# Lattice simulation of Stochastic inflation

## New Horizon in Primordial Black Hole PHYSICS

June 19th 2023

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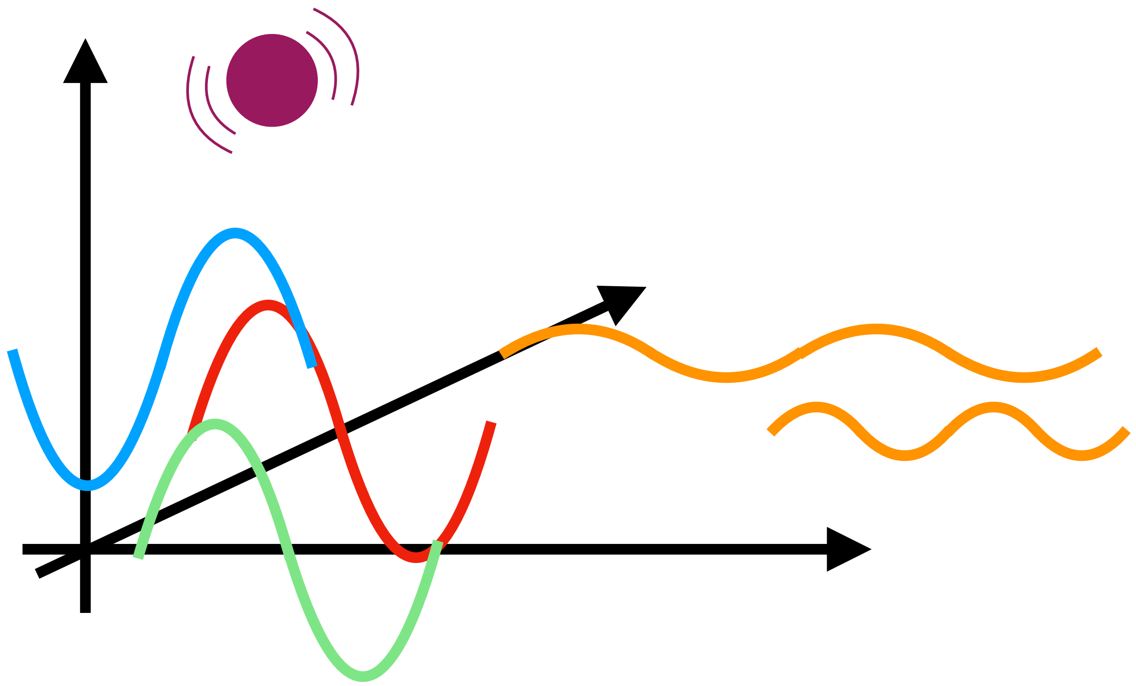
# PBH and Stochastic Inflation

***PBH***



LIGO-Virgo, LISA, SKA, etc.

***Stochastic Inflation***

A. A. Starobinsky, 1986

$$\begin{cases} g_{\mu\nu} = g_{\mu\nu(IR)} + g_{\mu\nu(UV)} \\ \phi = \phi_{IR} + \hat{\phi}_{UV} \; ( \; \hat{\phi}_{UV} \ll \phi_{IR} \; ) \end{cases}$$

We focus on

**the super-horizon mode!**

***Our goal*** **:** How accurate assumptions are in PBH formation

# Flow

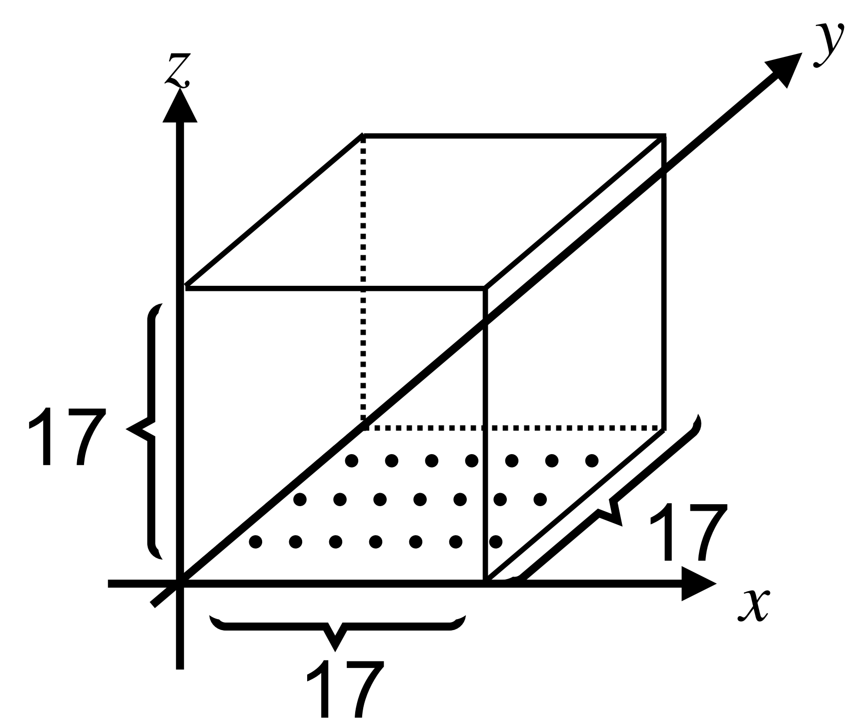
Equation of motion in scalar field

↓

Solving with { the stochastic approach / $\delta N$ formalism }

↓

Profile of curvature perturbation




We developed the original lattice simulation code of stochastic inflation

# Inflaton Potential

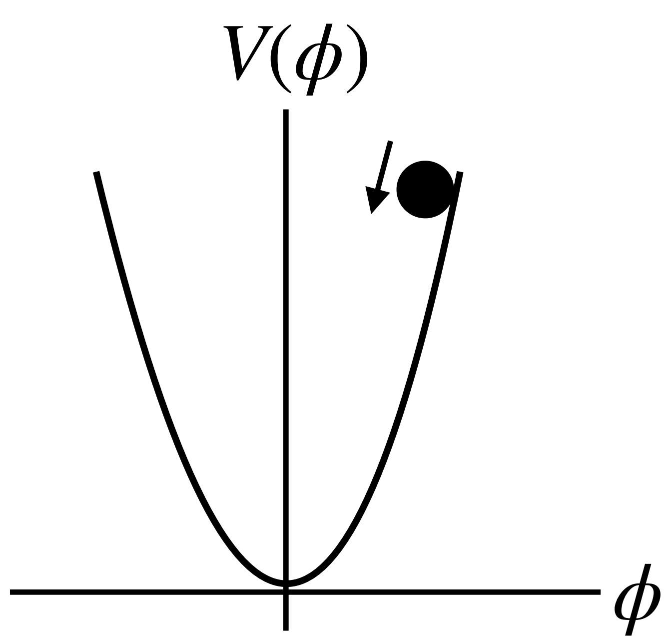
***Focused potential***

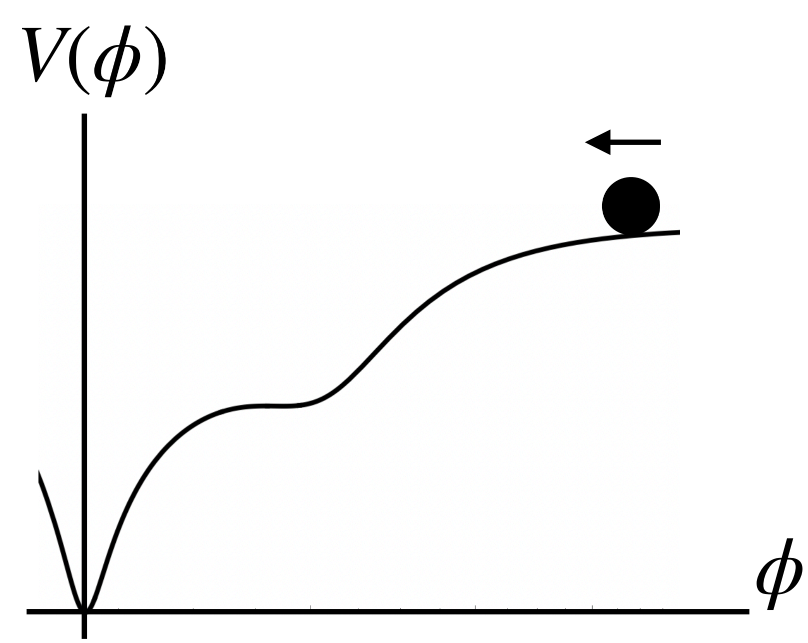
- Chaotic inflation (Linde, A. D.,1983) $V(\phi) = \frac{1}{2}m^2\phi^2$

- Inflection (M. Biagetti *et al,*2018)

$$V(\phi) = \frac{{W_0}^2}{\mathcal{V}^3}\left[\frac{c_{\rm up}}{\sqrt[3]{\mathcal{V}}} + \frac{a_{\rm w}}{e^{\frac{\phi}{\sqrt{3}}} - b_{\rm w}} - \frac{c_{\rm w}}{e^{\frac{\phi}{\sqrt{3}}}} + \frac{e^{\frac{2\phi}{\sqrt{3}}}}{\mathcal{V}}\left(d_{\rm w} - \frac{g_{\rm w}}{r_{\rm w}e^{\sqrt{3}\phi}/\mathcal{V} + 1}\right)\right]$$

| $a_{\rm w}$ | $b_{\rm w}$ | $c_{\rm w}$ | $d_{\rm w}$ | $g_{\rm w}$ | $r_{\rm w}$ | $\mathcal{V}$ | $W_0$ | $c_{\rm up}$ |
|---|---|---|---|---|---|---|---|---|
| 0.02 | 1 | 0.04 | 0 | $3.076278 \cdot 10^{-2}$ | $7.071067 \cdot 10^{-1}$ | 1000 | 12.35 | 0.0382 |

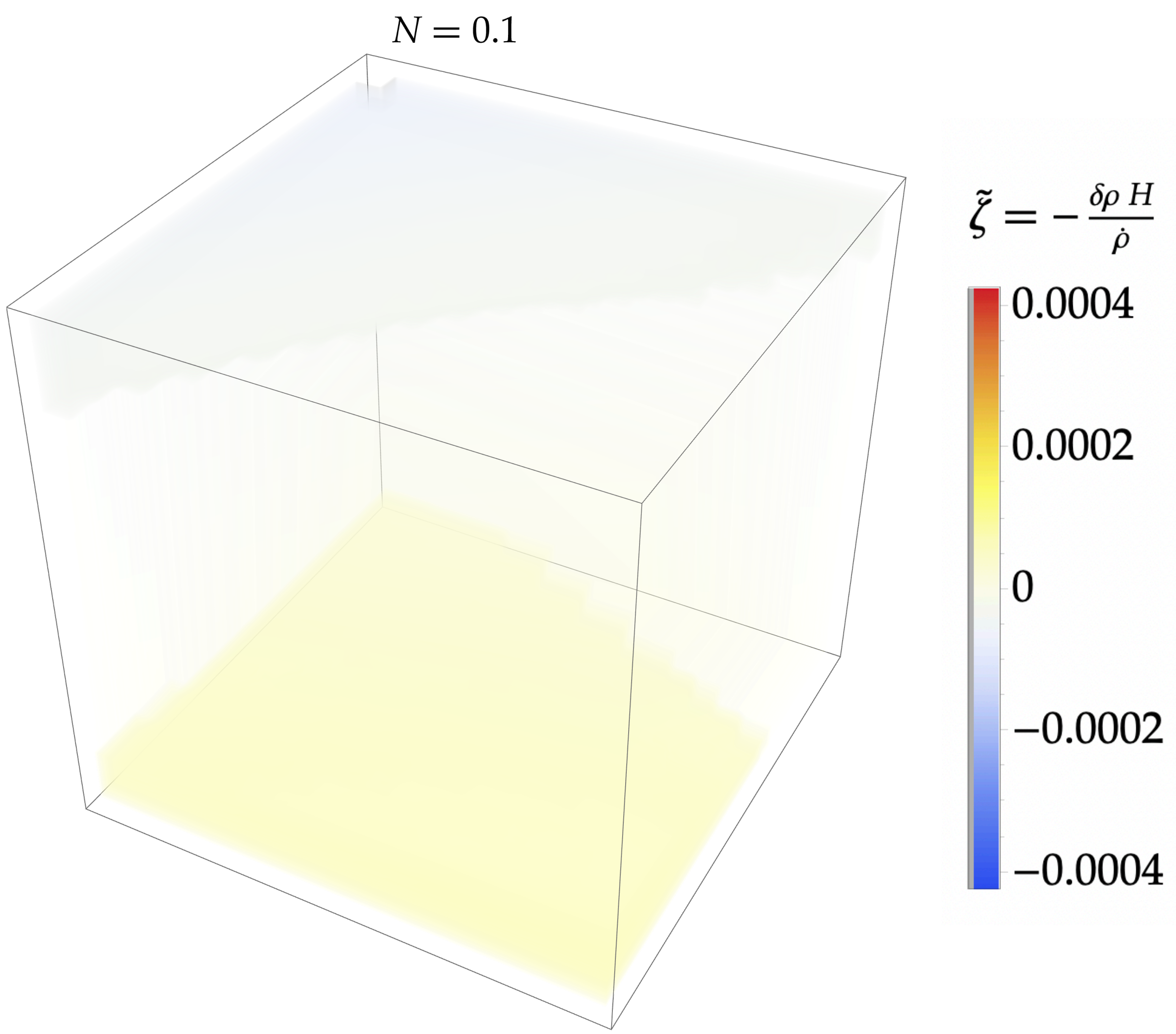


# Result1: Stochastic lattice simulation

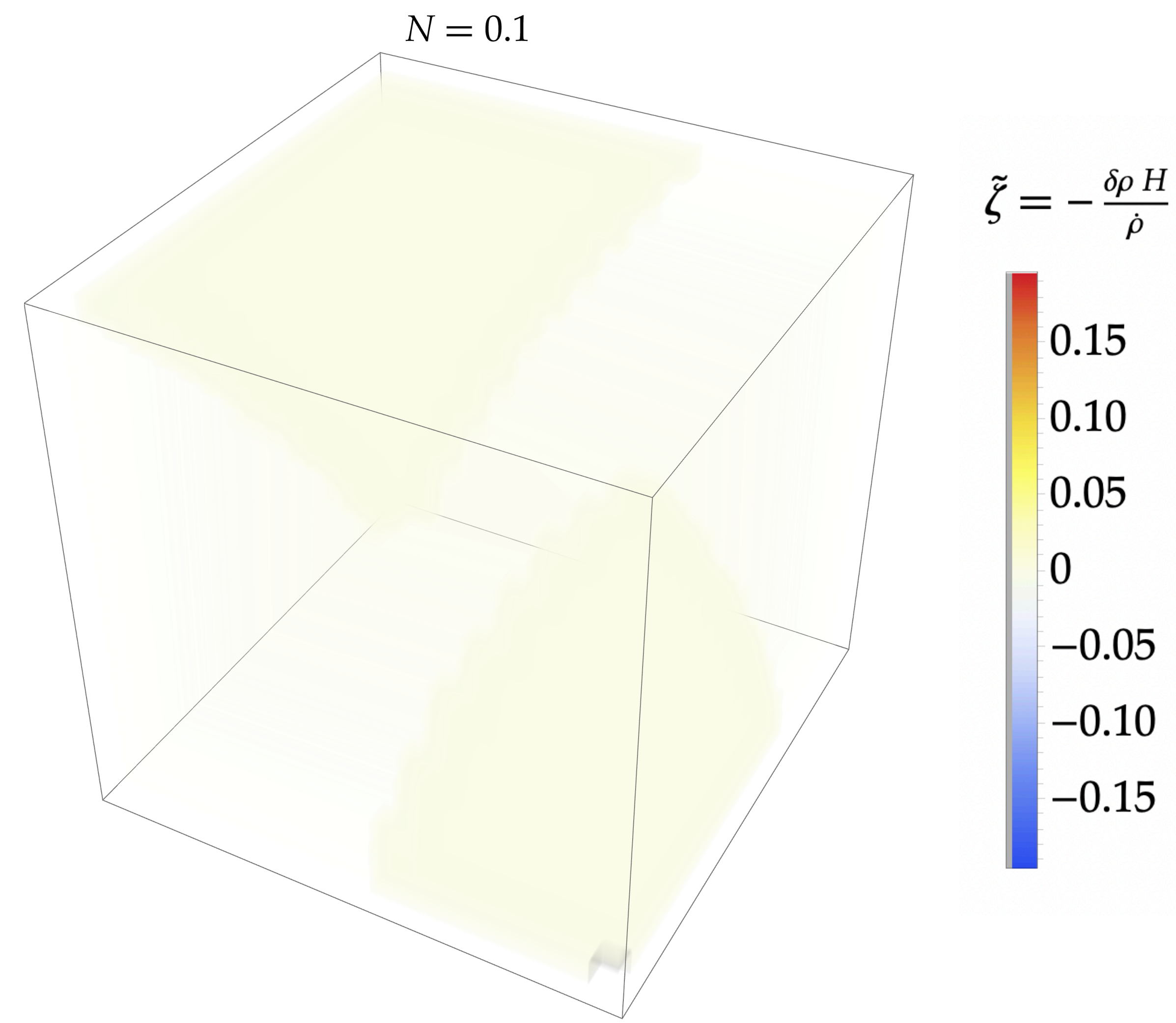
***Chaotic inflation***

$$m = 10^{-5}\,, \phi = 15.0, \pi = -10^{-11}$$




***Inflection***

$$\phi = 3.60547, \pi = -2.37409 \times 10^{-7}$$



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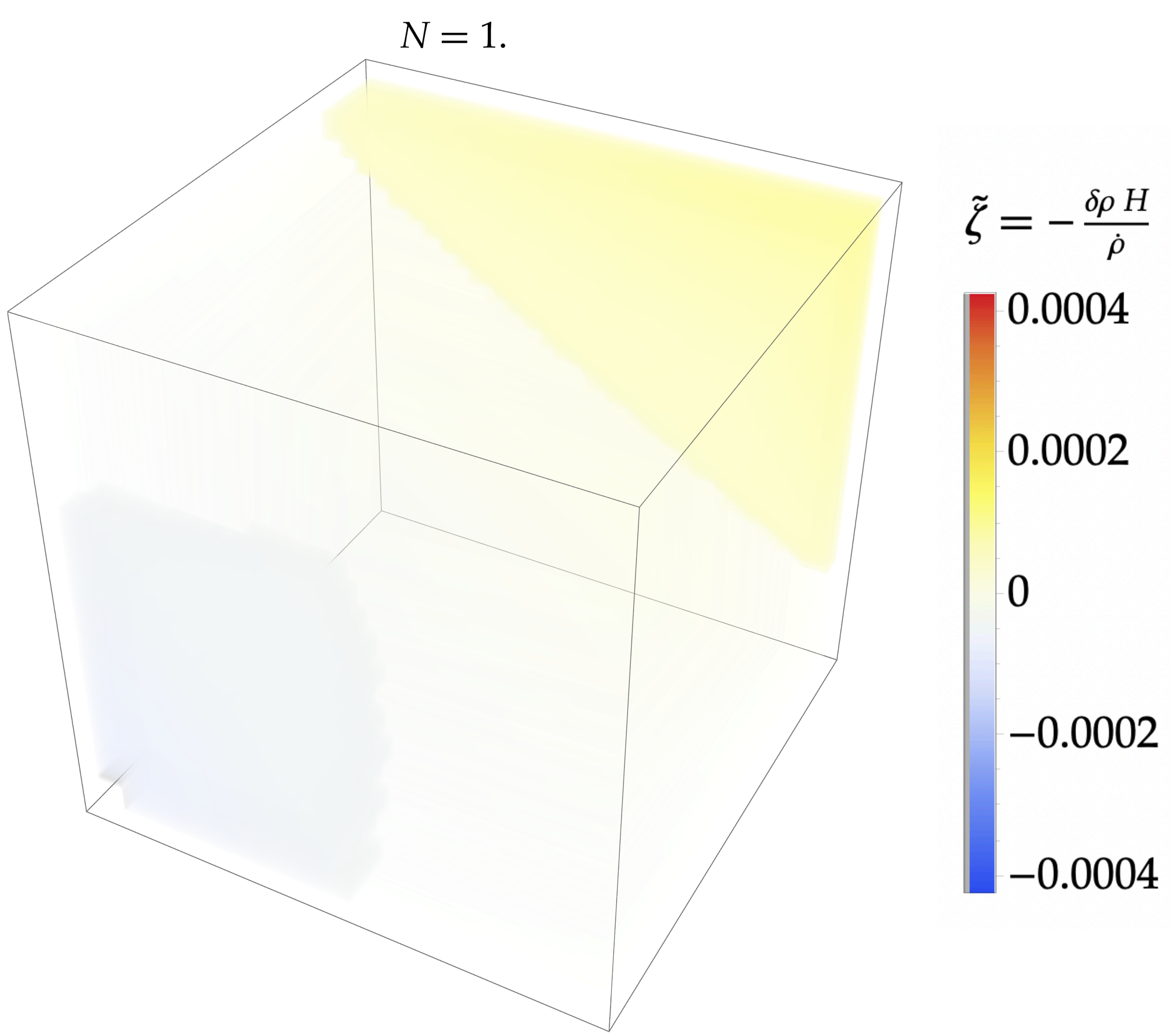


