\alpha_s \sum_{ij} \int \mathrm{d}^4x \int \mathrm{d}^4z \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{e^{\mathrm{i}q\cdot(x-z)}}{\boldsymbol{q}_\perp^2} \mathcal{O}_{n_i}^{iA}(x)\, \mathcal{O}_s^{j n_i A}(z)$$

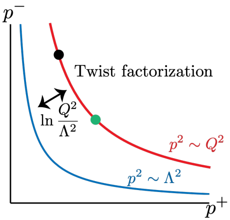


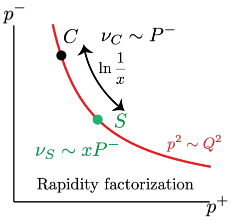

$$\mathcal{O}_s^{n_i,qA} = \overline{\psi}_S^{n_i} \mathbf{T}_i^A \frac{\not{n}_i}{2} \psi_S^{n_i} \,, \qquad \mathcal{O}_s^{n_i,gA} = \frac{1}{2} \mathcal{B}_{S\perp\mu}^{n_i B} (\mathrm{i} f^{ABC}) \frac{n_i}{2} \cdot (\mathcal{P} + \mathcal{P}^\dagger) \mathcal{B}_{S\perp}^{n_i C\mu} \,,$$

$$\mathcal{O}_{n_i}^{qA} = \overline{\chi}_{n_i} \mathbf{T}_i^A \frac{\bar{\not{n}}_i}{2} \chi_{n_i} \,, \qquad \mathcal{O}_{n_i}^{gA} = \frac{1}{2} \mathcal{B}_{n\perp\mu}^{B} (\mathrm{i} f^{ABC}) \frac{\bar{n}_i}{2} \cdot (\mathcal{P} + \mathcal{P}^\dagger) \mathcal{B}_{n\perp}^{C\mu} \,,$$

Dotted propagator represents insertion of operators from the Glauber Lagrangian.

# Twist vs. rapidity factorization




| | |
|---|---|
| > Hard matching at scale $Q$. | > No hard matching. The EFT at scale $Q$ reproduces QCD in the forward scattering limit. |
| > IR divergences in QCD $\leftrightarrow$ UV divergences in the low energy theory at $p^2 \sim \Lambda^2_{\text{QCD}}$. | > No rapidity divergences in QCD (but large rapidity logs). Rapidity divergences in EFT $\leftrightarrow$ an artifact of separating soft and collinear modes. |
| > IR divergences can be regulated in dimensional regularization. | > Rapidity divergences require new regulators. |

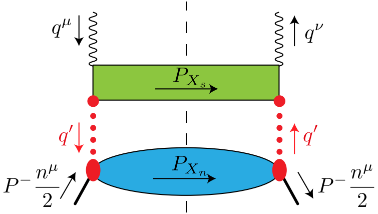
# Small-$x$ factorization formula

We include *two insertions* of the $ns$ Glauber action:

$$S_G^{ns} = 8\pi\alpha_s \sum_{i,j,A} \int \mathrm{d}^d y \int \mathrm{d}^d x \int \frac{\mathrm{d}^d q'}{(2\pi)^d} \frac{e^{\mathrm{i}(x-y)\cdot q'}}{\boldsymbol{q}_\perp'^2} \mathcal{O}_n^{iA}(x)\mathcal{O}_s^{j_n A}(y)\,.$$

Factorization formula at NLL:

$$W^{\alpha\beta}(q,P) = \int \mathrm{d}^{d-2}q'_\perp \; S^{\alpha\beta}\Big(q, q'_\perp, \frac{\nu}{x_b P^-}, \epsilon\Big) \; C\Big(q'_\perp, P, \frac{\nu}{P^-}, \epsilon\Big) + \dots\,.$$



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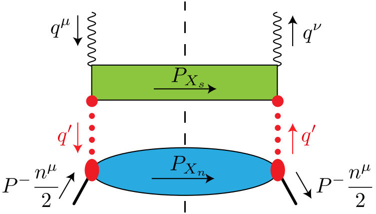


Factorization formula at NLL:

$$W^{\alpha\beta}(q,P) = \int \mathrm{d}^{d-2} q_\perp' \; S^{\alpha\beta}\Big(q, q_\perp', \frac{\nu}{x_b P^-}, \epsilon\Big) \; C\Big(q_\perp', P, \frac{\nu}{P^-}, \epsilon\Big) + \dots\,.$$

