Subleading Effects in Soft-Gluon Emission at One-Loop in Massless QCD

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QCD@LHC, September 6, 2023

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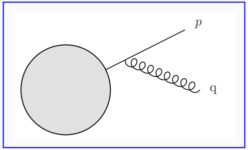
Infrared Divergences and Power Corrections

Infrared Divergences

- Amplitudes suffer from divergences when there is soft or collinear radiation, because the propagators of the external legs blow up $\frac{1}{(p+q)^2} = \frac{1}{2p\cdot q}$



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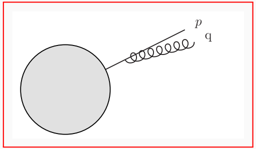

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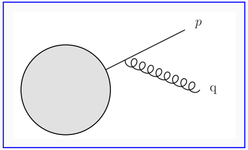

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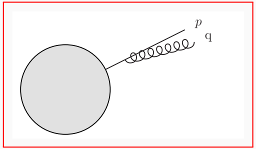

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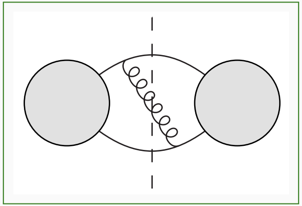
- Divergences cancel inclusively between real and virtual emissions



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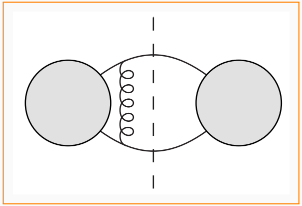

$$= \overbrace{\underbrace{\int_1 \langle M^{(0)}_{n+1}|M^{(0)}_{n+1}\rangle}_{\text{divergent}} + \underbrace{2\text{Re}\,\langle M^{(1)}_{n}|M^{(0)}_{n}\rangle}_{\text{divergent}}}^{\text{finite}}$$

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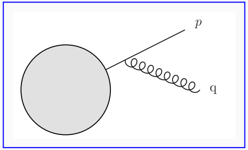

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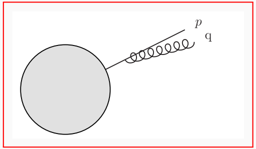

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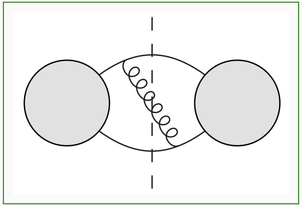

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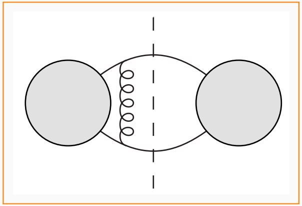

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- Divergences prevent direct numeric phase space integration.

Subtraction Schemes

$$\sigma_{\mathsf{NLO}} = \int_{m+1} \left(\mathrm{d}\sigma^{R}_{\mathsf{LO}}\right) + \int_{m} \left[\mathrm{d}\sigma^{V}_{\mathsf{NLO}}\right] = \int_{m+1} \left(\mathrm{d}\sigma^{R}_{\mathsf{LO}} - \mathrm{d}\sigma^{A}_{\mathsf{LO}}\right) + \int_{m} \left[\mathrm{d}\sigma^{V}_{\mathsf{NLO}} + \int_{1} \mathrm{d}\sigma^{A}_{\mathsf{LO}}\right]$$

Separately finite

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Separately finite

- Consider now **soft phase-space region**: "+1" momentum $q = \mathcal{O}(\lambda)$
- Laurent expansion: $\sigma^{\text{R}} = \frac{\sigma^{\text{R}}_{\text{LP}}}{\lambda^2} + \frac{\sigma^{\text{R}}_{\text{NLP}}}{\lambda} + \mathcal{O}(\lambda^0)$, $\frac{\sigma^{\text{R}}_{\text{LP}}}{\lambda^2} = \sigma^{A}$
- LP: leading power, NLP: next-to-leading (subleading) power

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Large cancellation Separately finite

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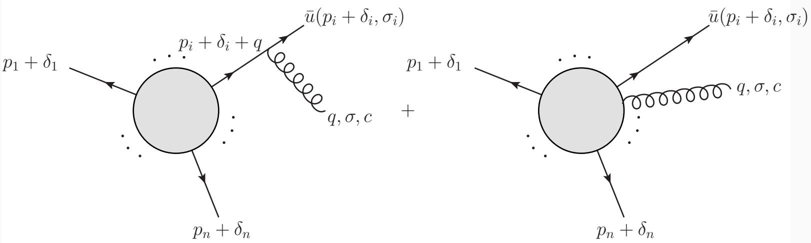
- LP: leading power, NLP: next-to-leading (subleading) power
- Calculating σ^{R} for very soft phase-space points can be numerically unstable, replacing $\mathrm{d}\sigma^{R} - \mathrm{d}\sigma^{A}$ with $\sigma^{\mathsf{R}}_{\mathsf{NLP}}$ for such points has been applied as *next-to-soft stabilization* in QED.