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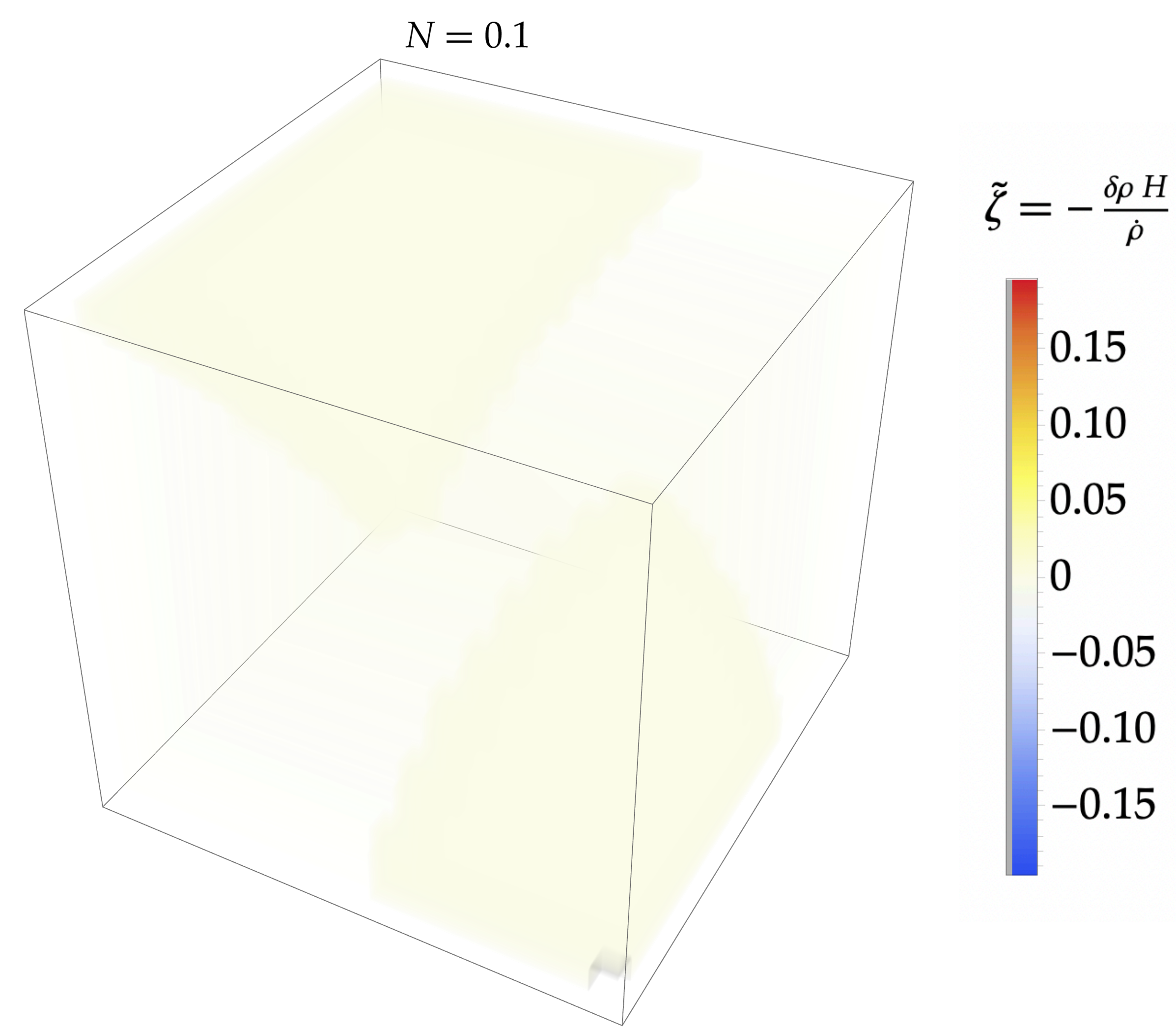

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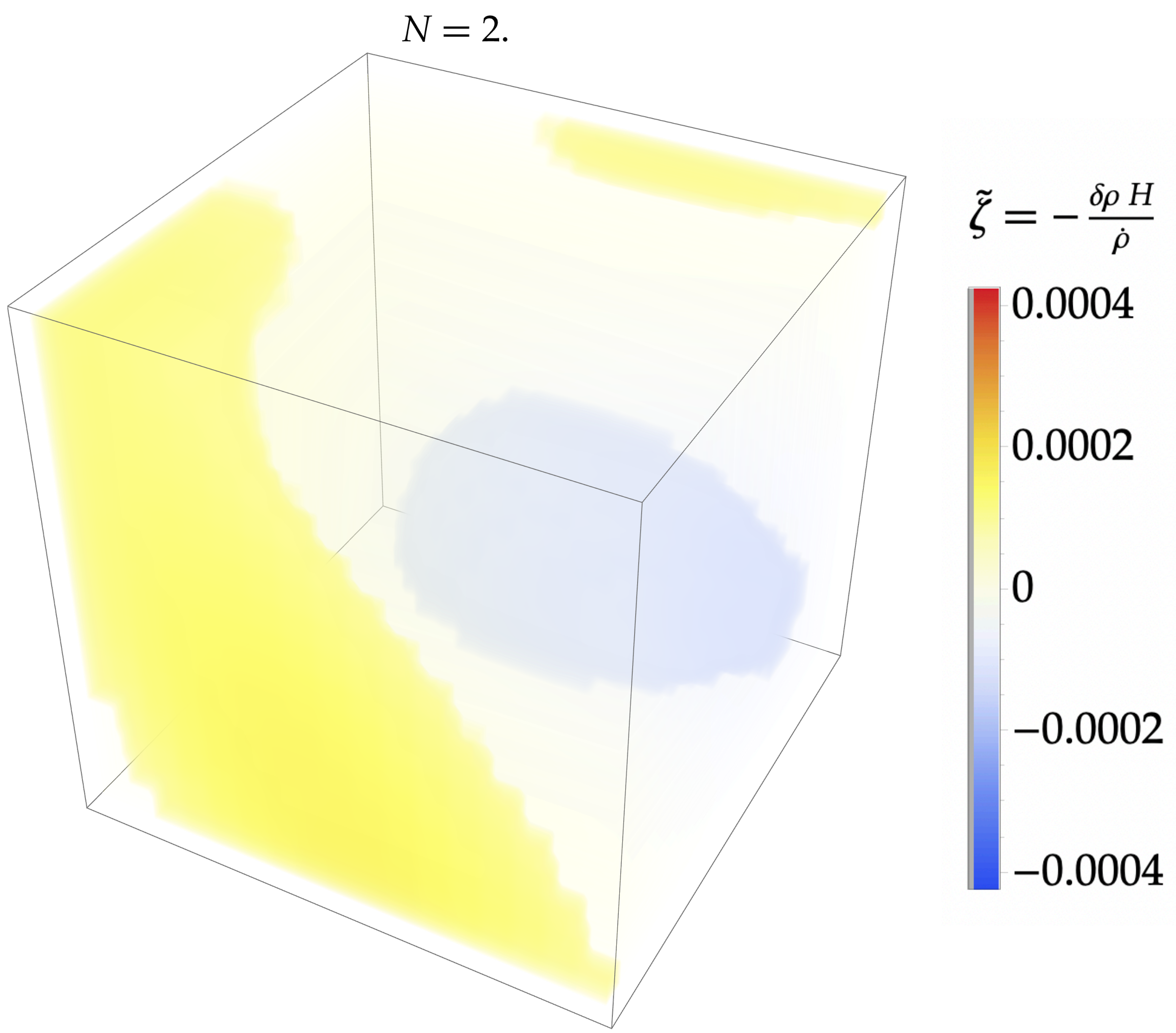


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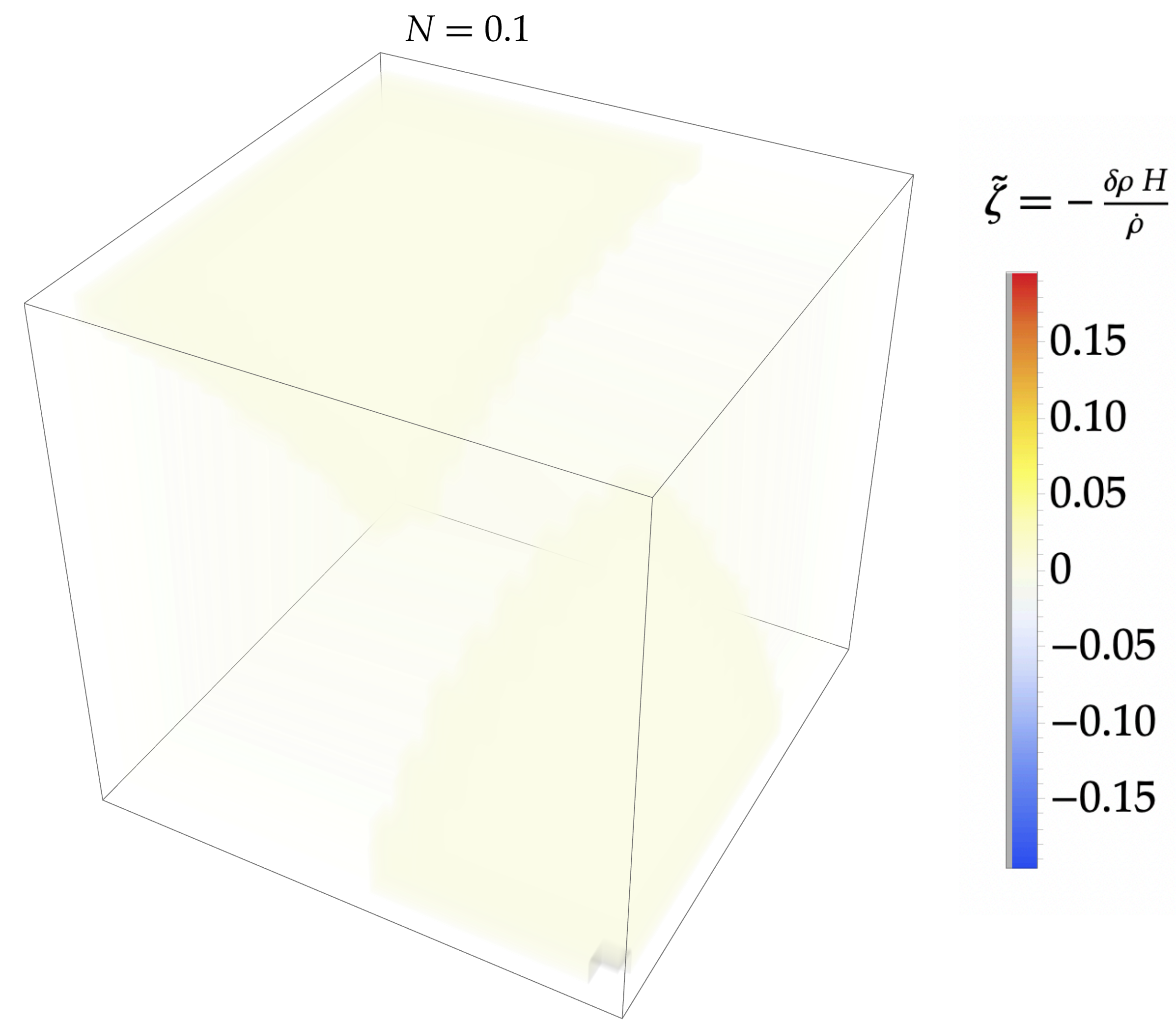

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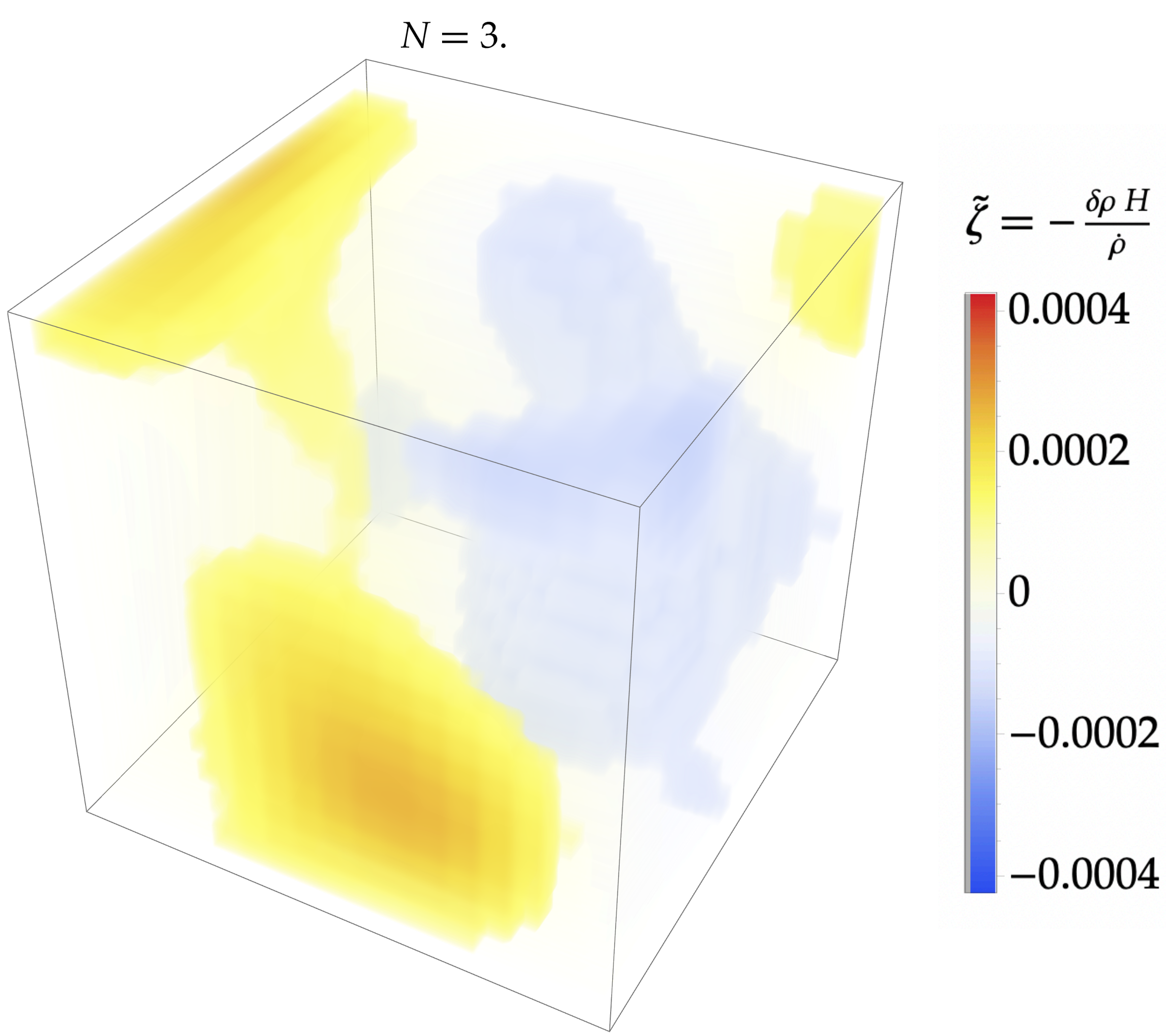