The collinear and soft functions are defined as

$$C \equiv \frac{1}{\pi\nu}\frac{1}{\boldsymbol{q}_\perp'^2} \sum_{i,j,A} \int \frac{\mathrm{d}q'^+}{2\pi} \int \mathrm{d}^d x\, e^{\mathrm{i}\frac{x^- q'^+}{2} + \mathrm{i}x_\perp \cdot q_\perp'} \langle P|\mathcal{O}_n^{i\,A}(x)\mathcal{O}_n^{j\,A}(0)|P\rangle_\nu\,,$$

$$S^{\alpha\beta} \equiv \frac{\nu}{\boldsymbol{q}_\perp'^2} \frac{(2\pi\iota\mu^2)^{4-d}\big(8\pi\alpha_s(\mu^2)\big)^2}{16\pi^2\,(N_c^2-1)} \sum_{i,j,A} \int \frac{\mathrm{d}q'^-}{4\pi} \int \mathrm{d}^d z\, e^{\mathrm{i}z\cdot q} \int \mathrm{d}^d y_L \mathrm{d}^d y_R$$

$$\times e^{-\mathrm{i}\frac{q'^-(y_L^+ - y_R^+)}{2} - \mathrm{i}q_\perp'\cdot(y_{L\perp} - y_{R\perp})} \langle 0|\bar{T}\{J^\alpha(z)\mathcal{O}_s^{i_n\,A}(y_L)\}T\{J^\beta(0)\mathcal{O}_s^{j_n\,A}(y_R)\}|0\rangle_\nu\,.$$

# Small-$x$ factorization formula

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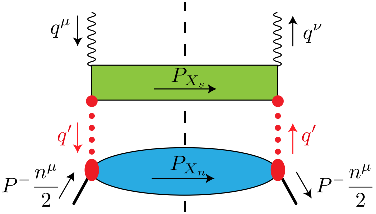



Here small-$x_b$ logs are resummed via *rapidity evolution* for $\nu_S \sim x_b P^-$ and $\nu_C \sim P^-$

$$\frac{\nu_S}{\nu_C} = x_b$$

# Collinear function at NLO

We computed the collinear function at NLO

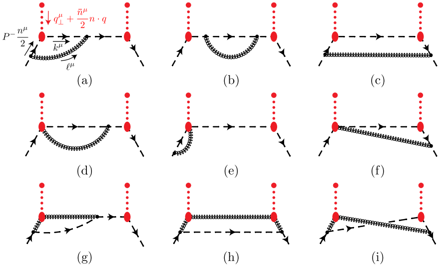
$$C_\kappa^{\text{LO}}(q'_\perp) = \frac{P^-}{\nu}\frac{c_\kappa}{\pi \boldsymbol{q}'^2_\perp}\,, \qquad c_\kappa = C_F, C_A$$

(bad boundary condition!)

$$C_q^{\text{NLO}} = \bar{\alpha}_s C_q^{\text{LO}} \times (-2\pi)\, I_\epsilon\left[\boldsymbol{q}'^2_\perp\right] \left( \frac{1}{\eta} + \ln\left(\frac{\nu}{P^-}\right) + \frac{3}{4} \right),$$

$$C_g^{\text{NLO}} = \bar{\alpha}_s C_g^{\text{LO}} \times (-2\pi)\, I_\epsilon\left[\boldsymbol{q}'^2_\perp\right]$$

$$\times \left( \frac{1}{\eta} + \ln\left(\frac{\nu}{P^-}\right) + \frac{11}{12} - \frac{n_f T_R}{4C_A}\left(1 - \frac{1}{3(1-\epsilon)}\right)\right),$$




$$(2\pi) I_\epsilon\left[\boldsymbol{r}^2_\perp\right] = -\frac{1}{\epsilon} + \ln\left(\frac{\boldsymbol{r}^2_\perp}{\mu^2}\right) + \mathcal{O}(\epsilon)\,, \qquad \bar{\alpha}_s \equiv \frac{\alpha_s C_A}{\pi}$$

We see that the one-loop contribution is IR divergent and exhibits a rapidity divergence.