(Banerjee et al., 2106.07469) (Banerjee et al., 2107.12311) (Broggio et al., 2212.06481)

Subleading Soft at Tree Level: LBK Theorem (Low, 1958), (Burnett and Kroll, 1968)

$$|M_g^{(0)}(\{p_i\},\{q_i\},q)\rangle = \sum_i$$



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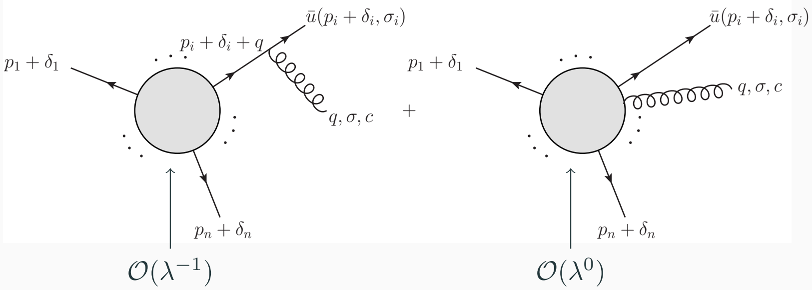

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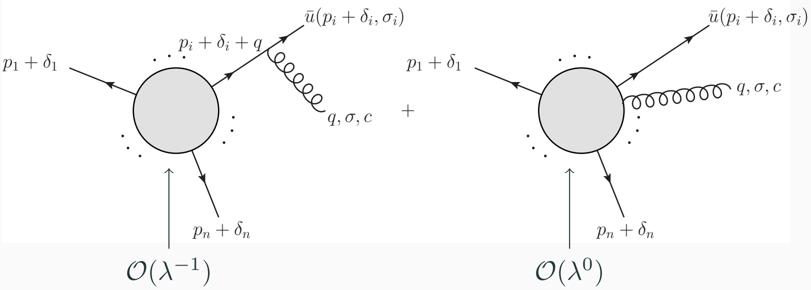

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$$|M_g^{(0)}(\{p_i+\delta_i\},q)\rangle = \mathbf{S}^{(0)}(\{p_i\},\{\delta_i\},q)\ |M^{(0)}(\{p_i\})\rangle + \mathcal{O}(\lambda)\ ,$$

$$\mathbf{S}_i^{(0)} = \frac{p_i\cdot\epsilon^*}{p_i\cdot q} + \frac{1}{p_i\cdot q}\left[\left(\epsilon^* - \frac{p_i\cdot\epsilon^*}{p_i\cdot q}q\right)\cdot\delta_i + p_i\cdot\epsilon^*\sum_j\delta_j\cdot\partial_j + \frac{1}{2}F_{\mu\nu}\left(J_i^{\mu\nu}-\mathbf{K}_i^{\mu\nu}\right)\right]$$

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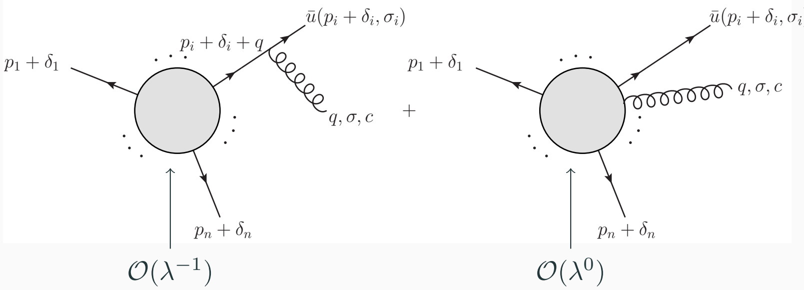

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Our Goal: Extend the LBK theorem to one loop!

State of the Art Power Corrections at One Loop

SCET:

- Very successful but process dependent (Larkoski, Neill, and Stewart, 1412.3108), (Beneke et al., 1912.01585), (Liu et al., 2112.00018)

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QCD:

- ???

Subleading Soft at One-Loop: Method of Regions (Beneke and Smirnov, hep-ph/9711391)

- Objective: Taylor expansion of loop integral in some small scale λ
- Decompose loop momentum $l = l_+ n + l_\perp + l_- \bar{n}$, $\quad l_\perp \cdot n = l_\perp \cdot \bar{n} = 0$, $\quad n \cdot \bar{n} = \frac{1}{2}$
- Assign **scaling behavior** to the components: $l_+ = \mathcal{O}(\lambda_+)$, $l_- = \mathcal{O}(\lambda_-)$, $l_\perp = \mathcal{O}(\lambda_\perp)$
- Identify **momentum regions** $(\lambda_+, \lambda_\perp, \lambda_-)$:
  - Hard region $(1, 1, 1)$
  - Soft region $(\lambda, \lambda, \lambda)$
  - i-collinear region: $n \propto p_i$, $(1, \sqrt{\lambda}, \lambda)$

$\longrightarrow$ Can expand integrand in λ *before* integration, as long as all possible regions are summed