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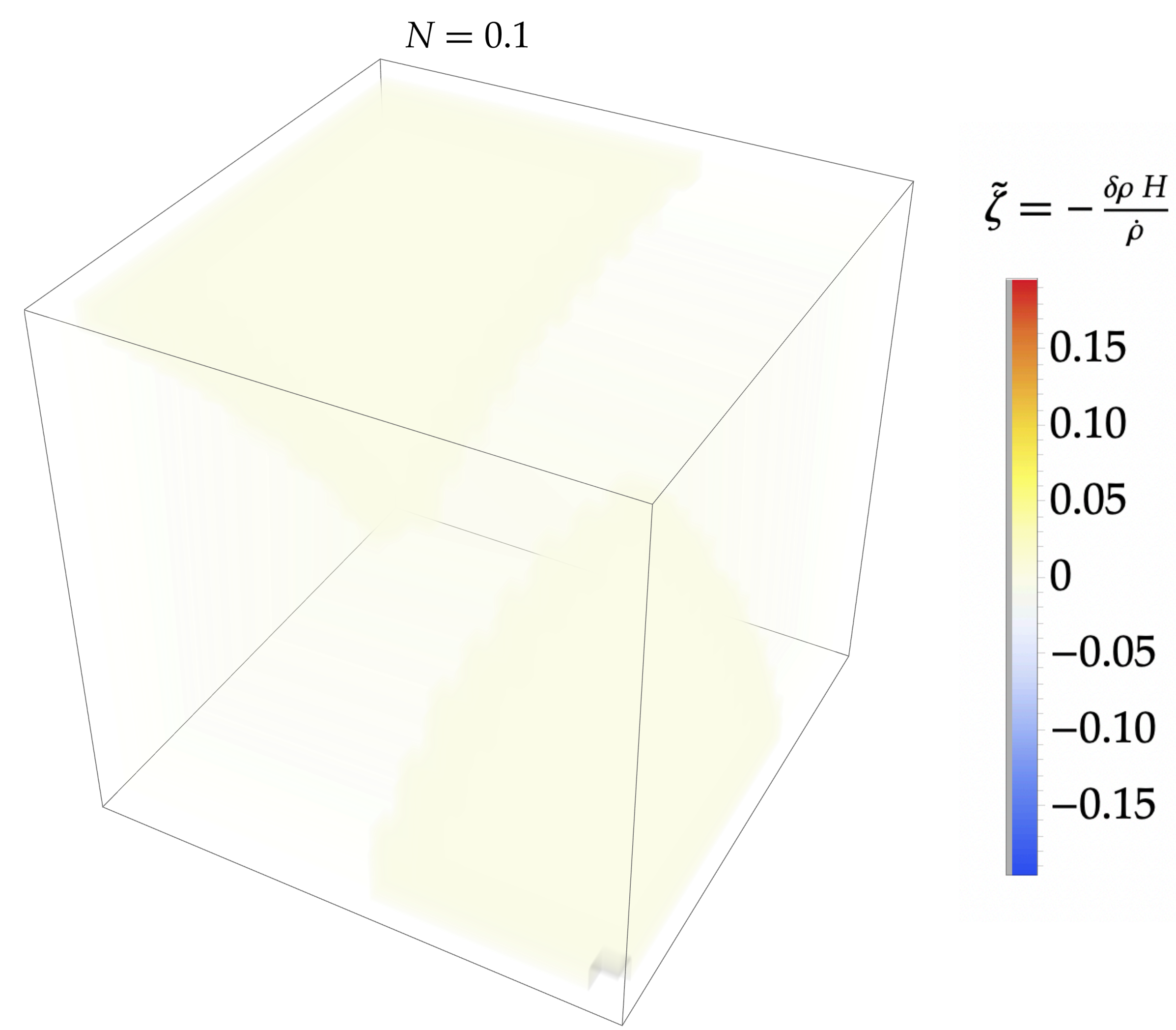

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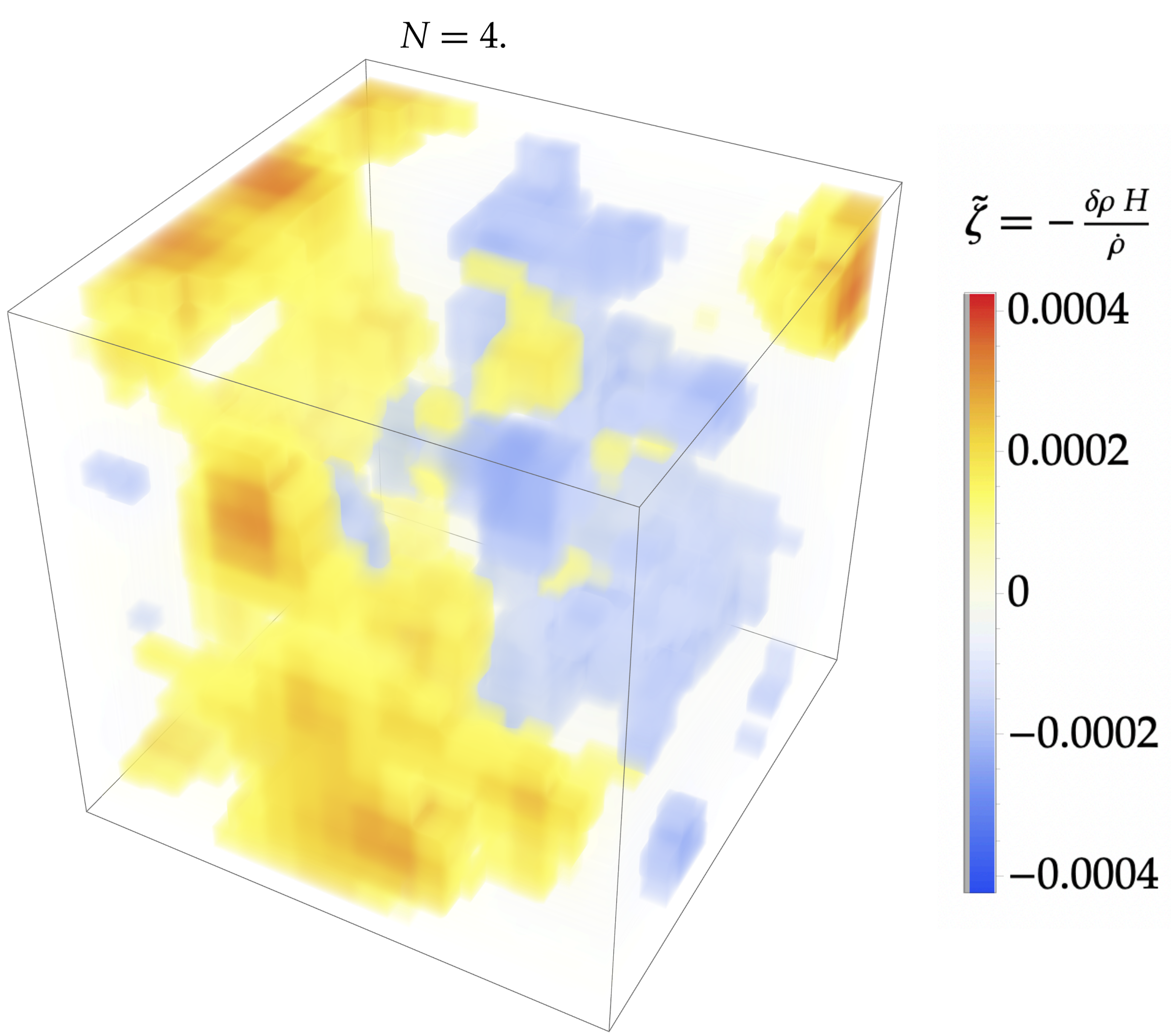


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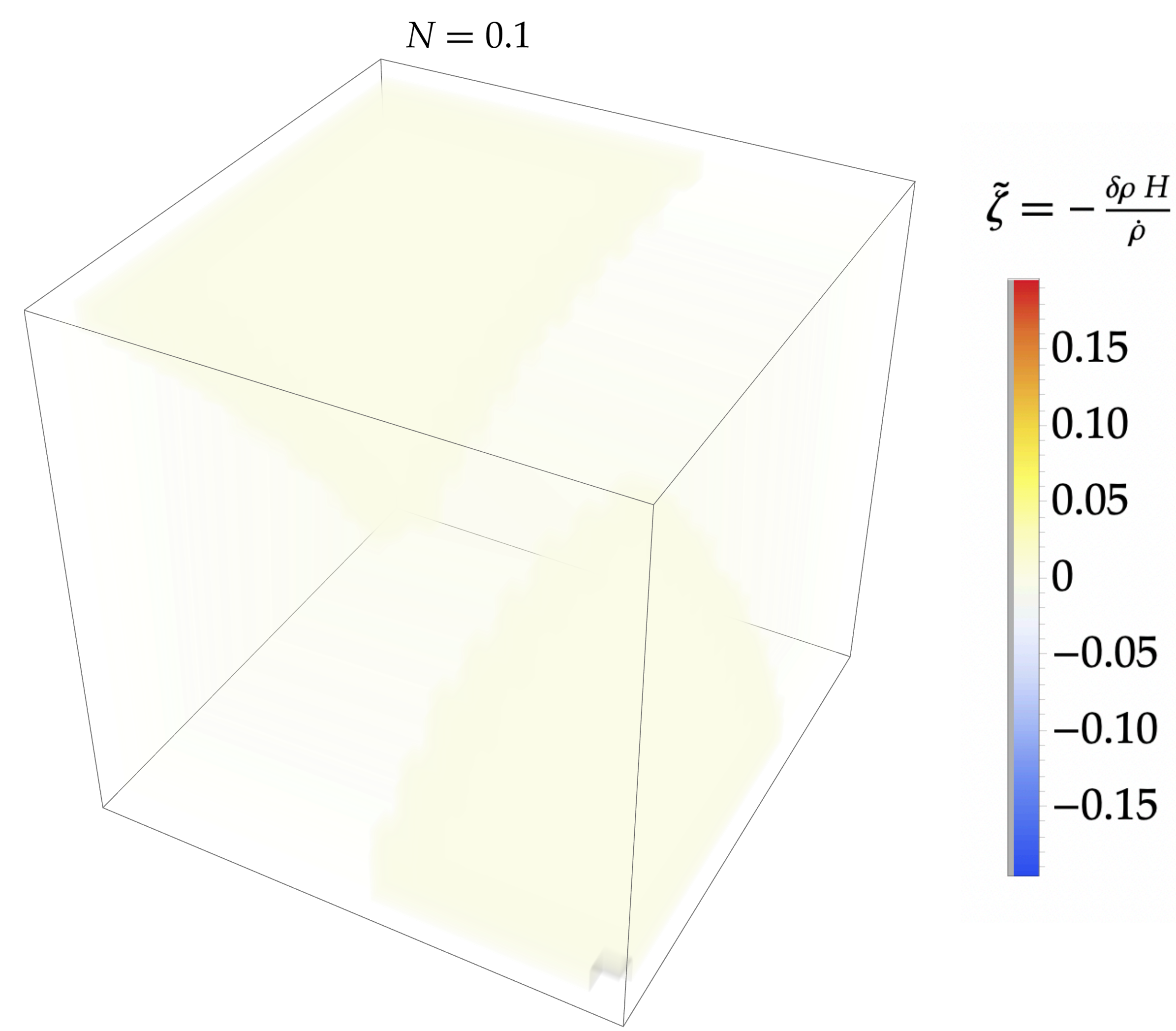

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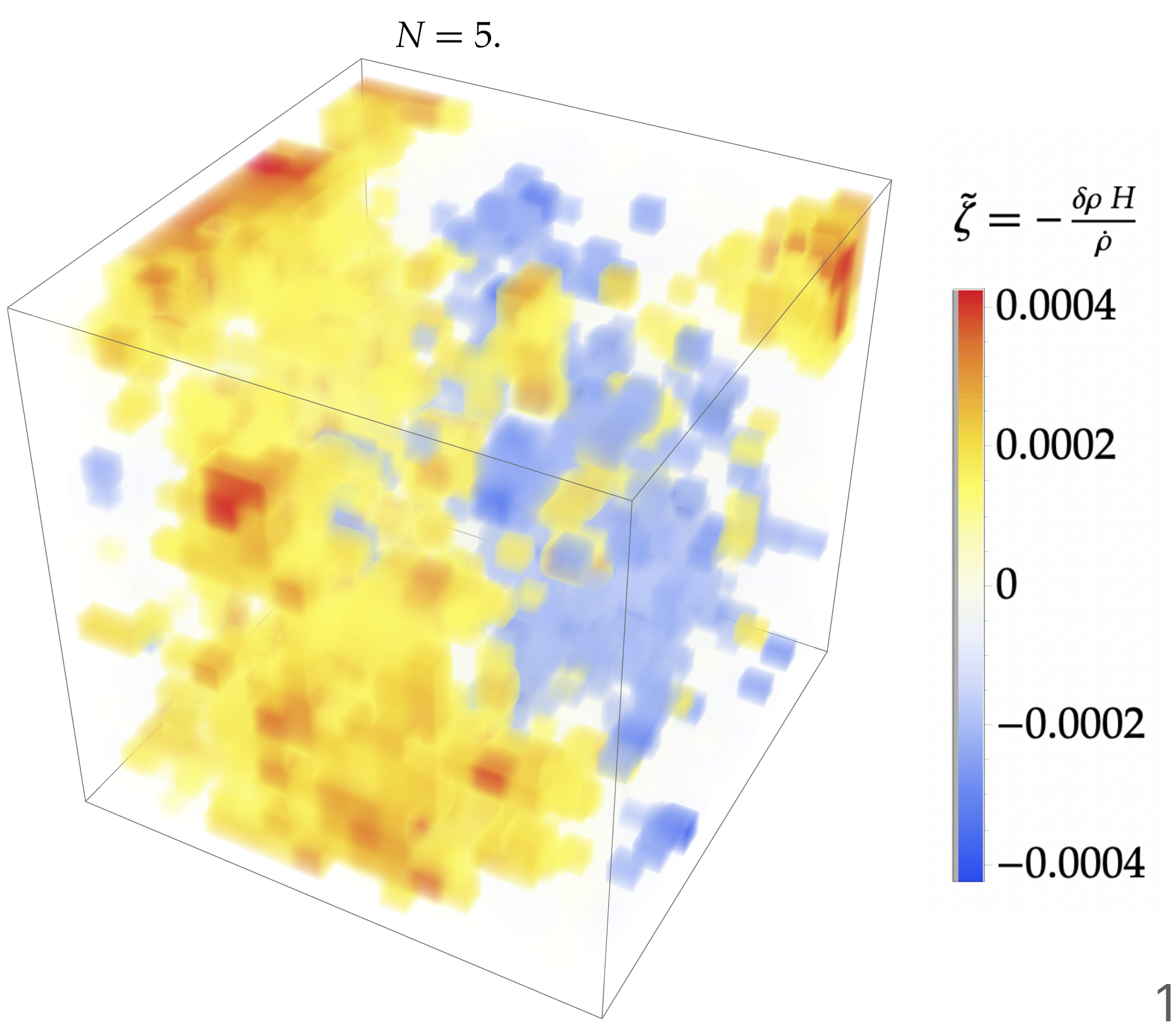


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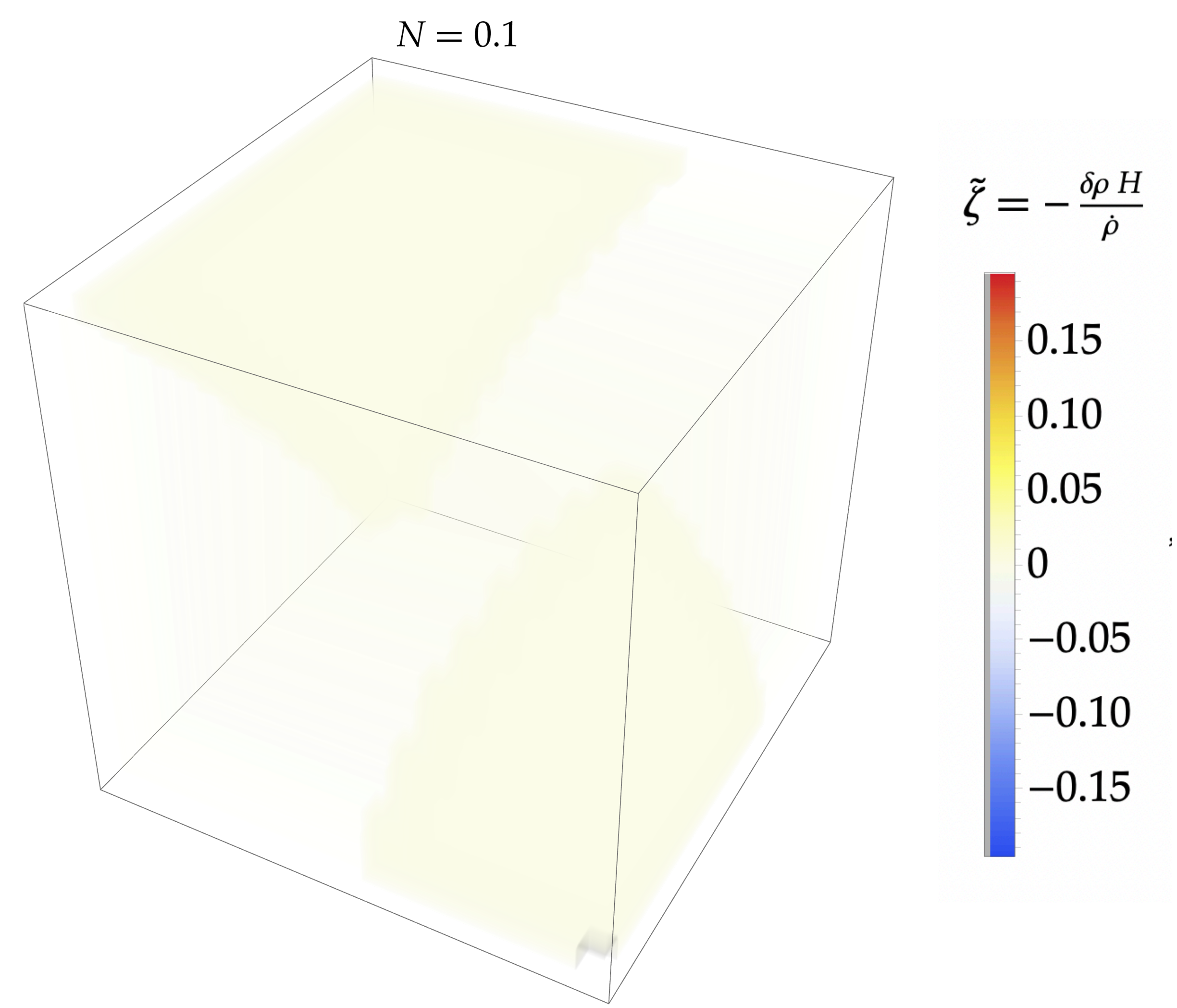