# Process independence and the BFKL equation

Rothstein and Stewart [RS16] showed that for $pp \to pp$ forward scattering

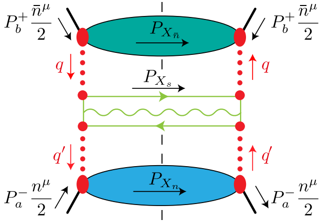
$$\sigma^{pp\to pp} \sim C_n \otimes S^{pp} \otimes C_{\bar{n}}$$

and $S^{pp}$ satisfies the BFKL equation:

$$\nu \frac{\mathrm{d}}{\mathrm{d}\nu} S^{pp} \sim +2\bar{\alpha}_s \iota^\epsilon K \otimes_\perp S^{pp}$$

The collinear function is process independent and is expected to satisfy the BFKL equation from RG consistency:

$$\nu \frac{d}{d\nu} C = -C - \bar{\alpha}_s \iota^\epsilon K \otimes_\perp C\,.$$




Drell-Yan

The predicted rapidity logarithm agrees with our NLO result:

$$C_{\kappa\,\mathrm{LL}} = \frac{\nu}{P^-}\frac{c_\kappa}{\pi \boldsymbol{q}_\perp'^2}\left(1 - \bar{\alpha}_s (2\pi) I_\epsilon [\boldsymbol{q}_\perp'^2] \ln\left(\frac{\nu}{P^-}\right)\right) + \mathcal{O}(\alpha_s^2)\,.$$

# More IR divergences

$$\frac{1}{x_b} F_a(q,P) = \int_0^\infty \mathrm{d}^{d-2} q'_\perp \, S_a\left(q, q'_\perp, \frac{\nu}{x_b P^-}, \epsilon\right) C\left(q'_\perp, \frac{\nu}{P^-}, \epsilon\right), \qquad [S^{\mu\nu}] = 4-d\,, \qquad [C] = -2\,.$$

The convolution itself generates IR divergences as nothing prevents $q'_\perp$ from entering the IR region. To see this explicitly, let us note that the $\mathrm{SCET_{II}}$ collinear function has the all-orders expansion:

$$C\left(q'_\perp, \frac{\nu}{P^-}, \alpha_s(\mu^2), \epsilon\right) = \frac{1}{\boldsymbol{q}'^2_\perp} \sum_{\ell=0}^\infty C^{(\ell)}\left(\alpha_s(\mu^2), \frac{\nu}{P^-}, \epsilon\right) \left(\frac{\boldsymbol{q}'^2_\perp}{\mu^2}\right)^{-\ell\epsilon}.$$

Alternative form of the factorization formula:

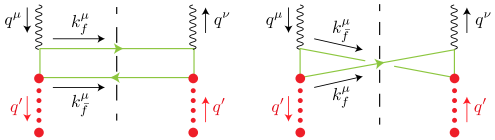
$$\frac{1}{x_b} F_a = \sum_{\ell=0}^\infty \left(\frac{\boldsymbol{q}^2_\perp}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a\left(\gamma = -\ell\epsilon\right).$$

The $\gamma$-transform of the soft function:

$$\tilde{S}_a(\gamma) \sim \int \frac{\mathrm{d}^{2-2\epsilon} q'_\perp}{\boldsymbol{q}'^2_\perp} \left(\frac{\boldsymbol{q}'^2_\perp}{\mu^2}\right)^\gamma S_a(q_\perp, q'_\perp, \epsilon)\,.$$

# More IR divergences

Leading order soft function calculated from



is IR finite for $\gamma \neq 0$:

$$\tilde{S}_2^{\text{LO}}(\gamma) = \alpha_s^2 n_f T_F \left(\frac{\nu}{x_b P^-}\right)\left(\frac{\pi^2\big(-3\gamma^2+3\gamma+2\big)\csc^2(\pi\gamma)}{8\Gamma(\frac{5}{2}-\gamma)\Gamma(\frac{3}{2}+\gamma)}\right) + \mathcal{O}(\epsilon)\,,$$

$$\tilde{S}_L^{\text{LO}}(\gamma) = \alpha_s^2 n_f T_F \left(\frac{\nu}{x_b P^-}\right)\left(\frac{\pi^2\big(-\gamma+1\big)\csc^2(\pi\gamma)}{4\Gamma(\frac{5}{2}-\gamma)\Gamma(\frac{3}{2}+\gamma)}\right) + \mathcal{O}(\epsilon)\,.$$