- Each region is **independently gauge invariant!**

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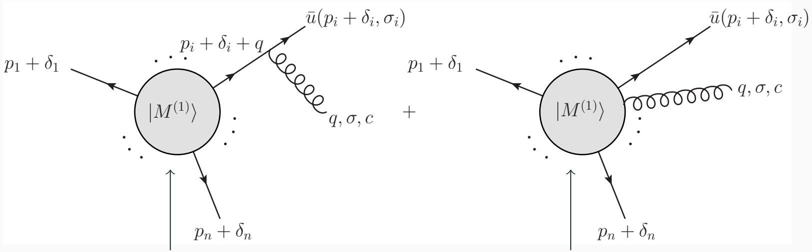
- **Idea: Apply the method of regions to soft radiation in a process independent manner!**

Subleading Effects at One Loop

Hard Region

$$|M_g^{(1)}(\{p_i\},\{q_i\},q)\rangle = \sum_i$$



$\mathcal{O}(\lambda^{-1})$ $\mathcal{O}(\lambda^{0})$

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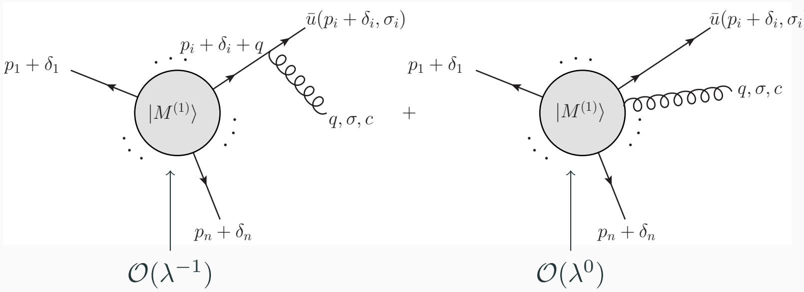
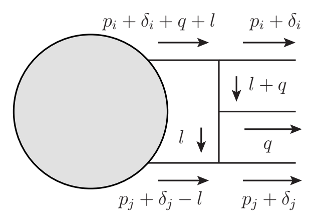


- Loop momentum is hard and all internal lines are far from the mass shell except for the emission from external lines $\longrightarrow$ soft divergences arise only from external emission
- We can directly apply tree-level results (LBK) on one-loop amplitudes

$$|M_g^{(1)}(\{p_i+\delta_i\},q)\rangle\,\big|_{\text{hard}} = \mathbf{S}^{(0)}(\{p_i\},\{\delta_i\},q)\,|M^{(1)}(\{p_i\})\rangle$$

Soft Region

- Loop momentum is soft $l = \mathcal{O}(\lambda), q = \mathcal{O}(\lambda)$
- Only non-vanishing diagram is



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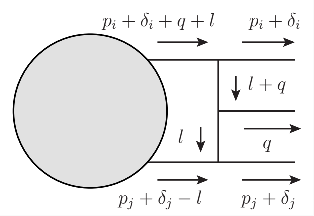
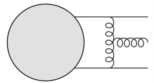
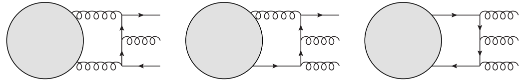


Flavor Diagonal





Flavor Off-diagonal

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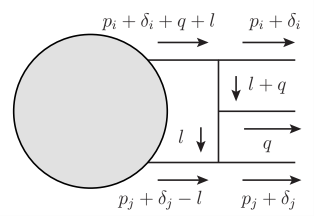

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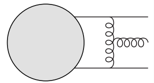
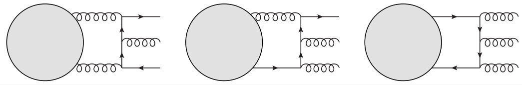




Flavor Off-diagonal

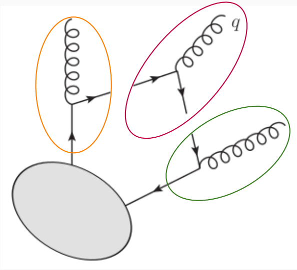
- Inside the blob, all lines are far from the mass shell, i.e. we can expand in the soft scale.

$$|M_g^{(1)}(\{p_i + \delta_i\}, q)\rangle\,\Big|_{\text{soft}} = \mathbf{S}^{(1)}(\{p_i\}, \{\delta_i\}, q)\,|M^{(0)}(\{p_i\})\rangle$$
$$+ \sum_{i \neq j} \sum_{\substack{\tilde{a}_i \neq a_i \\ \tilde{a}_j \neq a_j}} \tilde{\mathbf{S}}_{a_i a_j \leftarrow \tilde{a}_i \tilde{a}_j, ij}(p_i, p_j, q)\,|M^{(0)}(\{p_i\})|^{a_i \to \tilde{a}_i}_{a_j \to \tilde{a}_j}\rangle$$