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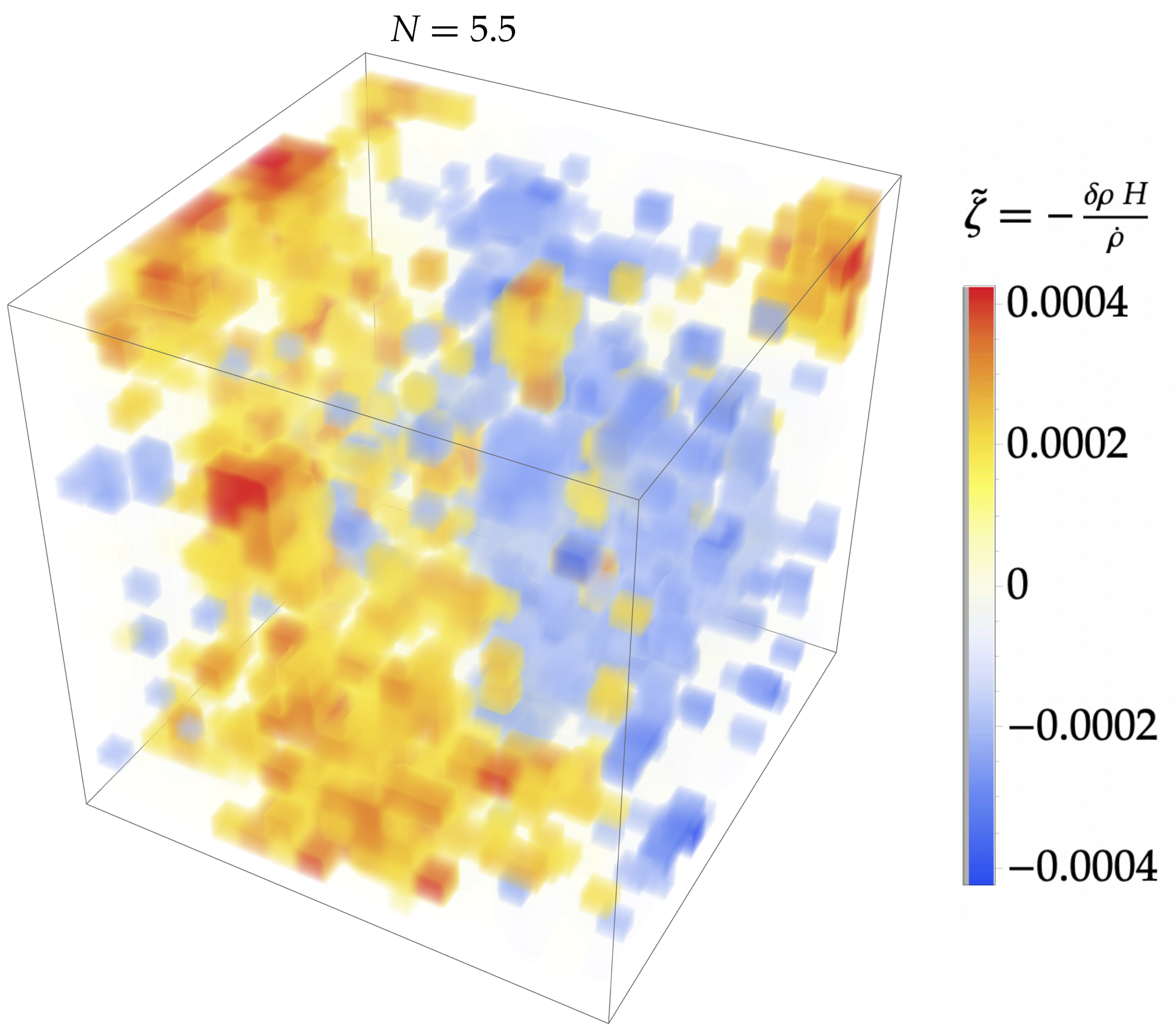


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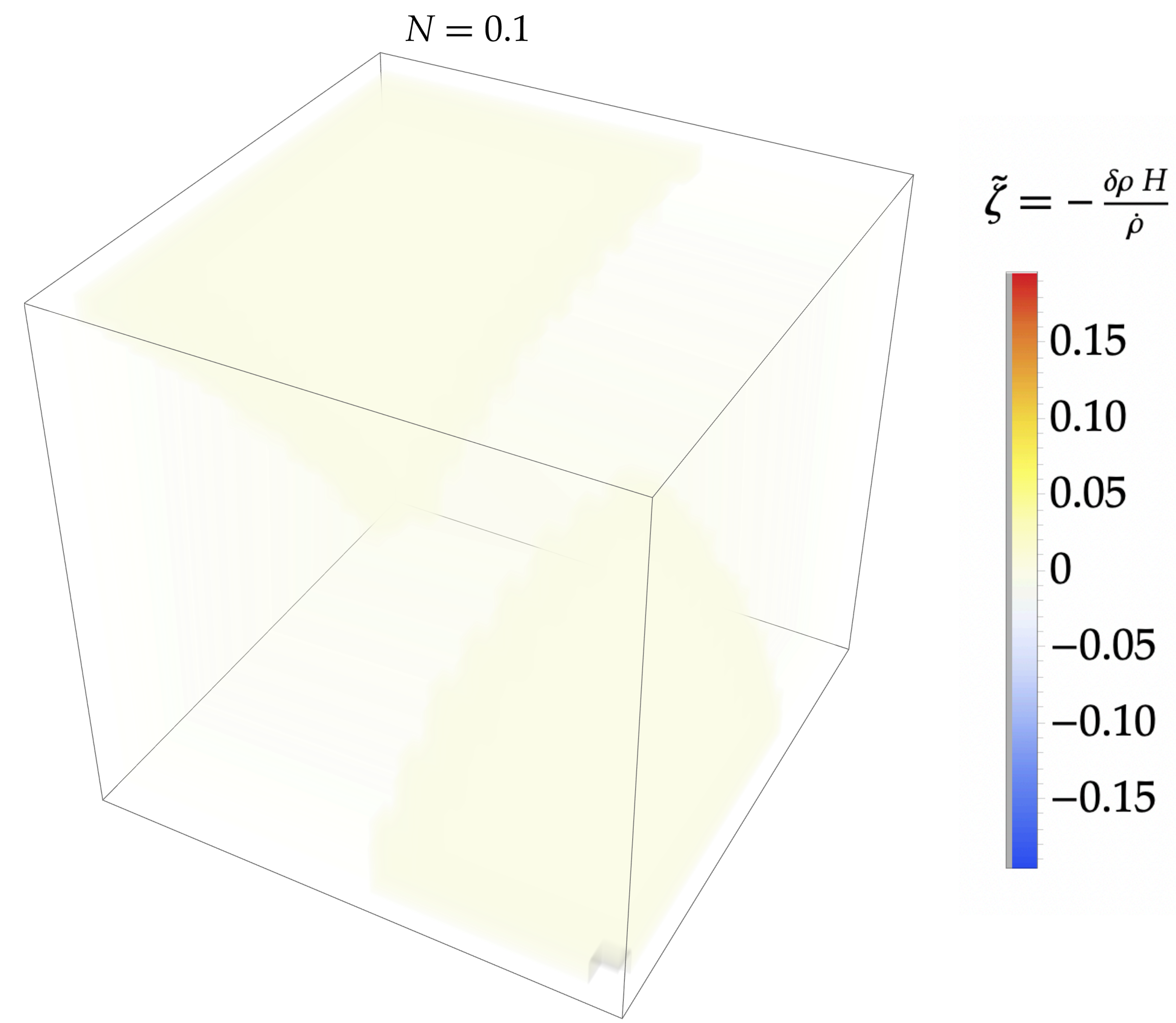

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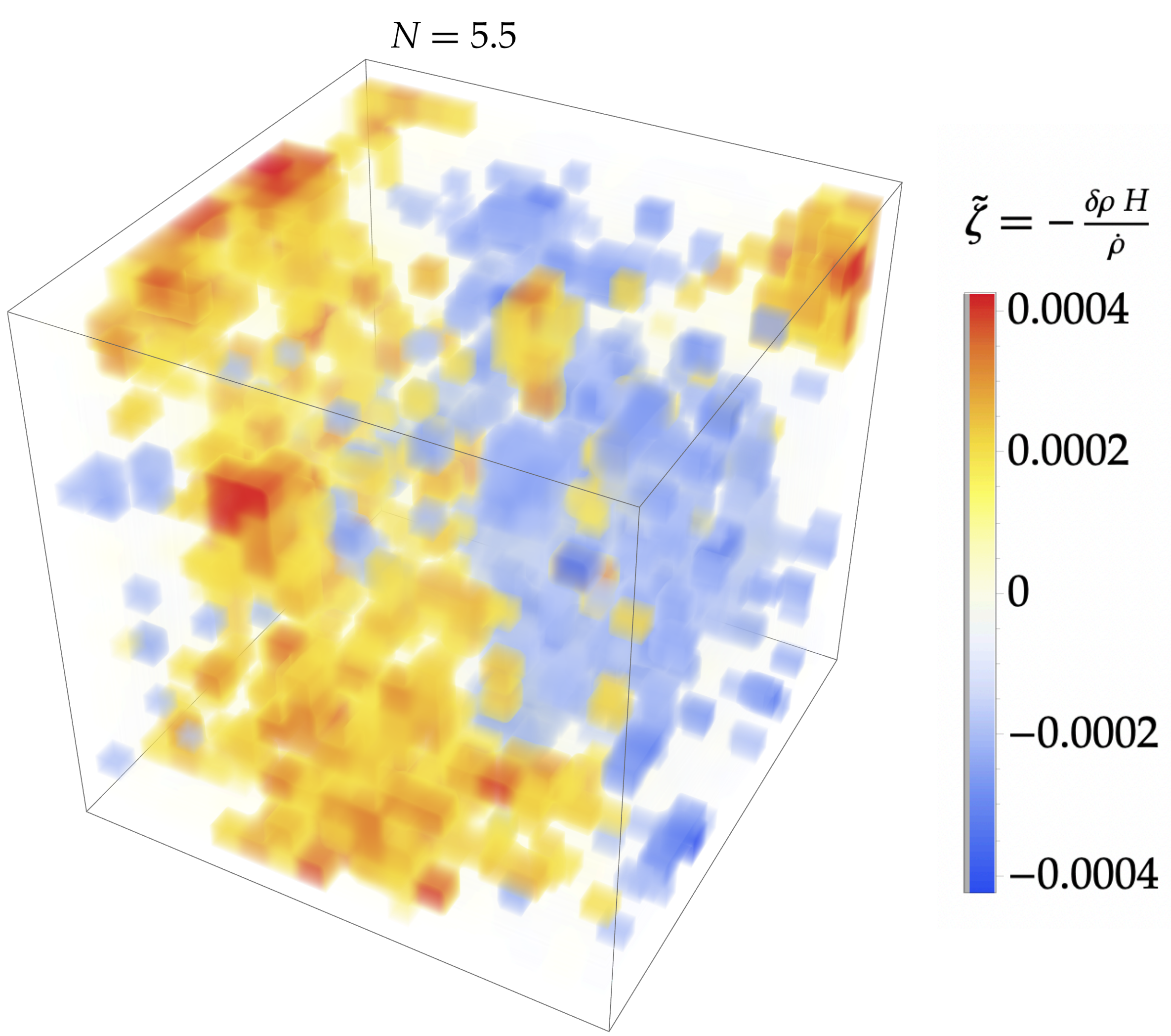


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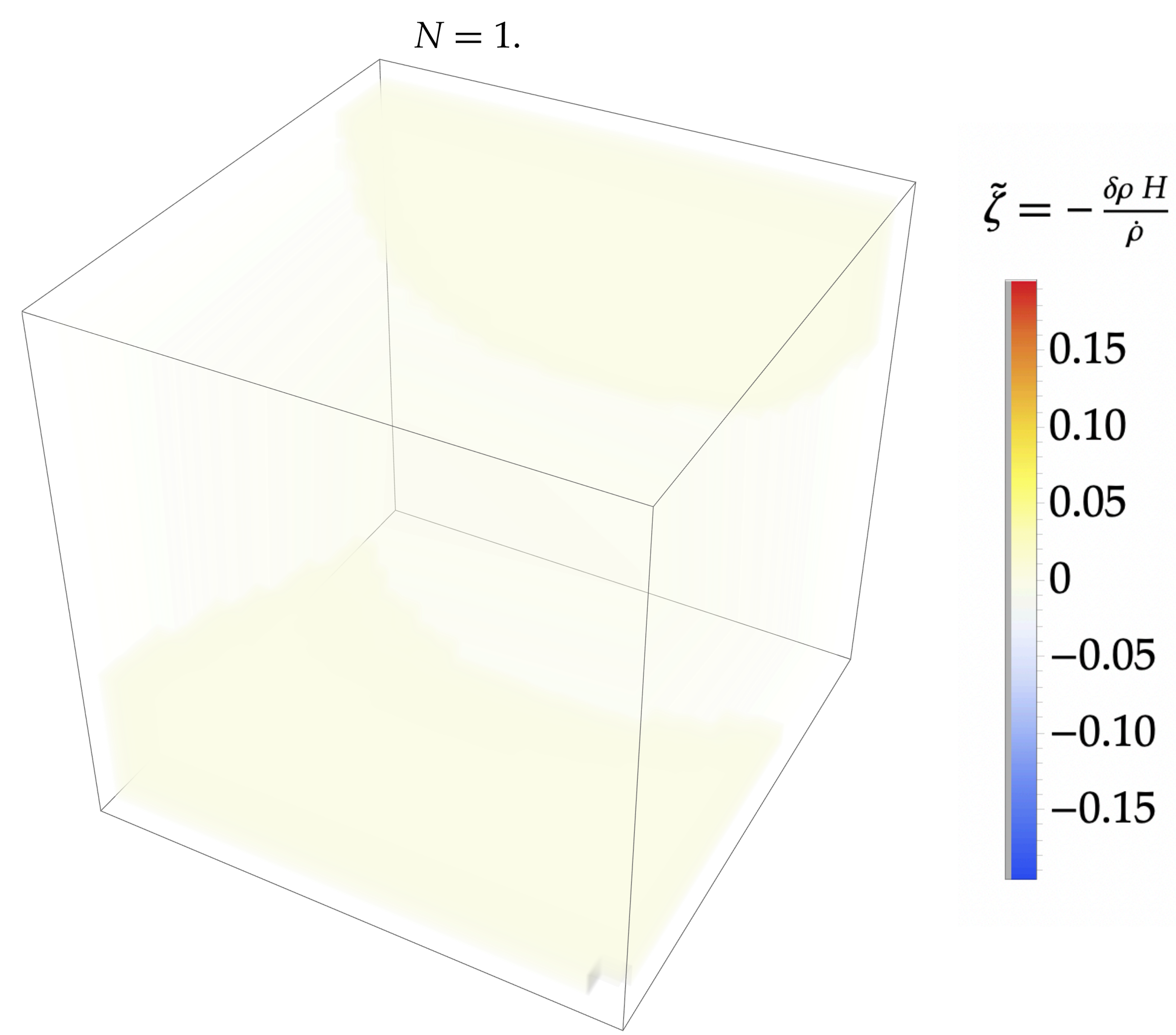

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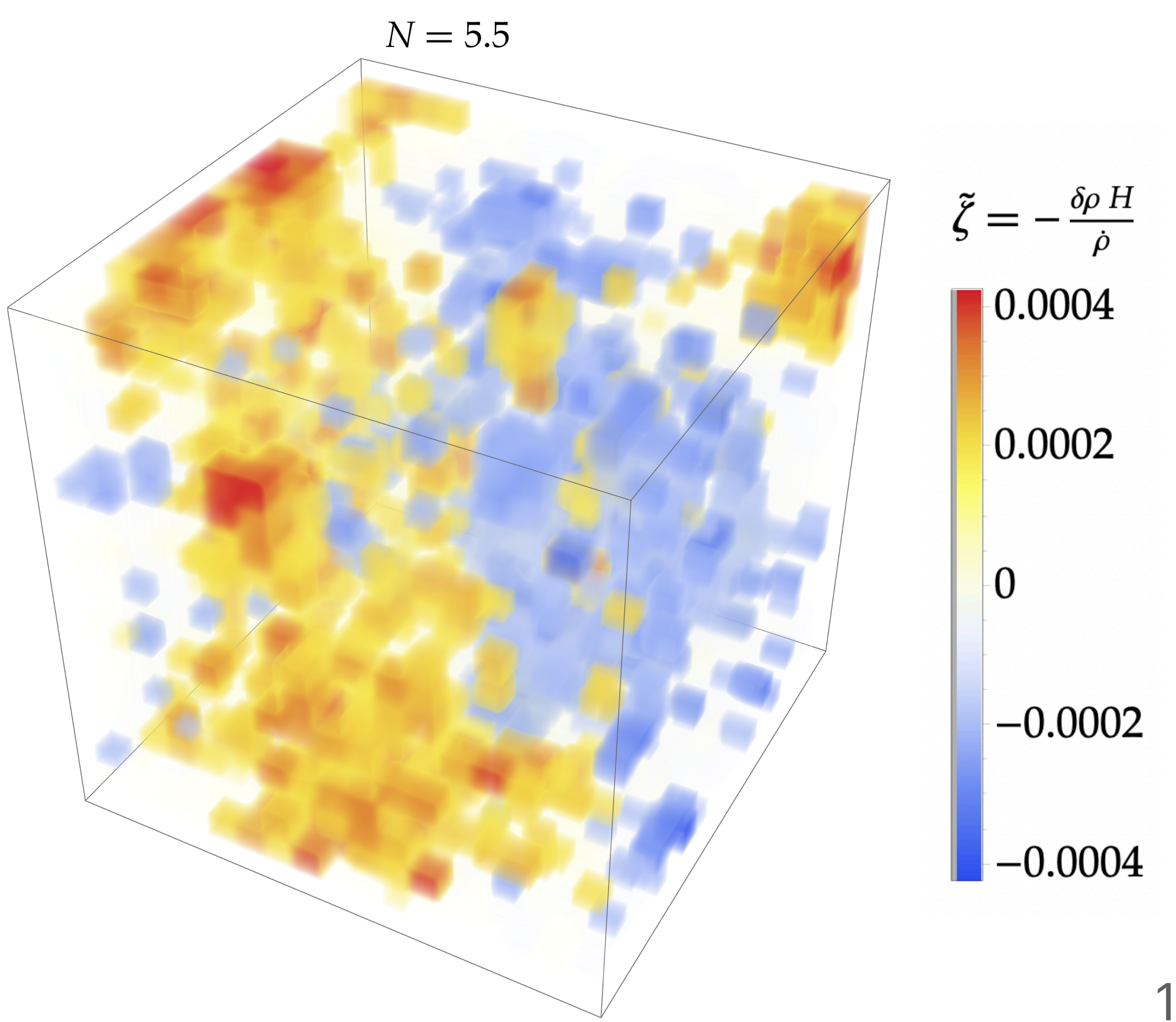