# More IR divergences

In the convolution the collinear function forces us to set $\gamma = -\ell\epsilon$,

$$\frac{1}{x_b} F_a = \sum_{\ell=0}^{\infty} \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a\big(\ \gamma = -\ell\epsilon\ \big)\,.$$

which implies

$$\lim_{\epsilon\to 0} \tilde{S}_2^{\text{LO}}(-\ell\epsilon) = \frac{2\alpha_s^2 n_f T_F}{3\pi} \frac{1}{(\ell+1)(\ell+2)} \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \mathcal{O}(\epsilon^0)\right),$$

$$\lim_{\epsilon\to 0} \tilde{S}_L^{\text{LO}}(-\ell\epsilon) = \frac{2\alpha_s^2 n_f T_F}{3\pi} \frac{1}{(\ell+1)} \left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right).$$

The $\tilde{S}_a$ soft function will also contribute to the PDF despite being a vacuum matrix element.

# More IR divergences

In the convolution the collinear function forces us to set $\gamma = -\ell\epsilon$,

$$\frac{1}{x_b} F_a = \sum_{\ell=0}^{\infty} \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a\left( \gamma = -\ell\epsilon \right) .$$

We find that for $\gamma \neq 0$, $\tilde{S}_L$ and $\tilde{S}_2$ are proportional to the off-shell cross section that appear in [CH94]:

$$\tilde{S}_2(\gamma, \epsilon = 0) = \left(\frac{\nu}{x_b P^-}\right) \alpha_s \frac{h_2(\gamma)}{\gamma^2} , \tag{1}$$

$$\tilde{S}_L(\gamma, \epsilon = 0) = \left(\frac{\nu}{x_b P^-}\right) \alpha_s \frac{h_L(\gamma)}{\gamma} .$$

This is not the right limit for us and the full $\epsilon$ dependence is needed to perform small-$x_b$ resummation.

> In [CH94] these IR divergences were separately captured in the gluon and quark Green's functions.

# Leading log small-$x_b$ resummation

Setting $\nu = \nu_S$ trivializes rapidity logs in the soft function:

$$\frac{1}{x_b} F_a^\kappa(x_b, Q^2) = \int \mathrm{d}^{d-2} q'_\perp S_a(1, q_\perp, q'_\perp, \epsilon) C_\kappa(x_b, q'_\perp, \epsilon)$$

Mellin space :

$$\bar{C}_\kappa(n, q'_\perp, \epsilon) = \frac{c_\kappa}{n\pi \boldsymbol{q}'^2_\perp} + \frac{\bar{\alpha}_s \iota^\epsilon}{n} K \otimes_\perp \bar{C}_\kappa(n, q'_\perp, \epsilon) \quad , \quad c_\kappa = C_F, C_A \, , \quad \bar{\alpha}_s = \frac{\alpha_s C_A}{\pi}$$

Solve for $\bar{C}_\kappa$ as a power series as before:

$$\bar{C}_{\kappa,\mathrm{LL}}(n, q'_\perp, \epsilon) = \frac{1}{n} \frac{c_\kappa}{\pi \boldsymbol{q}'^2_\perp} \sum_{\ell=0}^{\infty} c_{\ell+1}(\epsilon) \left( \frac{\bar{\alpha}_s}{n} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \left( \frac{\boldsymbol{q}'^2_\perp}{\mu^2} \right)^{-\epsilon} \right)^\ell , \qquad c_\ell(\epsilon) = \frac{1}{\ell!} \left( \frac{-1}{\epsilon} \right)^\ell \left( 1 + \mathcal{O}(\epsilon^2) \right)$$

Now include the soft contribution to arrive at small-$x_b$ resummed structure functions:

$$\bar{F}^\kappa_{a,\mathrm{LL}}(n, Q^2) = \frac{c_\kappa}{n\pi} \left( \frac{\boldsymbol{q}^2_\perp}{\mu^2} \right)^{-2\epsilon} \sum_{\ell=0}^{\infty} d_{a,\ell+1}(\epsilon) \left( \frac{\bar{\alpha}_s}{n} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \left( \frac{\boldsymbol{q}^2_\perp}{\mu^2} \right)^{-\epsilon} \right)^\ell \quad , \quad d_{a,\ell+1}(\epsilon) \equiv c_{\ell+1}(\epsilon) \tilde{S}_a(1, -\ell\epsilon, \alpha_s, \epsilon)$$

# Outline

# Small-$x$ vs. twist expansion

Here we are dealing with two different power expansions simultaneously:

$$\lambda \sim x_b \qquad \text{and} \qquad \lambda' \sim \frac{\Lambda_{\text{QCD}}}{Q} \,.$$

Key subtleties:

- Small-$x_b$ and twist expansions *do not commute*.
- Both expansions have terms that are leading power in one but subleading in the other.