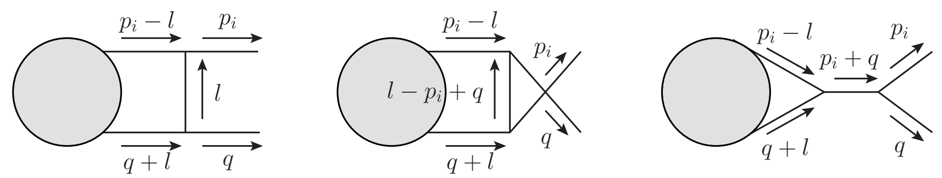
Soft Region results

$$\mathbf{P}_g(\sigma,c)\,\mathbf{S}^{(1)}(\{p_i\},\{\delta_i\},q)+\mathcal{O}(\lambda)=\frac{2\,r_{\text{Soft}}}{\epsilon^2}\sum_{i\neq j} i f^{abc}\mathbf{T}_i^a\mathbf{T}_j^b\otimes\left(-\frac{\mu^2 s_{ij}^{(\delta)}}{s_{iq}^{(\delta)}s_{jq}^{(\delta)}}\right)^{\epsilon}\left[\mathbf{S}_i^{(0)}(p_i,\delta_i,q,\sigma)+\frac{\epsilon}{1-2\epsilon}\frac{1}{p_i\cdot p_j}\left(\frac{p_i^\mu p_j^\nu-p_j^\mu p_i^\nu}{p_i\cdot q}+\frac{p_j^\mu p_j^\nu}{p_j\cdot q}\right)F_{\mu\rho}(q,\sigma)\,(J_i-\mathbf{K}_i)^{\nu\rho}\right]$$

$$\begin{aligned}
&\tilde{\mathbf{S}}^{(1)}_{gg\leftarrow q\bar{q},\,ij}(p_i,p_j,q)\,\left|\ldots,c_i',\ldots,c_j',\ldots;\ldots,\sigma_i',\ldots,\sigma_j',\ldots\right\rangle\\
&=-\frac{r_{\text{Soft}}}{\epsilon(1-2\epsilon)}\left(-\frac{\mu^2 s_{ij}}{s_{iq}s_{jq}}\right)^{\epsilon}\sum_{\sigma c}\sum_{\sigma_i c_i}\sum_{\sigma_j c_j}\sum_{\sigma_i'' c_i''}\sum_{\sigma_j'' c_j''}\\
&\times\; T^{c}_{c_i''c_j''}\,\bar{v}(p_i,\sigma_i'')\,\not{\epsilon}^*(q,p_i,\sigma)\,u(p_j,\sigma_j'')\\
&\times\; \langle c_i,c_j'';\sigma_i,\sigma_j''|\mathbf{Split}^{(0)}_{gq\leftarrow q}(p_i,p_j,p_i)|c_i';\sigma_i'\rangle\\
&\times\; \langle c_j,c_i'';\sigma_j,\sigma_i''|\mathbf{Split}^{(0)}_{g\bar{q}\leftarrow\bar{q}}(p_j,p_i,p_j)|c_j';\sigma_j'\rangle\\
&\times\; \left|\ldots,c_i,\ldots,c_j,\ldots,c;\ldots,\sigma_i,\ldots,\sigma_j,\ldots,\sigma\right\rangle
\end{aligned}$$



Collinear Regions

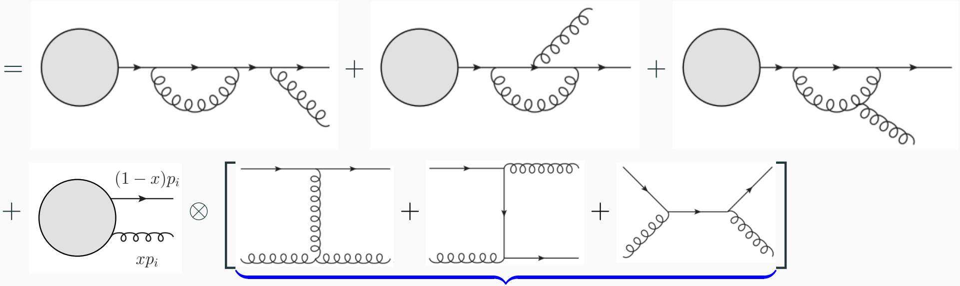


- $l = l_+ n + l_\perp + l_- \bar{n}, \quad n \propto p_i, \quad (l_+, l_\perp, l_-) \propto (1, \sqrt{\lambda}, \lambda)$
- Use light-cone gauge, because collinear vertices get power suppressed
- $\mathrm{d}^d l = \frac{1}{2} \mathrm{d}l_+ \mathrm{d}l_- \mathrm{d}^{d-2} l_\perp$, perform integrations separately
- Problem: large l_+ component flows into process-dependent blob $\rightarrow$ no Taylor expansion possible

$\longrightarrow$ While l_- and $l_\perp$ integrations can be performed independently of the hard process, a convolution over $x \equiv l_+ / p_{i+}$ remains.