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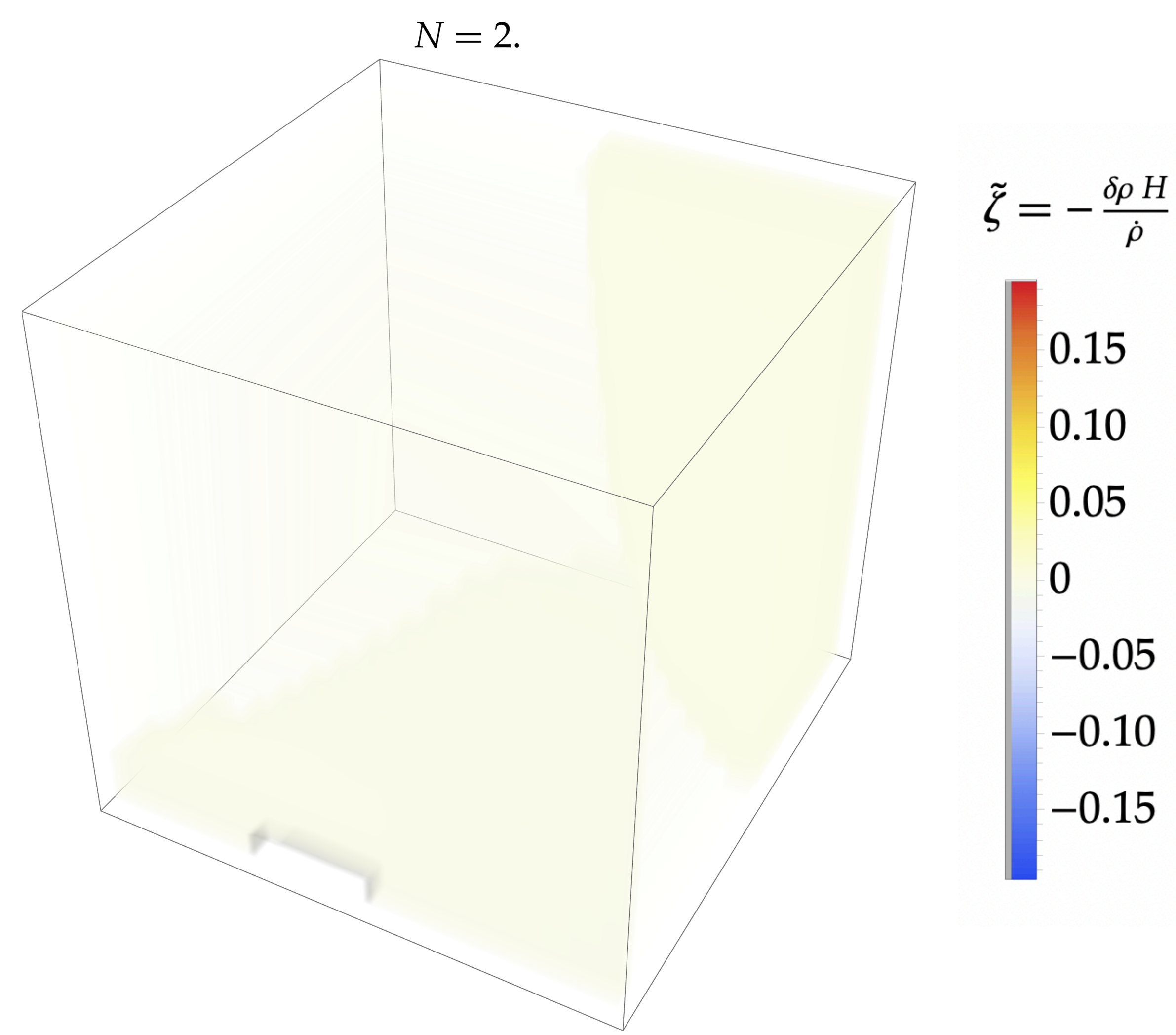

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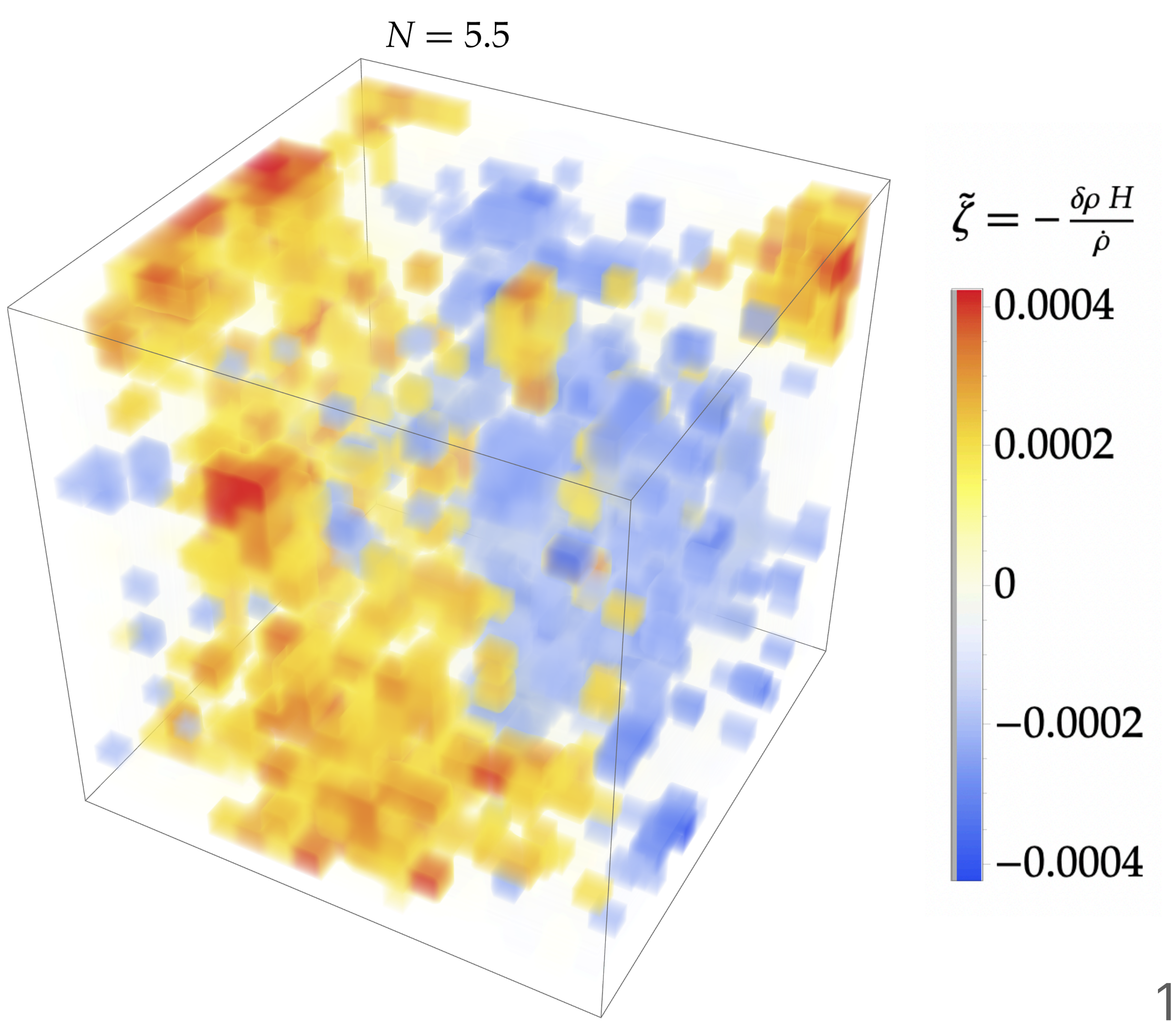


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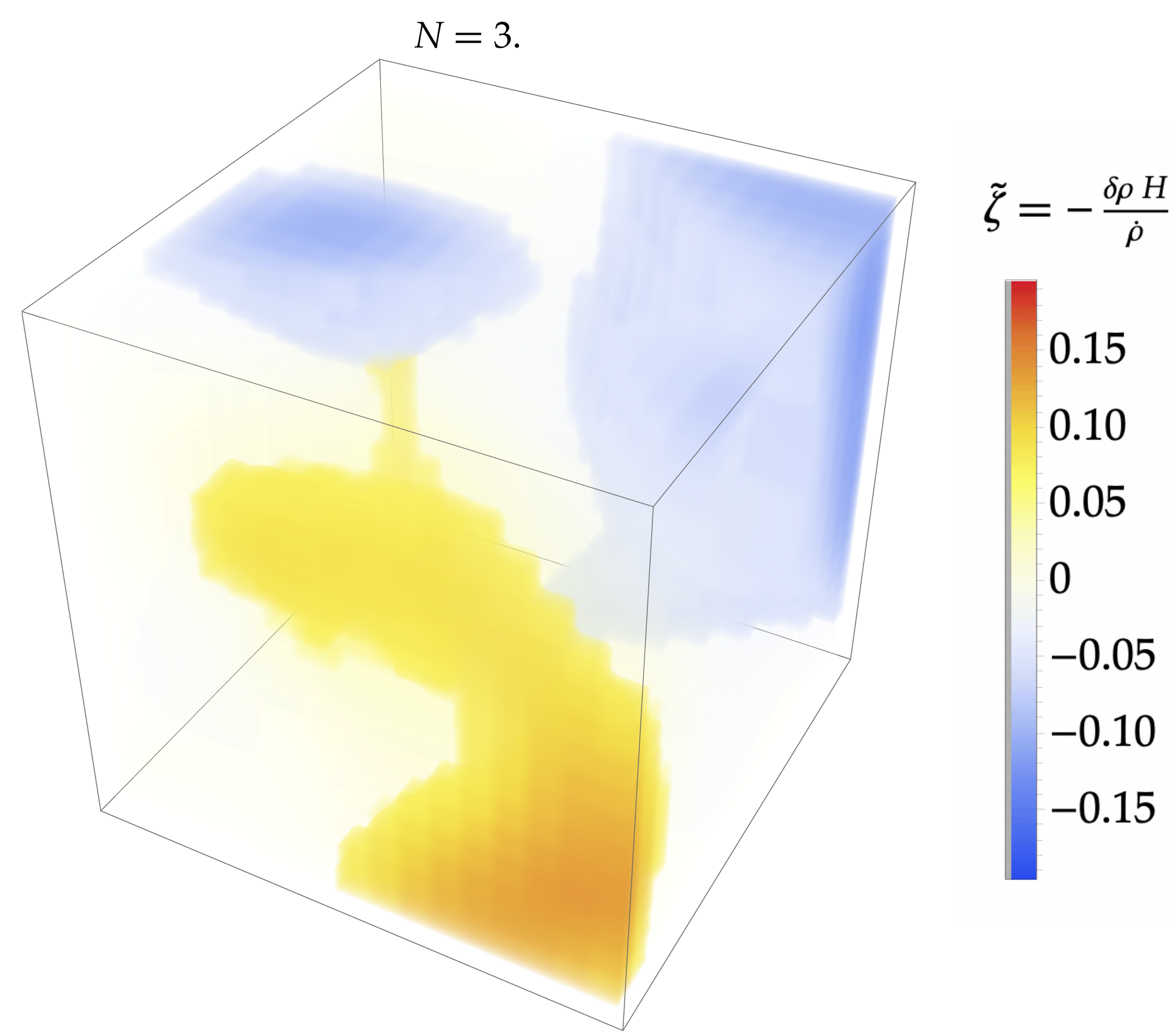

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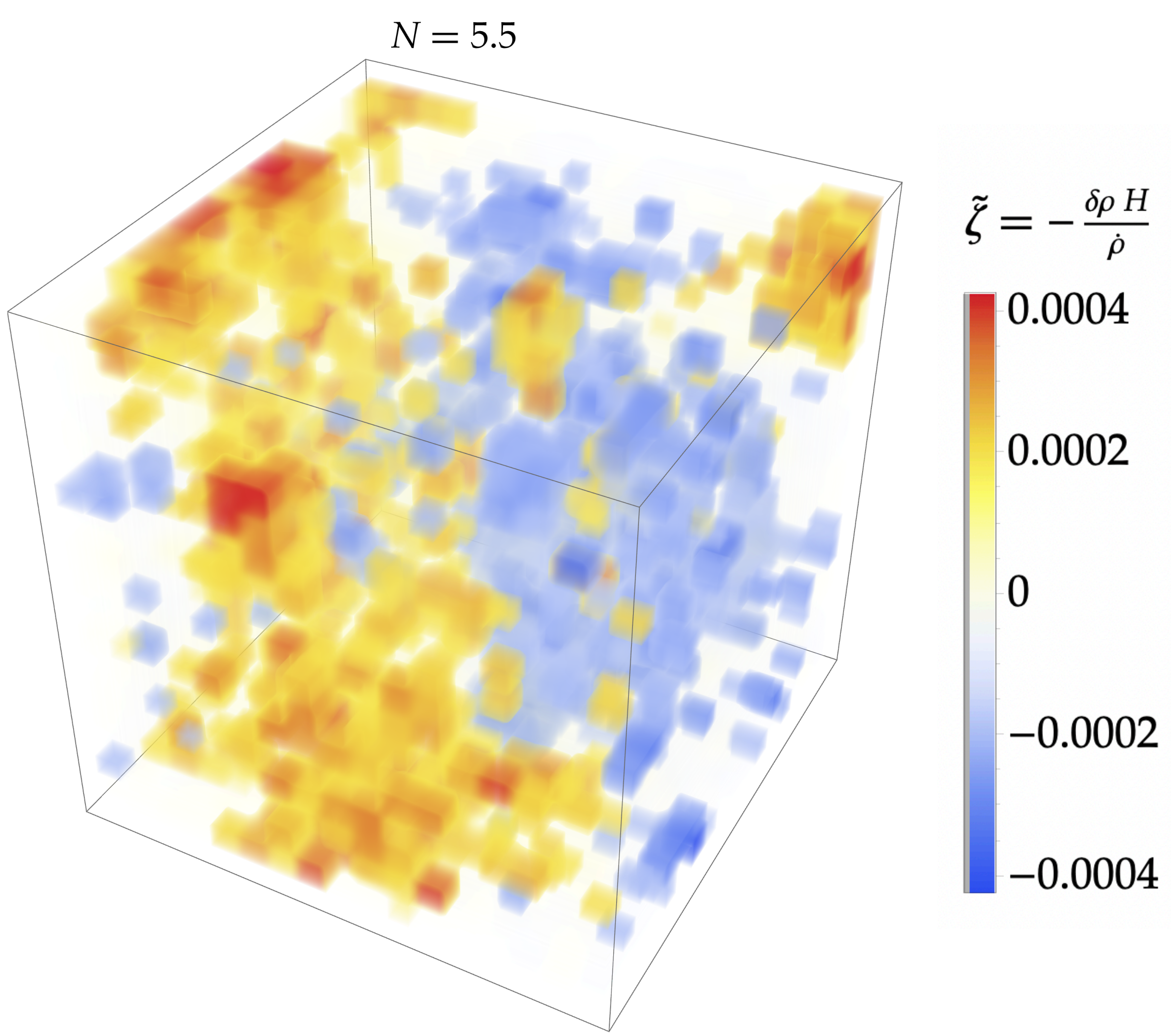


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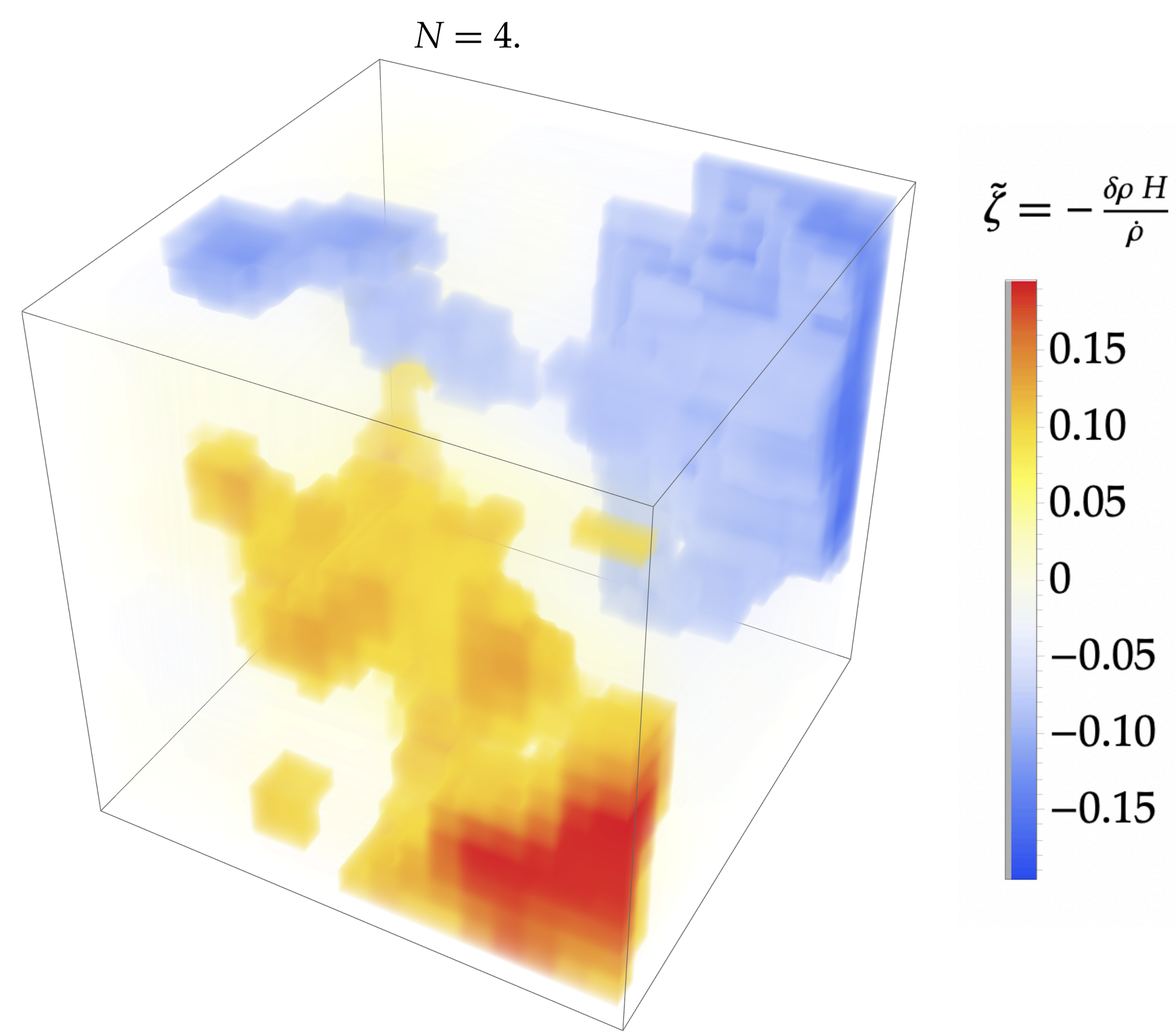