Consider the fixed order series: Leading twist-2 contributions at $\mathcal{O}(\alpha_s^0)$ and $\mathcal{O}(\alpha_s)$ are actually power suppressed in $x_b$-expansion. For example,

$$H_L^{(g)}(x) \sim \alpha_s\, x(1-x) + \mathcal{O}(\alpha_s^2) \qquad \Leftrightarrow \qquad \bar{H}_L^{(g)}(x) \sim \alpha_s \Big(\frac{1}{n+2} - \frac{1}{n+3}\Big) + \mathcal{O}(\alpha_s^2)$$

Thus in connecting with the twist expansion we will have to include power suppressed pieces.
(See an illustration in the backup.)

# BFKL Resummation of $F_L$

We set $\mu^2 = Q^2$ and start with formula involving unknown pieces (HP = higher power)

$$\bar{F}^g_{L,\text{HP}} + \bar{F}^g_{L,\text{LL}}(n) = \bar{H}^{(g)}_L\Big(n, \frac{Q^2}{\mu^2} = 1, \alpha_s\Big)\bar{\Gamma}_{gg}(\alpha_s, n) \ .$$

Parameterize the the terms we want to determine for LL results as

$$\bar{H}^{(g)}_L = \frac{\alpha_s}{\pi}\sum_{k=0}^{\infty}\epsilon^k h^{(0,k)}_{L,g} + \frac{\alpha_s}{\pi}\sum_{\ell=1}^{\infty}\Big(\frac{\alpha_s}{\pi n}\Big)^{\ell}\sum_{k=0}^{\infty}\epsilon^k h^{(\ell,k)}_{L,g} \,,$$

$$\gamma_{gg} = \sum_{\ell=1}^{\infty}\gamma_{gg,\ell-1}\Big(\frac{\bar{\alpha}_s}{\pi}\Big)^{\ell} \,,$$

$$\bar{F}^g_{L,\text{HP}} = \frac{\alpha_s}{\pi}\sum_{k=-1}^{\infty}\epsilon^k f^{(k)}_{L,g} \,.$$

We have truncated the higher power pieces to $\mathcal{O}(\alpha_s)$ which is sufficient for LL resummation in small-$x_b$.

# BFKL Resummation of $F_L$

We set $\mu^2 = Q^2$ and start with formula involving unknown pieces (HP = higher power)

$$\bar{F}^g_{L,\text{HP}} + \bar{F}^g_{L,\text{LL}}(n) = \bar{H}^{(g)}_L\Big(n, \frac{Q^2}{\mu^2} = 1, \alpha_s\Big)\bar{\Gamma}_{gg}(\alpha_s, n) \,.$$

By sequentially comparing the coefficients of $(\alpha_s/\epsilon)^\ell$, $\alpha_s(\alpha_s/\epsilon)^\ell, \ldots$ terms we find

$$\gamma_{gg} = \frac{\bar{\alpha}_s}{n} + 2\zeta_3\Big(\frac{\bar{\alpha}_s}{n}\Big)^4 + \ldots \,,$$

$$\bar{H}^{(g)}_L = \frac{2\alpha_s n_f T_F}{3\pi}\bigg(1 - \frac{1}{3}\frac{\bar{\alpha}_s}{n} + \Big(\frac{34}{9} - \zeta_2\Big)\Big(\frac{\bar{\alpha}_s}{n}\Big)^2 + \Big(-\frac{40}{27} + \frac{\pi^2}{18} + \frac{8}{3}\zeta_3\Big)\Big(\frac{\bar{\alpha}_s}{n}\Big)^3 + \ldots\bigg) \,,$$

$$\bar{F}^g_{L,\text{HP}} = \frac{2\alpha_s n_f T_F}{3\pi}\bigg(1 + 3\epsilon + \Big(6 - \frac{1}{2}\zeta_2\Big)\epsilon^2 + \Big(12 - \frac{\pi^2}{4} - \frac{7}{3}\zeta_3\Big)\epsilon^3 + \ldots\bigg) \,.$$