Collinear Regions

$$|M_g^{(1)}\rangle\Big|_{i\text{-collinear}}^{i\text{: quark}} =$$



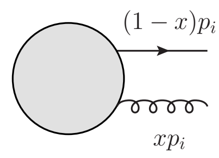
$$\mathbf{J}_i^{(1)}(x, p_i, q)$$

$$= \int_0^1 \mathrm{d}x\, \mathbf{J}_i^{(1)}(x, p_i, q) \left(\lim_{l_\perp \to 0} \frac{1}{\sqrt{1-x}} \left[|M_g^{(0)}\rangle - \mathbf{Split}_{i,n+1\leftarrow i}^{(0)} |M^{(0)}\rangle \right] - \frac{1}{x} \frac{q \cdot \epsilon^*(p_i)}{q \cdot p_i} \mathbf{T}_i \, |M^{(0)}(\{p_i\})\rangle \right)$$

$$\mathbf{P}_g(\sigma, c)\, \mathbf{J}_i^{(1)}(x, p_i, q) = \frac{\Gamma(1+\epsilon)}{1-\epsilon} \left(-\frac{\mu^2}{s_{iq}} \right)^{\epsilon} (x(1-x))^{-\epsilon} \epsilon^*(q, p_i, \sigma) \cdot \epsilon(p_i, -\sigma) \sum_{c'} \mathbf{P}_g(-\sigma, c') \left[\left(\mathbf{T}_i^c \mathbf{T}_i^{c'} + \frac{1}{x} i f^{cdc'} \mathbf{T}_i^d \right) \otimes \left(-2 + x(1 + \boldsymbol{\Sigma}_{g,i}) \right) \right]$$

Collinear Amplitudes

$$|H^{(0)}_{g,i}(x)\rangle \equiv \lim_{l_\perp \to 0} \frac{1}{\sqrt{1-x}}\left[|M^{(0)}_g\rangle - \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}\,|M^{(0)}\rangle \right] - \frac{1}{x}\frac{q\cdot\epsilon^*(p_i)}{q\cdot p_i}\mathbf{T}_i\,|M^{(0)}(\{p_i\})\rangle$$



- Describes subleading collinear behavior of tree-level amplitude.
- Gauge invariant and fulfills Ward identity

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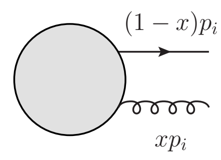
$$|H_{g,i}^{(0)}(x)\rangle \equiv \lim_{l_\perp \to 0} \frac{1}{\sqrt{1-x}} \left[|M_g^{(0)}\rangle - \mathbf{Split}_{i,n+1 \leftarrow i}^{(0)} |M^{(0)}\rangle \right] - \frac{1}{x} \frac{q \cdot \epsilon^*(p_i)}{q \cdot p_i} \mathbf{T}_i |M^{(0)}(\{p_i\})\rangle$$

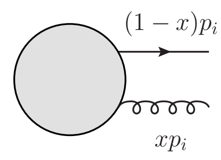


- Describes subleading collinear behavior of tree-level amplitude.
- Gauge invariant and fulfills Ward identity
- Dependence on x (for i (anti-)quark):

$$|H_{g,i}^{(0)}(x)\rangle = \left(\frac{1}{x} + \frac{1}{2}\right) |S_{g,i}^{(0)}\rangle + |C_{g,i}^{(0)}\rangle + \frac{x}{1-x} |\bar{S}_{g,i}^{(0)}\rangle + \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I}\right) |R_{g,i,I}^{(0)}\rangle$$

Collinear Amplitudes

$$|H_{g,i}^{(0)}(x)\rangle \equiv \lim_{l_\perp \to 0} \frac{1}{\sqrt{1-x}} \left[|M_g^{(0)}\rangle - \mathbf{Split}_{i,n+1 \leftarrow i}^{(0)} |M^{(0)}\rangle \right] - \frac{1}{x} \frac{q \cdot \epsilon^*(p_i)}{q \cdot p_i} \mathbf{T}_i |M^{(0)}(\{p_i\})\rangle$$



- Describes subleading collinear behavior of tree-level amplitude.
- Gauge invariant and fulfills Ward identity
- Dependence on x (for i (anti-)quark):

$$|H_{g,i}^{(0)}(x)\rangle = \underbrace{\left(\frac{1}{x} + \frac{1}{2}\right) |S_{g,i}^{(0)}\rangle + |C_{g,i}^{(0)}\rangle}_{\text{Obtainable with LBK theorem}} + \frac{x}{1-x} |\bar{S}_{g,i}^{(0)}\rangle + \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I}\right) |R_{g,i,I}^{(0)}\rangle$$

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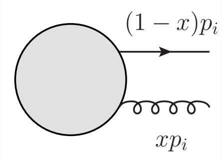