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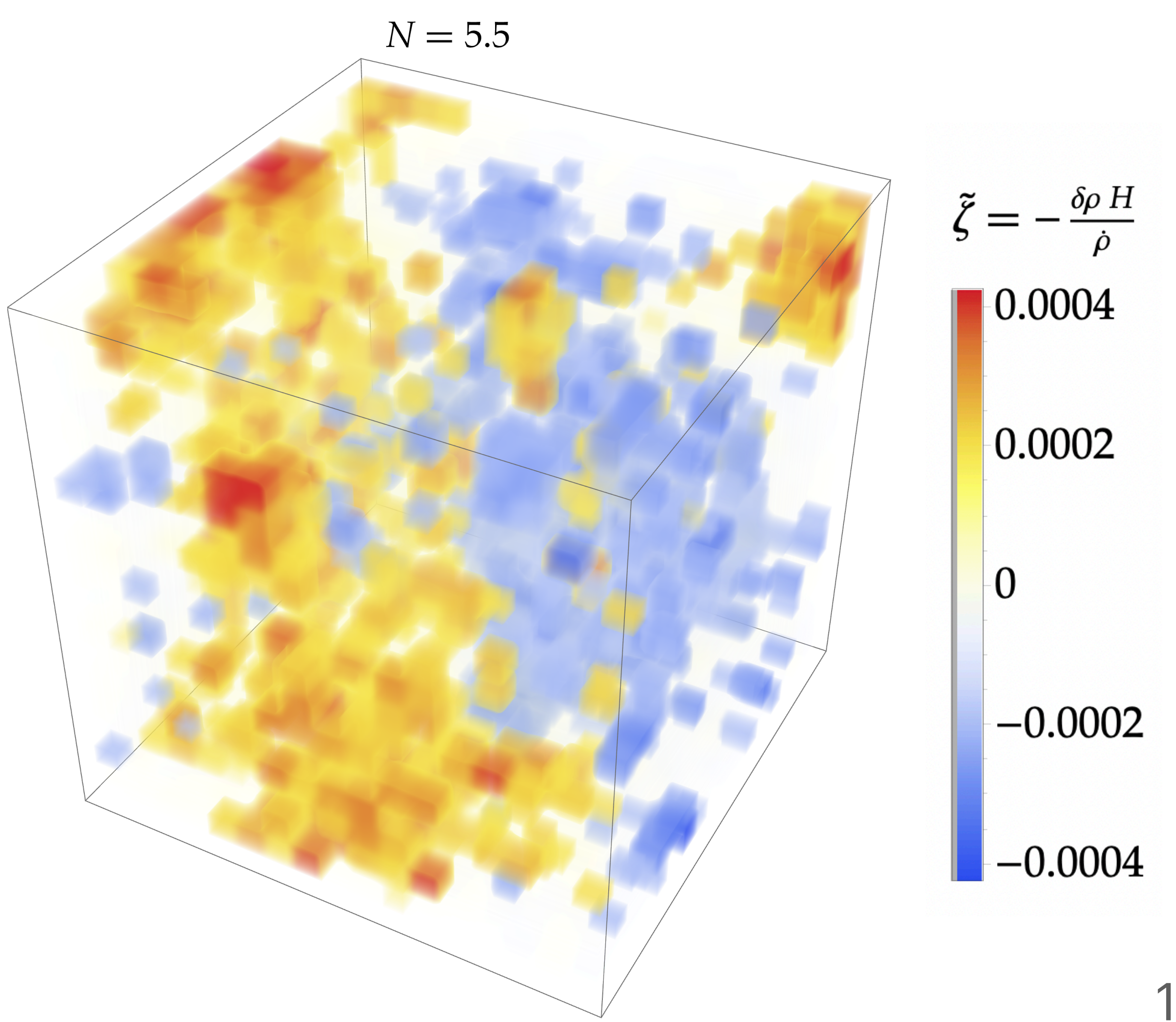


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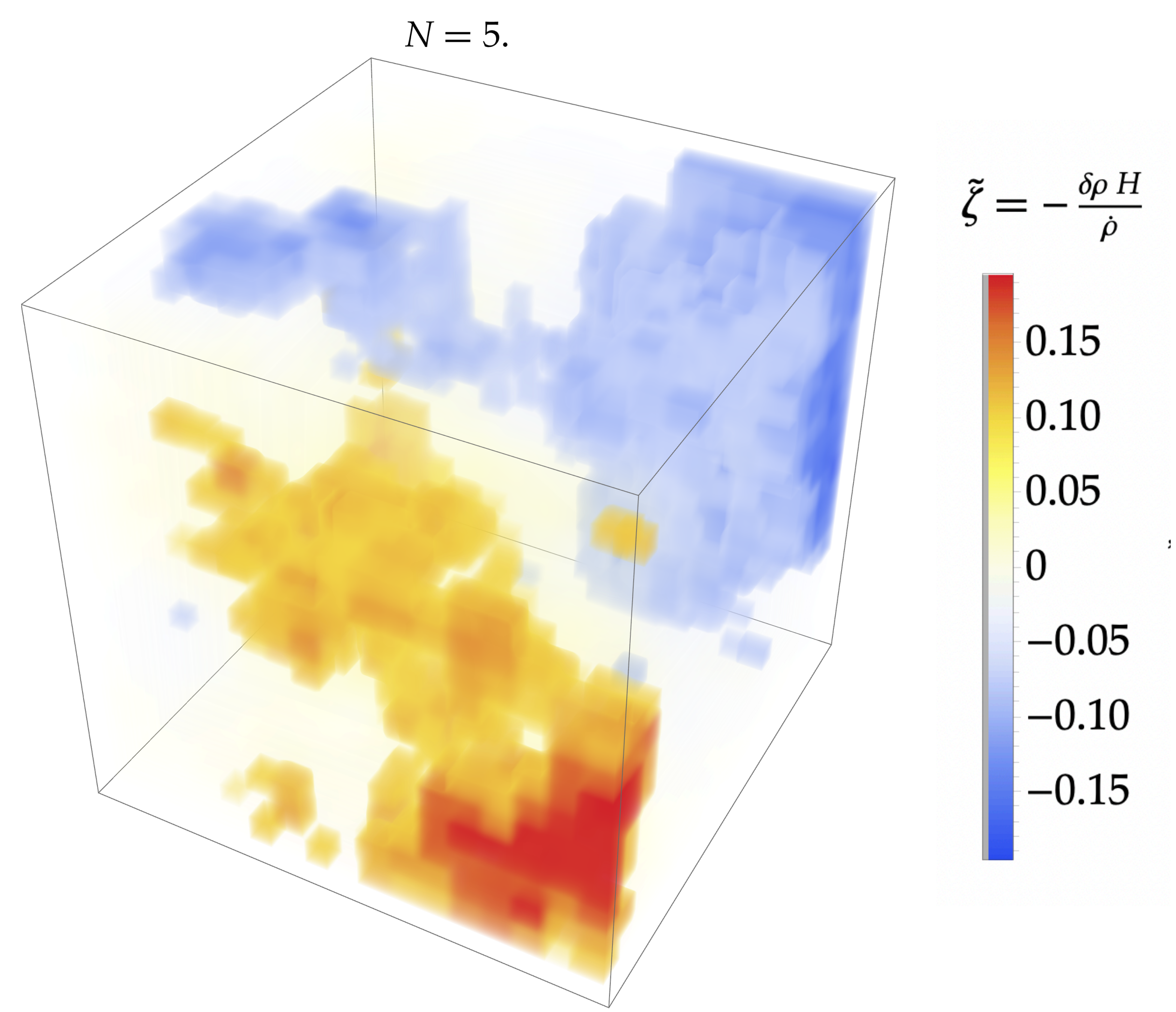

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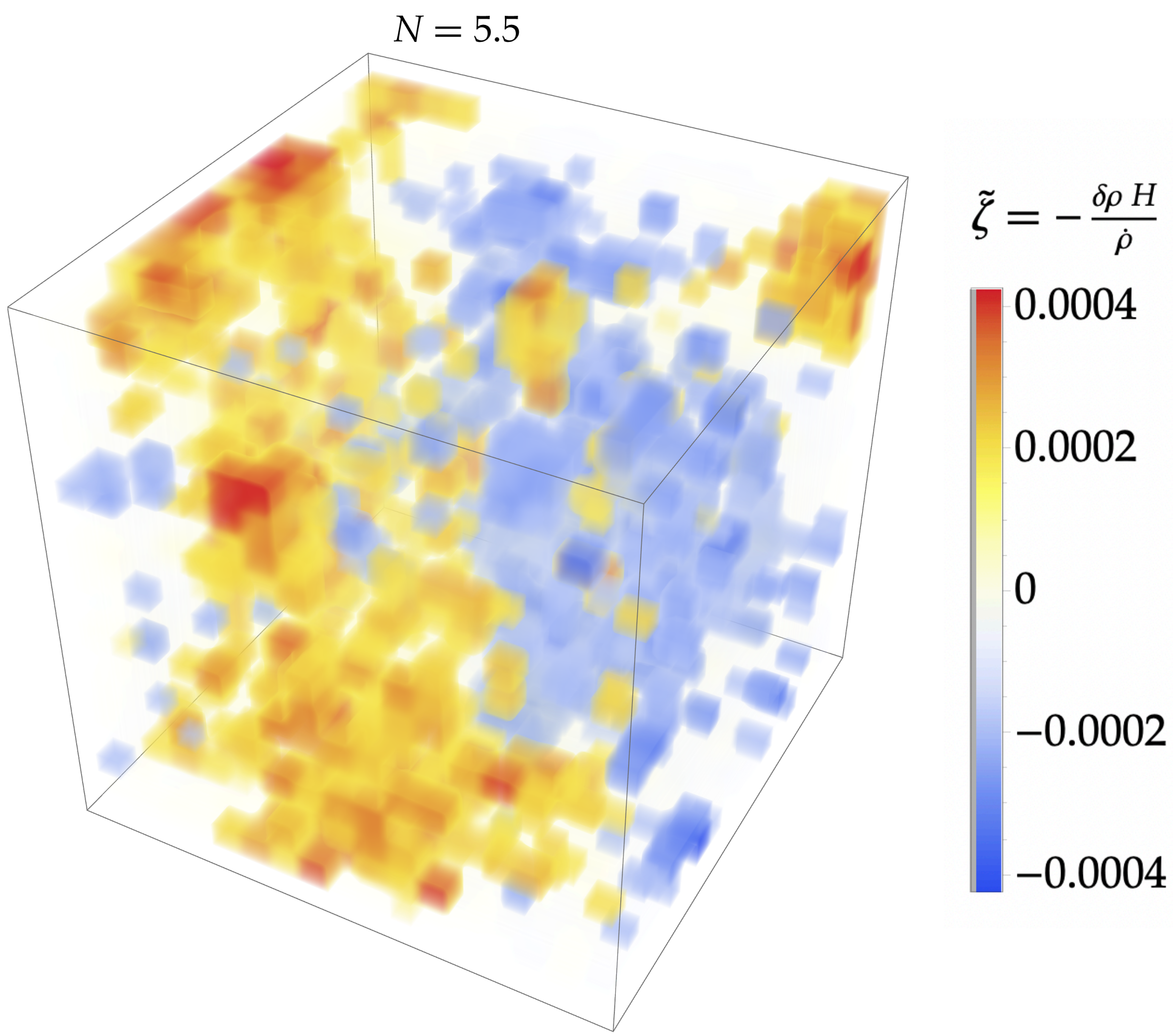


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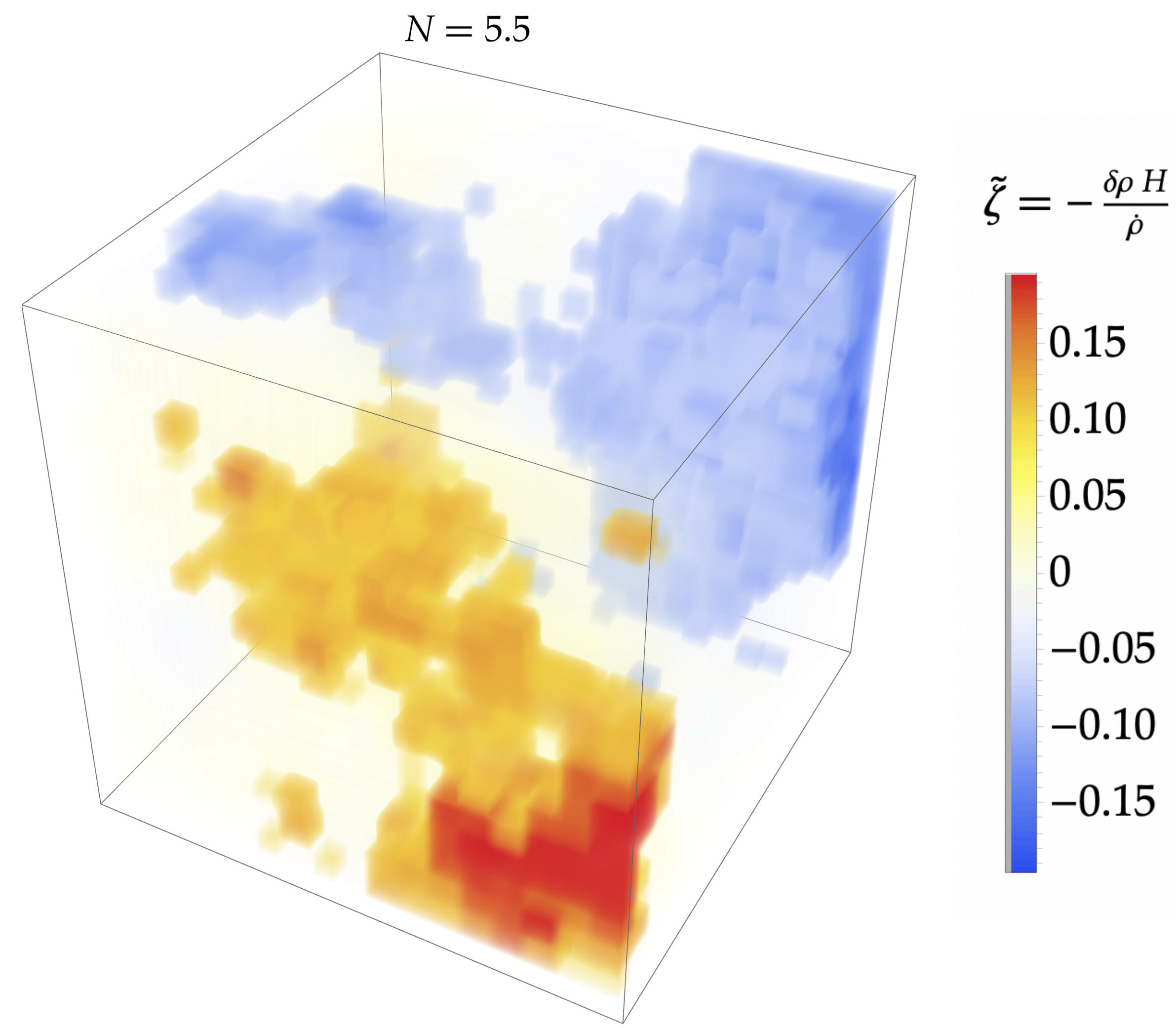

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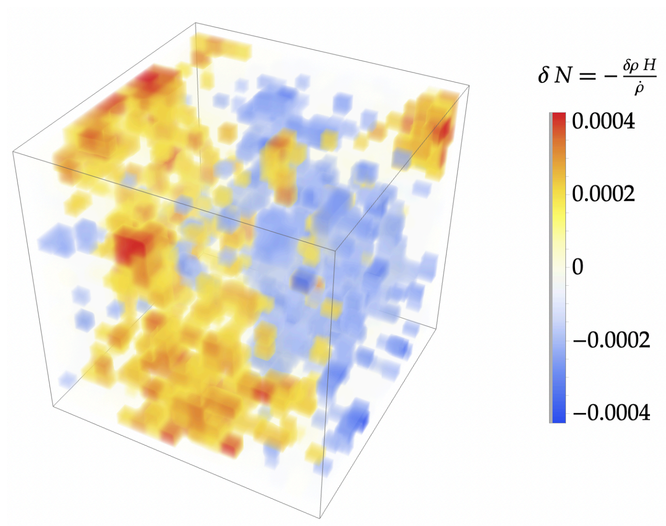
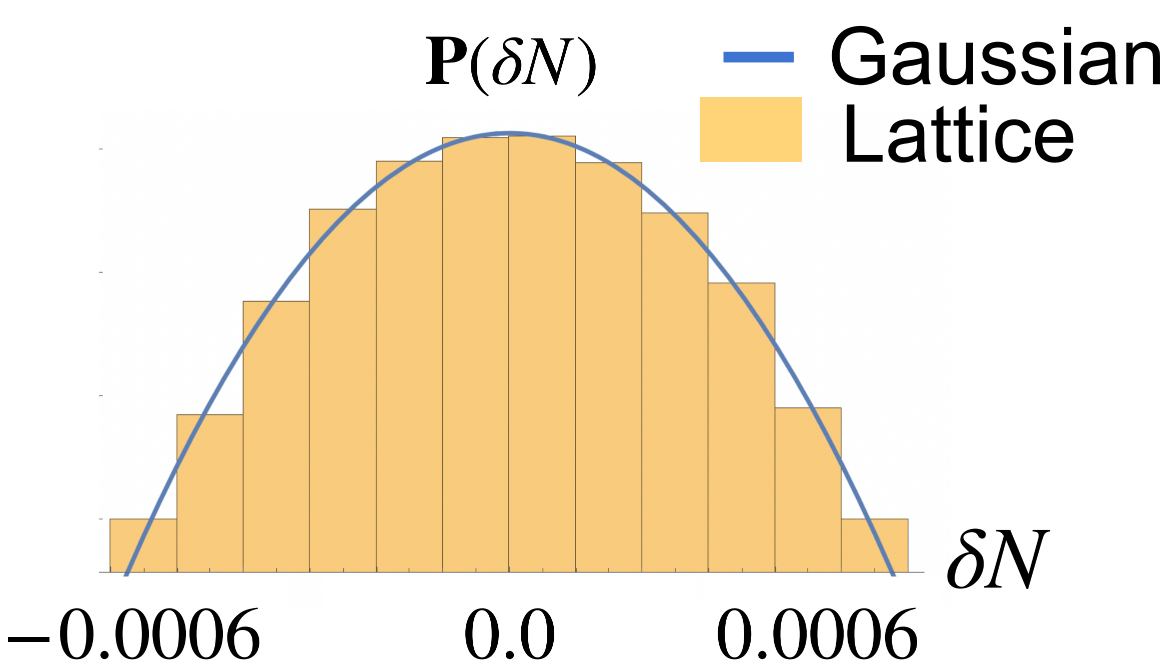
At the end of Inflation