- ✓ Series agree with LL results in Catani and Hautmann [CH94]. Interestingly, we simultaneously determine the LL results for $\gamma_{gg}$ and $\bar{H}^{(g)}_L$.
- ✓ We determined the unknown power suppressed pieces self-consistently!
- ✓ $F^g_{L,\text{HP}}$ has no IR poles $\rightarrow$ All the poles in $F_L$ channel generated through BFKL evolution.

# Resummation of $F_2$

For $F_2$, we write

$$\bar{F}^g_{2,\text{HP}} + \bar{F}^g_{2,\text{LL}}(n) = 2n_f\,\bar{\Gamma}_{qg} + \bar{H}^{(g)}_2\,\bar{\Gamma}_{gg}$$

Following the same steps as before, we find

$$\gamma_{qg} = \frac{\alpha_s T_F}{3\pi}\left(1 + \frac{5}{3}\frac{\bar{\alpha}_s}{n} + \frac{14}{9}\left(\frac{\bar{\alpha}_s}{n}\right)^2 + \left(\frac{82}{81} + 2\zeta_3\right)\left(\frac{\bar{\alpha}_s}{n}\right)^3 + \dots\right),$$

$$\bar{H}^{(g)}_2 = \frac{\alpha_s n_f T_F}{3\pi}\left(1 + \left(\frac{43}{9} - 2\zeta_2\right)\frac{\bar{\alpha}_s}{n} + \left(\frac{1234}{81} - \frac{13}{3}\zeta_2 + \frac{4}{3}\zeta_3\right)\left(\frac{\bar{\alpha}_s}{n}\right)^3 + \dots\right),$$

$$\bar{F}^g_{2,\text{HP}} = \frac{\alpha_s n_f T_F}{3\pi}\left(-\frac{2}{\epsilon} + 1 + (1+\zeta_2)\epsilon + \left(1 - \frac{1}{2}\zeta_2 + \frac{14}{3}\zeta_3\right)\epsilon^2 + \dots\right).$$

The IR pole in $\bar{F}^g_{2,\text{HP}}$ does not result from BFKL evolution. This required [CH94] to introduce a new auxiliary object, *the quark Green's function* (see backup). For us it results straightforwardly from our soft function $\tilde{S}_2$.

# Comparison with previous work

> ***Objects in factorization:***

[CH94] Made use of off-shell cross sections which can only be guaranteed to be gauge invariant at leading order.

here Employed individually gauge invariant (to all orders) collinear and soft functions.

> ***Resummation of $F_L$ vs. $F_2$:***

[CH94] Needed to define a separate quark Green's function for $F_2$

here Resummation of both $F_2$ and $F_L$ follow from the same soft function.

> ***Manifest power counting***

[CH94] Included $\mathcal{O}(\alpha_s)$ higher power pieces from the beginning.

here The resummed structure function $\bar{F}^{\kappa}_{a,\mathrm{LL}}$ is manifestly leading power. We could self-consistently determine the power suppressed pieces by demanding consistency with twist factorization.

> ***NLO computation***

[CC99] Calculated *impact factor* analogous to our collinear function, but required a careful subtraction of Green's function pieces, inducing factorization scheme dependencies.

here Computation of factorized functions in our formalism follow straightforwardly from operator definitions. No process or factorization scheme dependence.

# Conclusion

- We have shown how to construct from the SCET framework with Glauber interactions
  - small-$x_b$ factorization to NLL,
  - and resummation done explicitly to LL.
- Factorization involves a universal collinear function. Such universality is not obvious in the traditional approach.
- Advantages of the EFT approach:
  - Factorization functions gauge invariant to all orders.
  - No separate Green's functions needed to be calculated.
  - Off-shell cross sections replaced by one soft function $S^{\alpha\beta}$ for all DIS channels.
  - Manifest power counting.
  - No factorization or scheme dependencies.
  - Universal, process independent, collinear-function.
- This work provides a new framework for extending resummed calculations for coefficient functions and anomalous dimensions to higher logarithmic orders.