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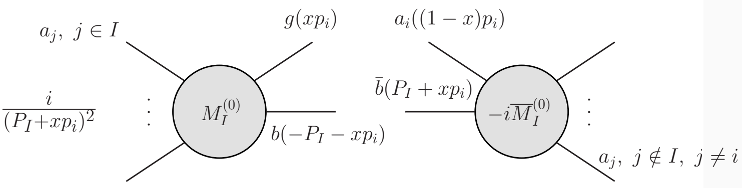
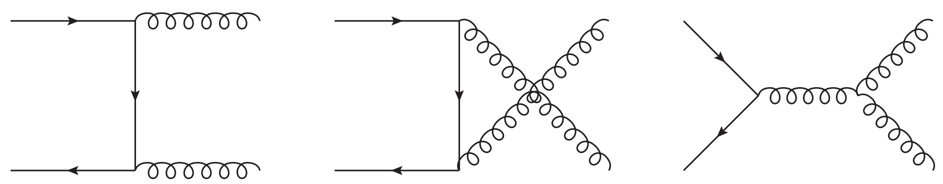


$$x_I \equiv -\frac{P_I^2 + i0^+}{2p_i \cdot P_I},$$

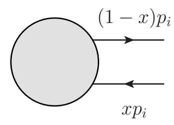
$$P_I \equiv \sum_{j \in I} p_j$$

Collinear Regions: Gluons

- More diagrams
- No conceptual difference to quark case, in fact, one obtains identical expression for jet operator $\mathbf{J}_i^{(1)}(x, p_i, q)$
- Additional flavor-off-diagonal jet operator:



$\longrightarrow$ Corresponding hard function $|H_{q,i}^{(0)}(x)\rangle$ leads to one yet unsolved complication: Formula for *soft-quark* emission at tree-level unknown $\implies$ $|C_{q,i}^{(0)}\rangle$ has to be obtained by evaluating $n+1$-particle process at tree-level for any x.



Subleading Soft Expansion: Summary

$$\left|M_g^{(1)}(\{p_i+\delta_i\},q)\right\rangle = \boxed{\mathbf{S}^{(0)}(\{p_i\},\{\delta_i\},q)\,|M^{(1)}(\{p_i\})\rangle} \quad \textbf{Hard}$$

$$+ \boxed{\mathbf{S}^{(1)}(\{p_i\},\{\delta_i\},q)\,|M^{(0)}(\{p_i\})\rangle + \sum_{i\neq j}\sum_{\substack{\tilde{a}_i\neq a_i\\ \tilde{a}_j\neq a_j}} \tilde{\mathbf{S}}^{(1)}_{a_i a_j \leftarrow \tilde{a}_i\tilde{a}_j,\,ij}(p_i,p_j,q)\,|M^{(0)}(\{p_i\})\Big|_{\substack{a_i\to\tilde{a}_i\\ a_j\to\tilde{a}_j}}\rangle} \quad \textbf{Soft}$$

$$+ \boxed{\int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q)\,|H_{g,i}^{(0)}(x,\{p_i\},q)\rangle + \int_0^1 \mathrm{d}x \sum_{\substack{i\\ a_i=g}} \tilde{\mathbf{J}}_i^{(1)}(x,p_i,q)\,|H_{\bar{q},i}^{(0)}(x,\{p_i\},q)\rangle} \quad \textbf{Collinear}$$

$$+\,\mathcal{O}(\lambda)$$

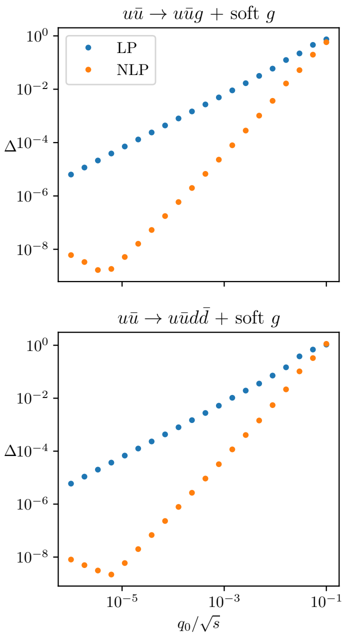
- BONUS: **Subleading collinear** behavior of tree-level amplitudes in terms of **gauge-invariant building blocks** given through LBK theorem or simpler sub-amplitudes (Exception: $g\to q\bar{q}$ splitting due to unknown subleading behavior of soft-quark emission).

Validation

- Check that no momentum regions are missing: ϵ-poles of result agree with expectation (obtainable from tree-level results through I-operator (Catani, Dittmaier, and Trocsanyi, hep-ph/0011222))
- Numerical tests:

$$\Delta_{\text{LP/NLP}} \equiv \frac{1}{N} \sum_{\substack{\text{singular} \\ \text{colour flows } \{c\} \\ \text{helicities } \{\sigma\}}} \left| \frac{\left[\left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle - \left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle_{\text{LP/NLP}}\right]_{\mathcal{O}(\epsilon^0)}}{\left[\left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle\right]_{\mathcal{O}(\epsilon^0)}} \right|$$

- Numerical values for amplitudes obtained with Recola (Actis et al., 1605.01090), CutTools (Ossola, Papadopoulos, and Pittau, 0711.3596), and OneLOop (van Hameren, 1007.4716)



Conclusions

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Summary

- Universal description of **subleading soft** behavior of one-loop amplitudes
- Bonus: **Subleading collinear** behavior of tree-level amplitudes

Outlook

- Generalization to **massive** case
- Incorporation of next-to-soft stabilization into existing NNLO subtraction scheme

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Summary

- Universal description of **subleading soft** behavior of one-loop amplitudes
- Bonus: **Subleading collinear** behavior of tree-level amplitudes

Outlook

- Generalization to **massive** case
- Incorporation of next-to-soft stabilization into existing NNLO subtraction scheme

Thank you!