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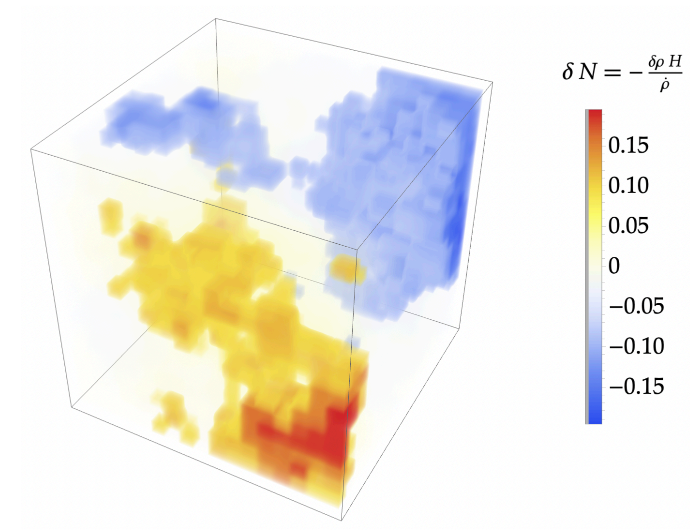


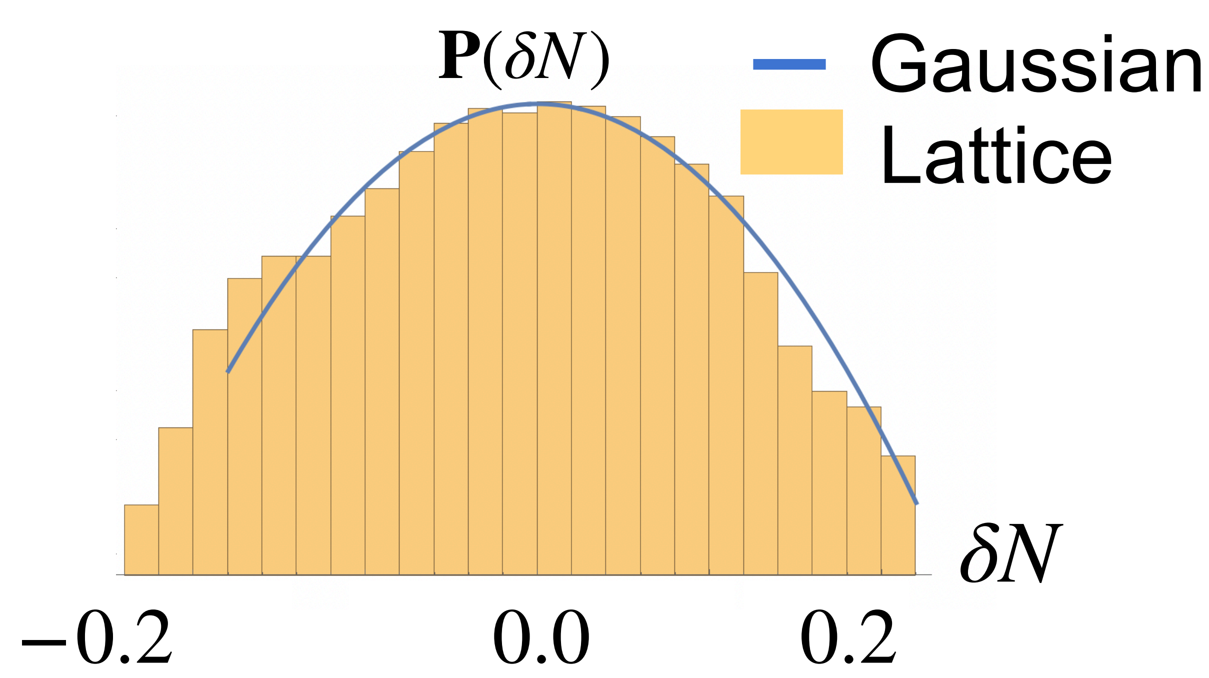

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Jackson, Joseph H. P. *et al*,2022

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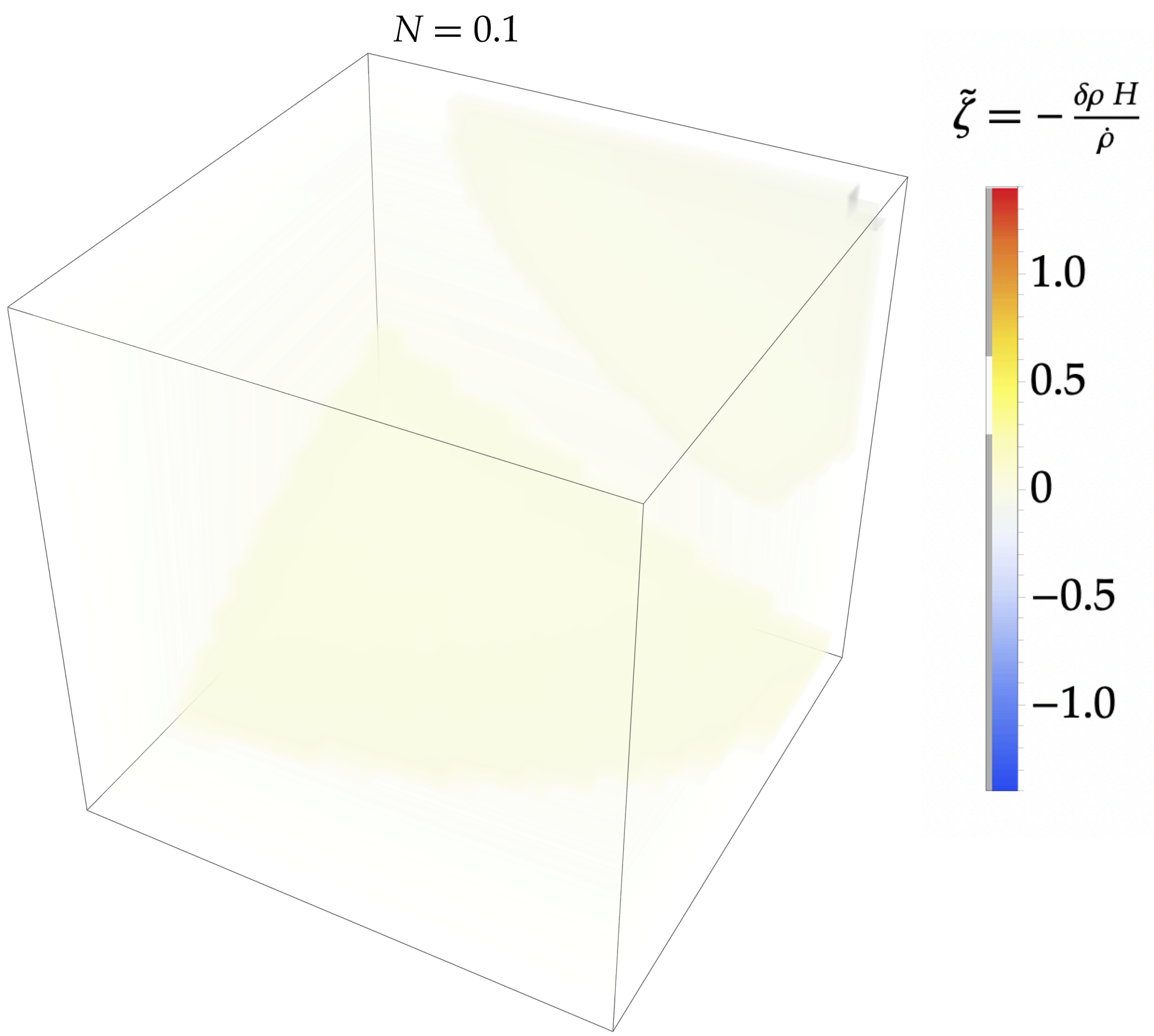


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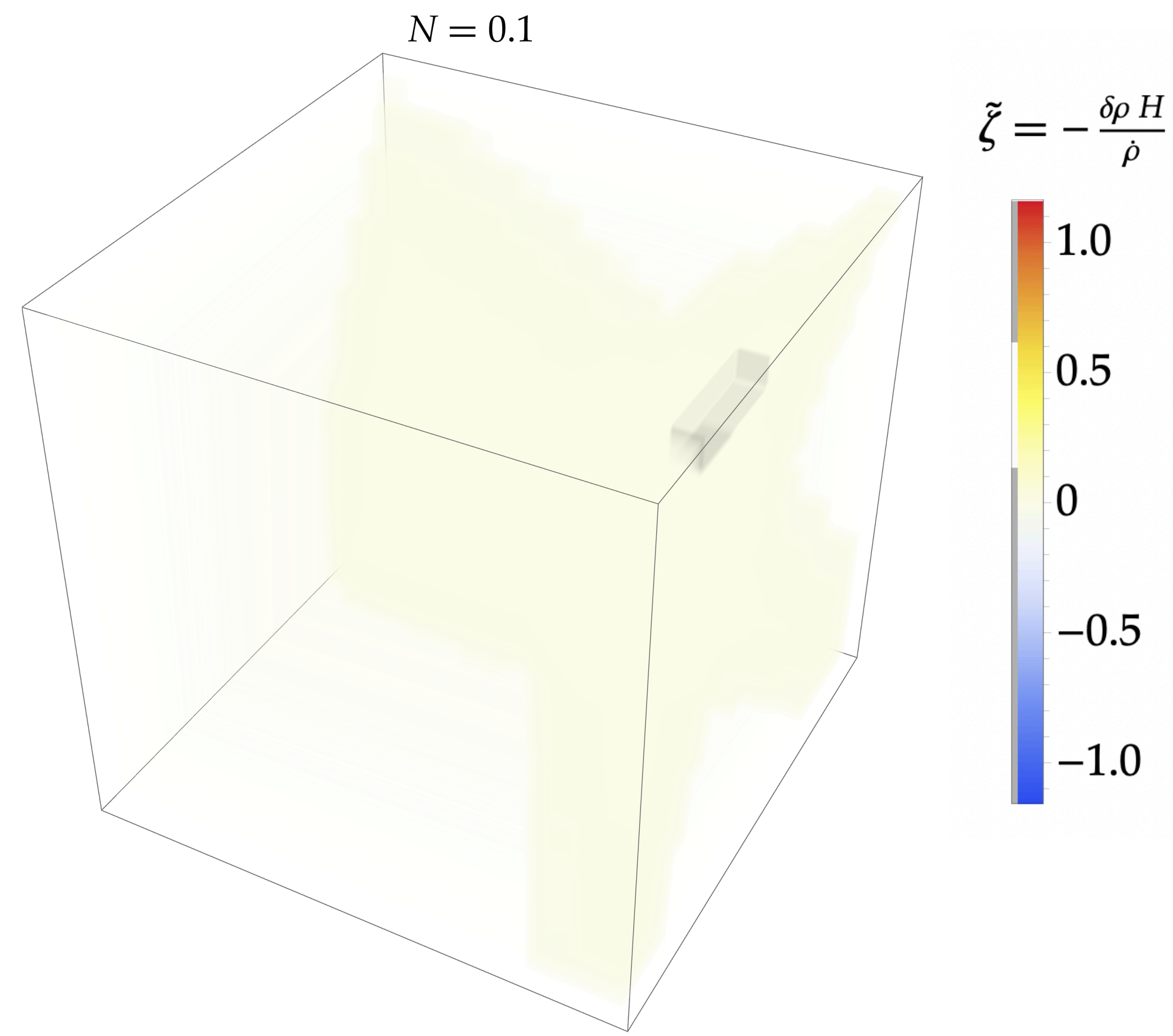

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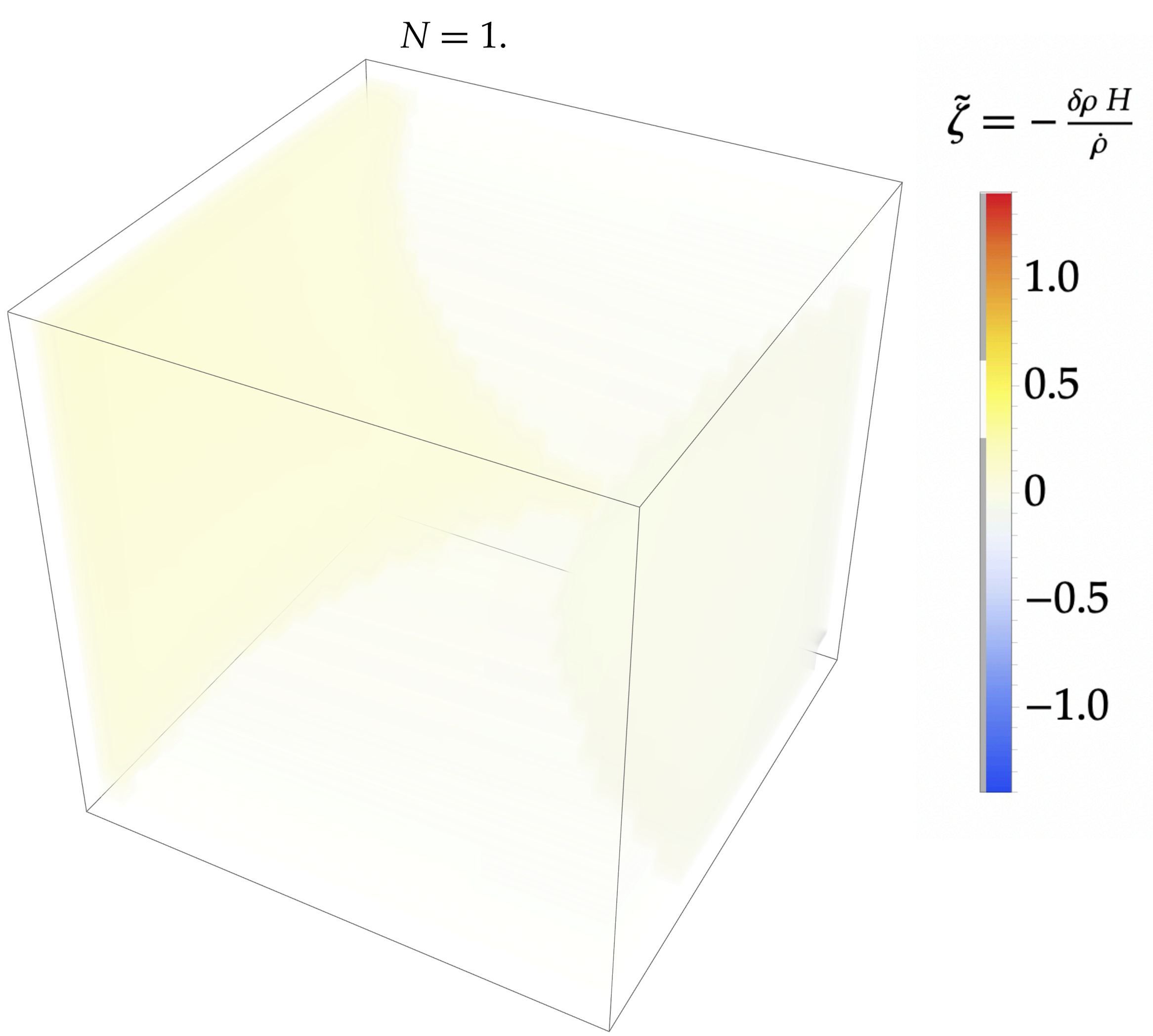


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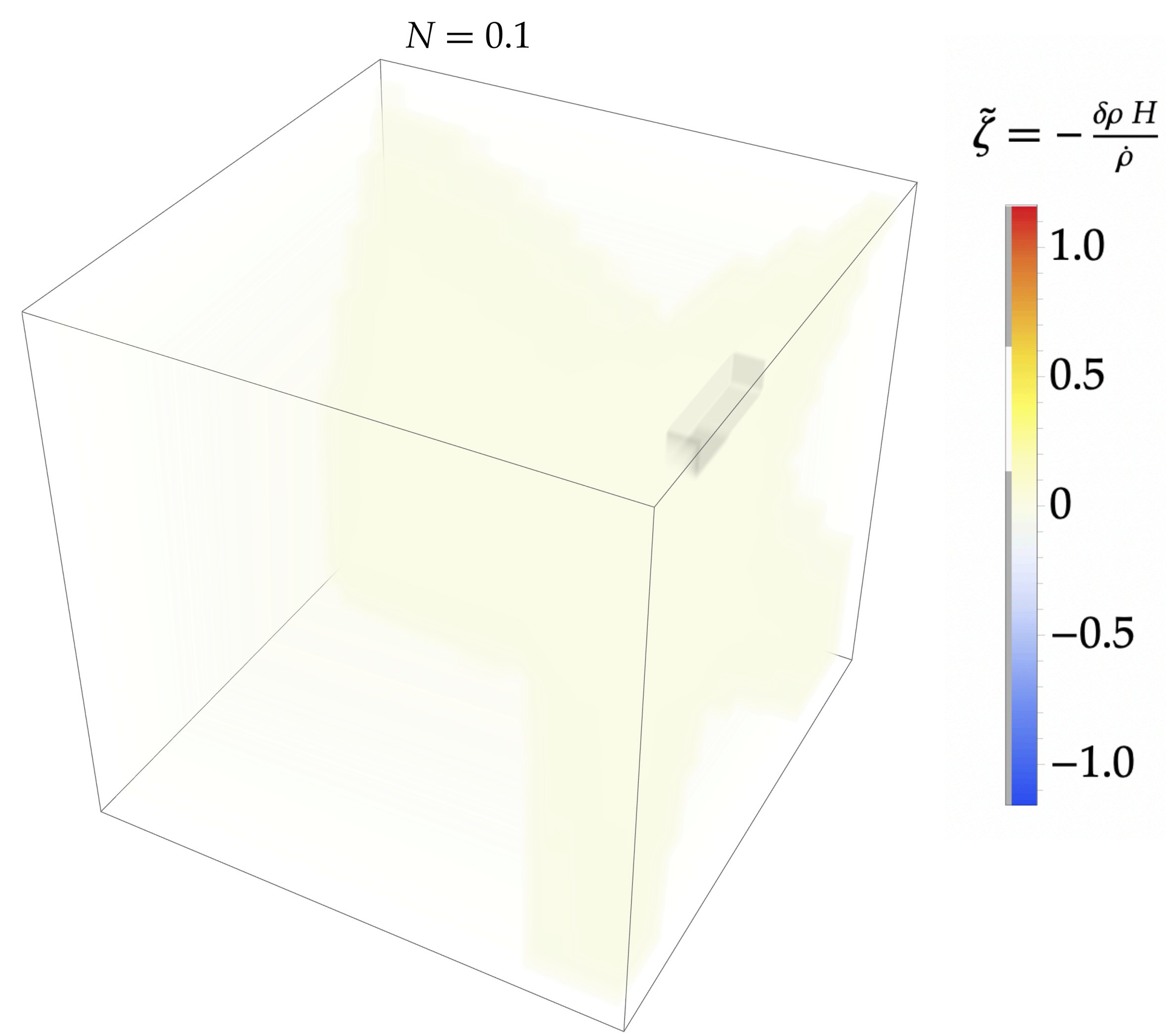