# Thank you!

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# Outline

# Backup

# Resummation of $\gamma_{qg}$ by Catani and Hautmann [CH94]

For $F_2$ structure function, they showed

$$\gamma_{gg}(N,\alpha_s)\bar{H}_2^{(g)}(n, Q^2/\mu^2 = 1, \alpha_s) + 2n_f\gamma_{qg}(\alpha_s, n) = h_2(\gamma)R(n,\alpha_s)\,,$$

where

$$h_2(\gamma) = \gamma\int_0^\infty \frac{\mathrm{d}\boldsymbol{k}_\perp^2}{\boldsymbol{k}_\perp^2}\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}\right)^\gamma \frac{\partial}{\partial \ln Q^2}\hat{\sigma}_2^g\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}, \alpha_s, \epsilon = 0\right).$$

Notice that they needed to take $\ln Q^2$ derivative as $\hat{\sigma}_2^g$ is not collinear safe. The structure of IR divergences in $\gamma_{qg}$ gets polluted by $1/\epsilon$ divergence in $\hat{\sigma}_2^g$, so define a new *quark Green function*:

$$G_{qg}^{(0)}(n,\alpha_s,\epsilon) = \int \mathrm{d}^{d-2}\boldsymbol{k}_\perp\, \hat{K}_{qg}\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}, \alpha_s, \mu, \epsilon\right)\mathcal{F}_g^{(0)}(n, \boldsymbol{k}_\perp, \alpha_s, \mu, \epsilon)\,.$$

$K_{qg}$ includes the $1/\epsilon$ pole associated with $\hat{\sigma}_2^g$ (same as what we saw in $\bar{F}^g_{2,\mathrm{HP}}$ above). Consistency with DGLAP resummation then enables determination of $\gamma_{qg}$ anomalous dimension using $G_{qg}^{(0)}$, although not in a closed form as in $\gamma_{gg}$.

# How do IR poles exponentiate?

After resumming the leading $(\bar{\alpha}_s/n)^\ell$ terms:

$$\bar{F}^{\kappa}_{a,\mathrm{LL}}(n,Q^2) = \frac{c_\kappa}{n\pi}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-2\epsilon}\sum_{\ell=0}^{\infty} d_{a,\ell+1}(\epsilon)\left(\frac{\bar{\alpha}_s}{n}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\right)^{\ell}$$

In twist expansion the bare structure function (in dim-reg) factorizes as

$$\bar{F}^{\kappa}_{p}(n,Q^2) = \sum_{\kappa'}\bar{H}_p^{(\kappa')}\left(n,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right)\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2),n) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right).$$

In the fixed coupling approximation the partonic PDF is

$$\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2),n) = \mathrm{P}\exp\left(-\frac{1}{\epsilon}\int_0^{\alpha_s(\mu^2)}\frac{\mathrm{d}\alpha}{\alpha}\gamma^s(\alpha,n)\right)_{\kappa'\kappa}.$$

For parton $\kappa \to \kappa'$ it captures the infra-red divergences of the perturbative calculation.

# How do IR poles exponentiate?

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Let us illustrate how the leading $(\alpha_s/\epsilon)^\ell$ IR poles exponentiate. The $d_{a,\ell}$ coefficients for $a = L$ behave as

$$\frac{1}{n}\left(\frac{\bar{\alpha}_s}{n}\right)^{\ell} d_{L,\ell+1}(\epsilon) = \frac{2\alpha_s n_f T_F}{3\pi}\left[\frac{1}{(\ell+1)!}\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)^{\ell+1} + \mathcal{O}(\epsilon^{-\ell})\right]$$

Thus,

$$\bar{F}^{g}_{L,\mathrm{LL}}(n) + \frac{2\alpha_s n_f T_F}{3\pi} = \frac{2\alpha_s n_f T_F}{3\pi}\left[\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)^{\ell}\left(1+\mathcal{O}(\epsilon)\right)\right]$$

$$= \frac{2\alpha_s n_f T_F}{3\pi}\,\exp\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)\left(1+\mathcal{O}\left(\frac{\bar{\alpha}_s}{n}\right)\right) + \mathcal{O}\left(\frac{1}{\epsilon}\left(\frac{\bar{\alpha}_s}{n}\right)^2\right)$$

Necessary to add by hand the $\mathcal{O}(\alpha_s)$ term to factorize IR divergences.

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