Subleading Collinear Effects at Tree-level: $q \longrightarrow qg$, $\bar{q} \longrightarrow \bar{q}g$

$$\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1})\,|M^{(0)}(\{k_i\}_{i=1}^{n+1})\rangle = \mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1})\Bigg[$$

$$\mathbf{Split}^{(0)}_{i,n+1\,\leftarrow\,i}(k_i, k_{n+1}, p_i)\,|M^{(0)}(\{p_i\})\rangle$$

$$+\sqrt{1-x}\Bigg(\left(\frac{1}{x}+\frac{1}{2}\right)|S^{(0)}_{g,i}(\{p_i\},q)\rangle + |C^{(0)}_{g,i}(\{p_i\},q)\rangle + \frac{x}{1-x}\,|\bar{S}^{(0)}_{g,i}(\{p_i\},q)\rangle$$

$$+\sum_I\left(\frac{1}{x_I-x}-\frac{1}{x_I}\right)|R^{(0)}_{g,i,I}(\{p_i\})\rangle\Bigg)\Bigg] + \frac{\sqrt{1-x}}{x}\,\frac{q\cdot\epsilon^*(p_i,\sigma_{n+1})}{q\cdot p_i}\,\mathbf{T}_i^{c_{n+1}}\,|M^{(0)}(\{p_i\})\rangle$$

$$+\,\mathcal{O}(l_\perp)\;.$$

$$k_{n+1} = xp_i + l_\perp - \frac{l_\perp^2}{2x}\frac{q}{p_i\cdot q},\quad k_i = (1-x)p_i - l_\perp - \frac{l_\perp^2}{2(1-x)}\frac{q}{p_i\cdot q},\quad l_\perp\cdot p_i = l_\perp\cdot q = 0,\quad k_j = p_j + \mathcal{O}(l_\perp^2)$$

Subleading Collinear Effects at Tree-level: $g \longrightarrow q\bar{q}$

$$
\begin{aligned}
|M^{(0)}(\{k_i\}_{i=1}^{n+1})\rangle = \mathbf{Split}^{(0)}_{i,n+1 \leftarrow i}(k_i, k_{n+1}, p_i)\, |M^{(0)}(\{p_i\})\rangle \\
+ \sqrt{x(1-x)}\Bigg(\frac{1}{x}\, |S^{(0)}_{\bar{q},i}(\{p_i\})\rangle + |C^{(0)}_{\bar{q},i}(\{p_i\}, q)\rangle + \frac{x}{1-x}\, |\bar{S}^{(0)}_{\bar{q},i}(\{p_i\})\rangle \\
+ \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I} \right) |R^{(0)}_{\bar{q},i,I}(\{p_i\})\rangle \Bigg) + \mathcal{O}(l_\perp)\,.
\end{aligned}
$$

Subleading Collinear Effects at Tree-level: $g \longrightarrow gg$

$$\mathbf{P}_i(\sigma_i, c_i)\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1})\,|M^{(0)}(\{k_i\}_{i=1}^{n+1})\rangle = \mathbf{P}_i(\sigma_i, c_i)\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1})\Bigg[$$

$$\mathbf{Split}^{(0)}_{i,n+1\,\leftarrow\,i}(k_i, k_{n+1}, p_i)\,|M^{(0)}(\{p_i\})\rangle$$

$$+\left(\frac{1-x^2}{x} + \frac{1-(1-x)^2}{1-x}\mathbf{E}_{i,n+1}\right)|S^{(0)}_{g,i}(\{p_i\}, q)\rangle + \big((1-x) + x\mathbf{E}_{i,n+1}\big)\,|C^{(0)}_{g,i}(\{p_i\}, q)\rangle$$

$$+\frac{1}{2}\sum_I \frac{x(1-x)}{x_I(1-x_I)}\left(\frac{1}{x_I - x} + \frac{1}{x_I-(1-x)}\mathbf{E}_{i,n+1}\right)|R^{(0)}_{g,i,I}(\{p_i\})\rangle\Bigg]$$

$$+\left[\frac{1}{x}\frac{q\cdot\epsilon^*(p_i,\sigma_{n+1})}{q\cdot p_i}\mathbf{P}_i(\sigma_i, c_i)\mathbf{T}_i^{c_{n+1}} + \frac{1}{1-x}\frac{q\cdot\epsilon^*(p_i,\sigma_i)}{q\cdot p_i}\mathbf{P}_i(\sigma_{n+1}, c_{n+1})\mathbf{T}_i^{c_i}\right]|M^{(0)}(\{p_i\})\rangle$$

$$+\,\mathcal{O}(l_\perp)\,,$$

Hard Functions I

$$|H_{g,i}^{(0)}(x,\{p_i\},q)\rangle = \left(\frac{1}{x}+\dim(a_i)\right)|S_{g,i}^{(0)}(\{p_i\},q)\rangle + |C_{g,i}^{(0)}(\{p_i\},q)\rangle + \frac{x}{1-x}\,|\bar{S}_{g,i}^{(0)}(\{p_i\},q)\rangle$$

$$+\sum_I\left(\frac{1}{x_I-x}-\frac{1}{x_I}\right)|R_{g,i,I}^{(0)}(\{p_i\})\rangle + x\,|L_{g,i}^{(0)}(\{p_i\},q)\rangle \ ,$$

$$\mathbf{P}_g(\sigma,c)\ |S_{g,i}^{(0)}(\{p_i\},q)\rangle = -\sum_{j\neq i}\mathbf{T}_j^c\left(\frac{p_j}{p_j\cdot p_i}-\frac{q}{q\cdot p_i}\right)\cdot\epsilon^*(p_i,\sigma)\,|M^{(0)}(\{p_i\})\rangle \ ,$$

$$\mathbf{P}_g(\sigma,c)\ |C_{g,i}^{(0)}(\{p_i\},q)\rangle =$$

$$-\sum_{j\neq i}\mathbf{T}_j^c\otimes\left(\frac{p_{i\mu}\epsilon_\nu^*(p_i,\sigma)}{p_j\cdot p_i}\left(p_j^\mu\partial_i^\nu-p_j^\nu\partial_i^\mu+iJ_j^{\mu\nu}-i\mathbf{K}_j^{\mu\nu}\right)+\frac{q_\mu\epsilon_\nu^*(p_i,\sigma))}{q\cdot p_i}\,i\mathbf{K}_i^{\mu\nu}\right)|M^{(0)}(\{p_i\})\rangle$$

Hard Function II

$$\langle c_1,\ldots,c_{n+1};\sigma_1,\ldots,\sigma_{n+1}|R^{(0)}_{g,i,I}(\{p_i\})\rangle =$$

$$\left(1-x_I\right)^{-\dim(a_i)}\,\frac{1}{2p_i\cdot P_I}\sum_{\sigma c} M^{(0)}_I(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)\,\bar{M}^{(0)}_I(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)$$

$$|\bar{S}^{(0)}_{g,i}(\{p_i\},q)\rangle = \mathbf{E}_{i,n+1}\begin{cases}\sum_{j\neq i}\mathbf{Split}^{(0)}_{j,n+1\leftarrow j}(p_j,p_i,p_j)\ |M^{(0)}(\{p_i\})\Big|^{a_i\to g}_{a_j\to\tilde{a}_j}\rangle & \text{for } a_i\in\{q,\bar{q}\}\\ |S^{(0)}_{g,i}(\{p_i\},q)\rangle & \text{for } a_i=g\end{cases}$$

$$|L^{(0)}_{g,i}(\{p_i\},q)\rangle = |\bar{S}^{(0)}_{g,i}(\{p_i\},q)\rangle - |S^{(0)}_{g,i}(\{p_i\},q)\rangle + |\bar{C}^{(0)}_{g,i}(\{p_i\},q)\rangle - |C^{(0)}_{g,i}(\{p_i\},q)\rangle$$
$$+\frac{1}{2}\sum_I\left(\frac{1}{x_I}+\frac{1}{1-x_I}\right)\Big(|R^{(0)}_{g,i,I}(\{p_i\})\rangle - |\bar{R}^{(0)}_{g,i,I}(\{p_i\})\rangle\Big)$$

Offdiagonal Hard Function

$$|H^{(0)}_{\bar{q},i}(x,\{p_i\},q)\rangle = \frac{1}{x}\,|S^{(0)}_{\bar{q},i}(\{p_i\})\rangle + |C^{(0)}_{\bar{q},i}(\{p_i\},q)\rangle + \frac{x}{1-x}\,|\bar{S}^{(0)}_{\bar{q},i}(\{p_i\})\rangle$$

$$+ \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I}\right)|R^{(0)}_{\bar{q},i,I}(\{p_i\})\rangle$$

$$|S^{(0)}_{\bar{q},i}(\{p_i\})\rangle = \sum_{j\neq i} \mathbf{Split}^{(0)}_{j,n+1\leftarrow j}(p_j,p_i,p_j)\;|M^{(0)}(\{p_i\})\Big|^{a_i\to q}_{a_j\to\tilde{a}_j}\rangle$$

$$|\bar{S}^{(0)}_{\bar{q},i}(\{p_i\})\rangle = \mathbf{E}_{i,n+1}\sum_{j\neq i} \mathbf{Split}^{(0)}_{j,n+1\leftarrow j}(p_j,p_i,p_j)\;|M^{(0)}(\{p_i\})\Big|^{a_i\to \bar{q}}_{a_j\to\tilde{a}_j}\rangle$$

$$\langle c_1,\ldots,c_{n+1};\sigma_1,\ldots,\sigma_{n+1}|R^{(0)}_{\bar{q},i,I}(\{p_i\})\rangle =$$

$$\big(x_I(1-x_I)\big)^{-1/2}\,\frac{1}{2p_i\cdot P_I}\sum_{\sigma c} M^{(0)}_I(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)\,\bar{M}^{(0)}_I(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)$$