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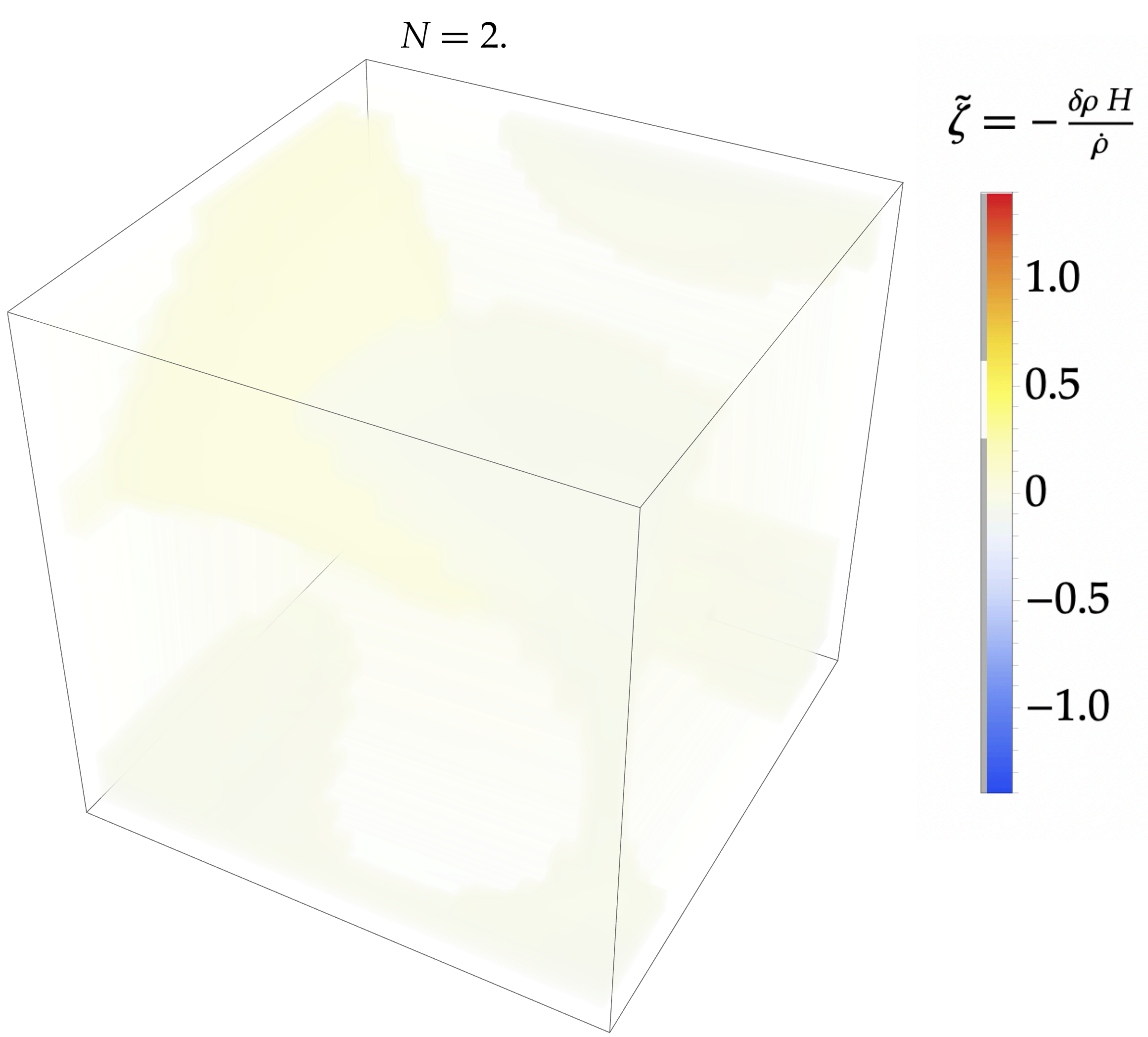


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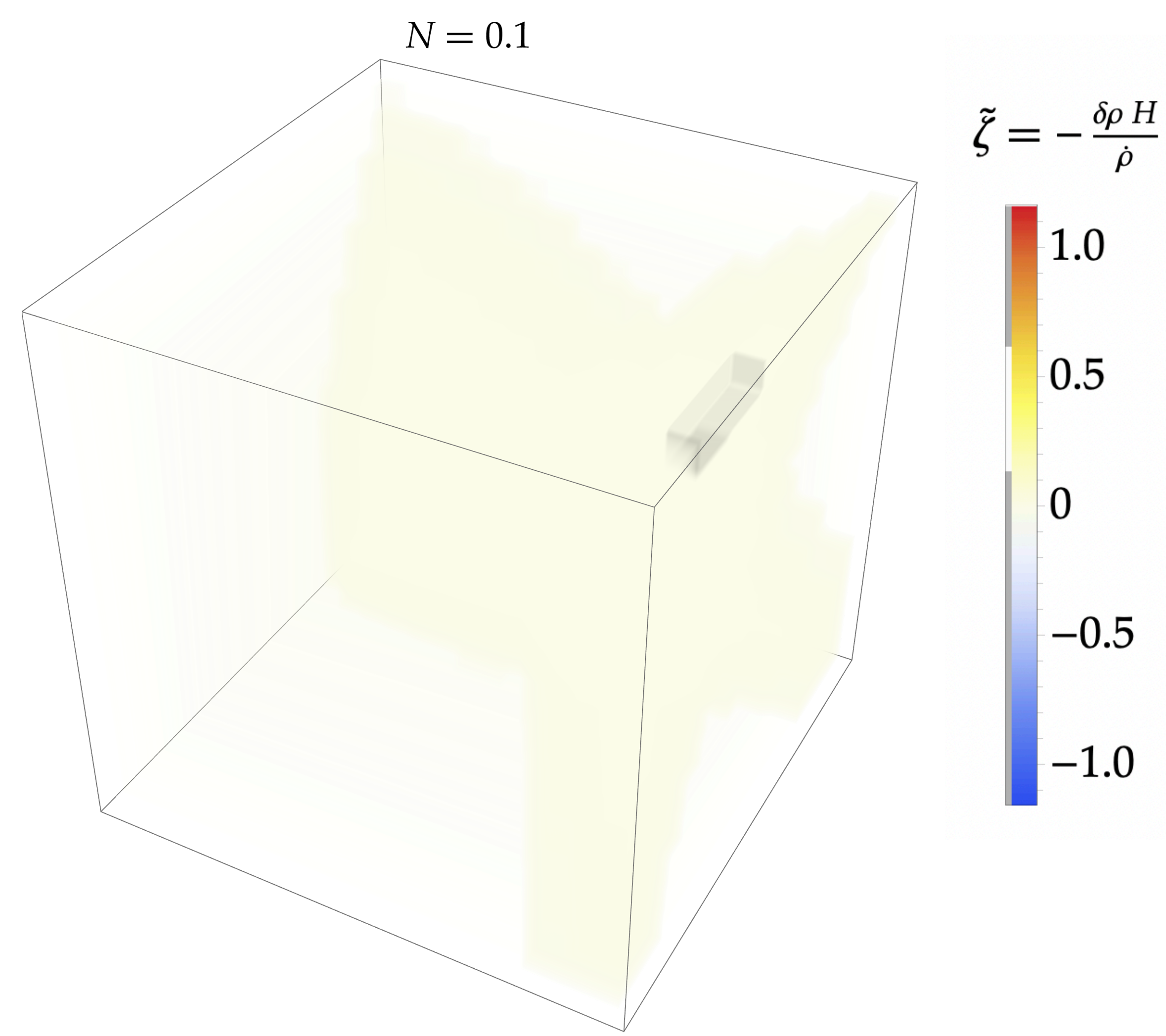

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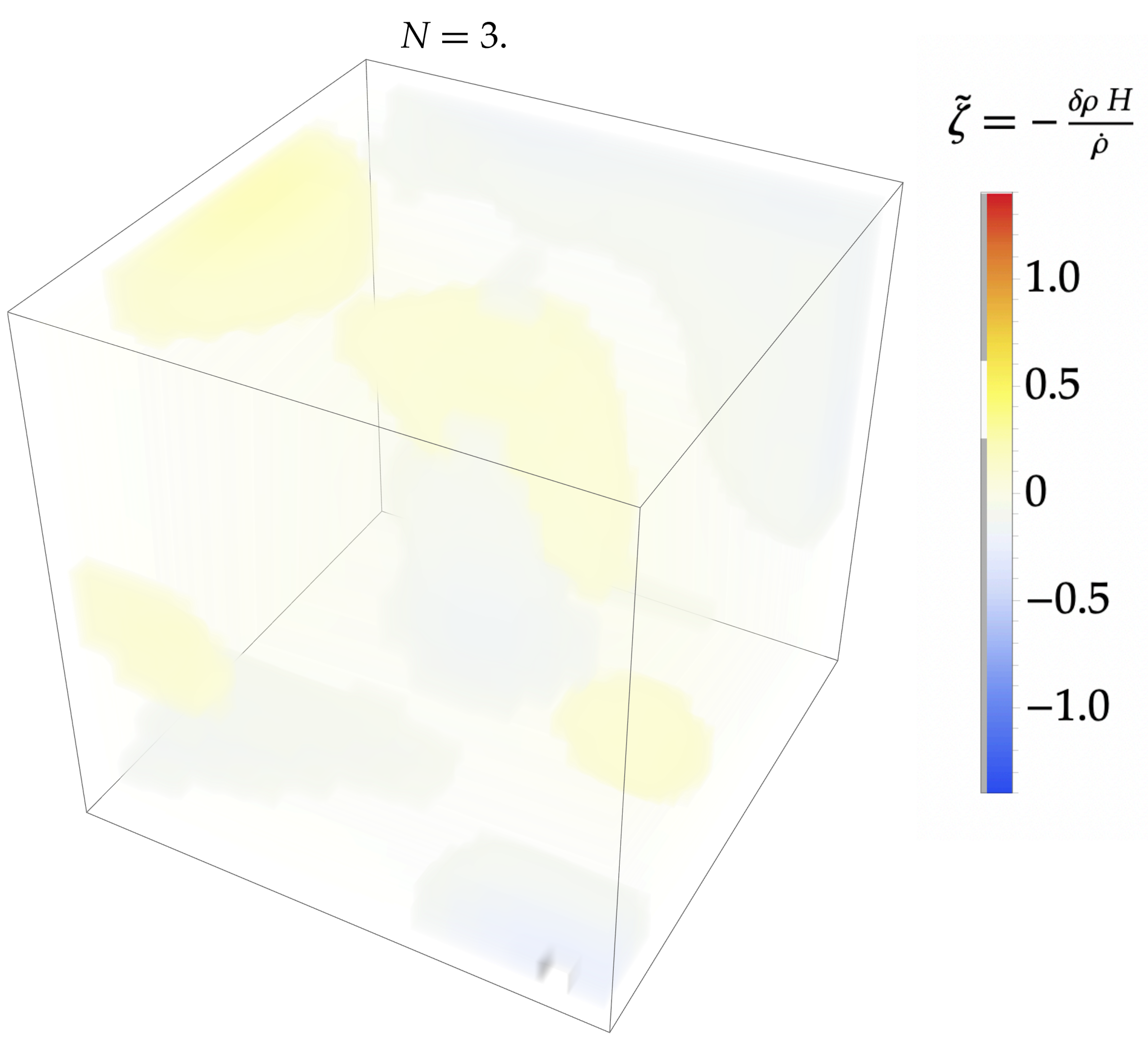


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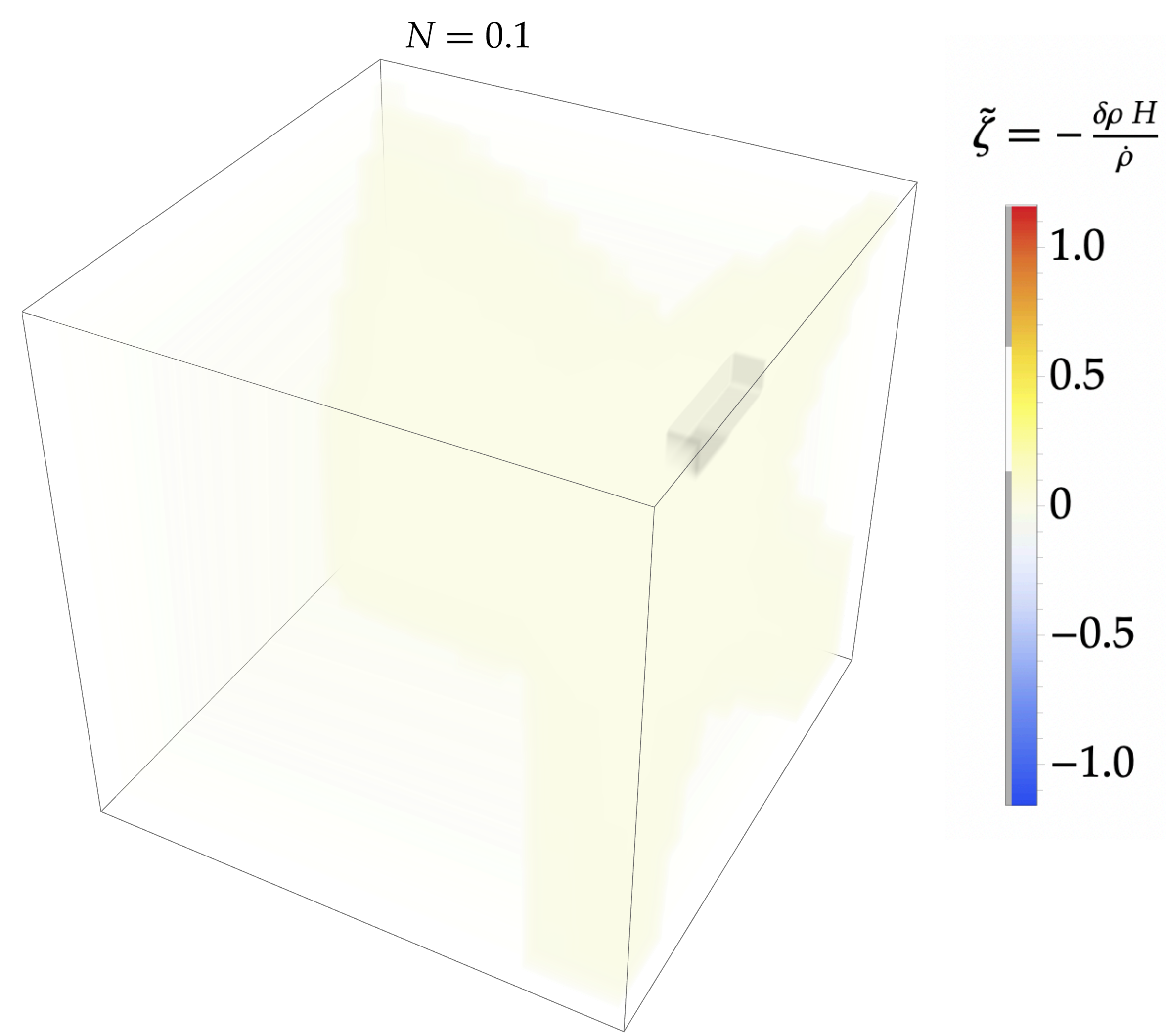

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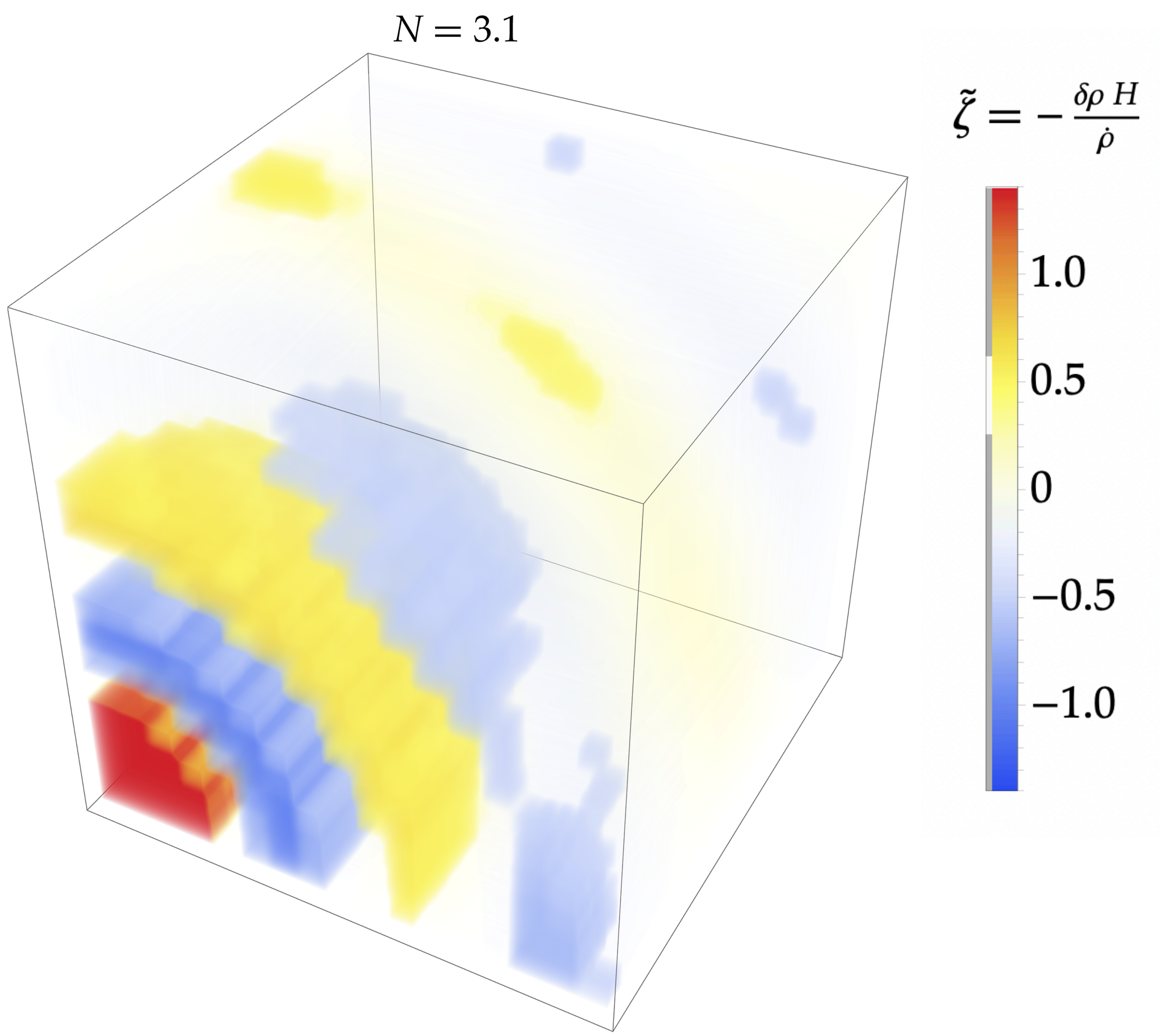


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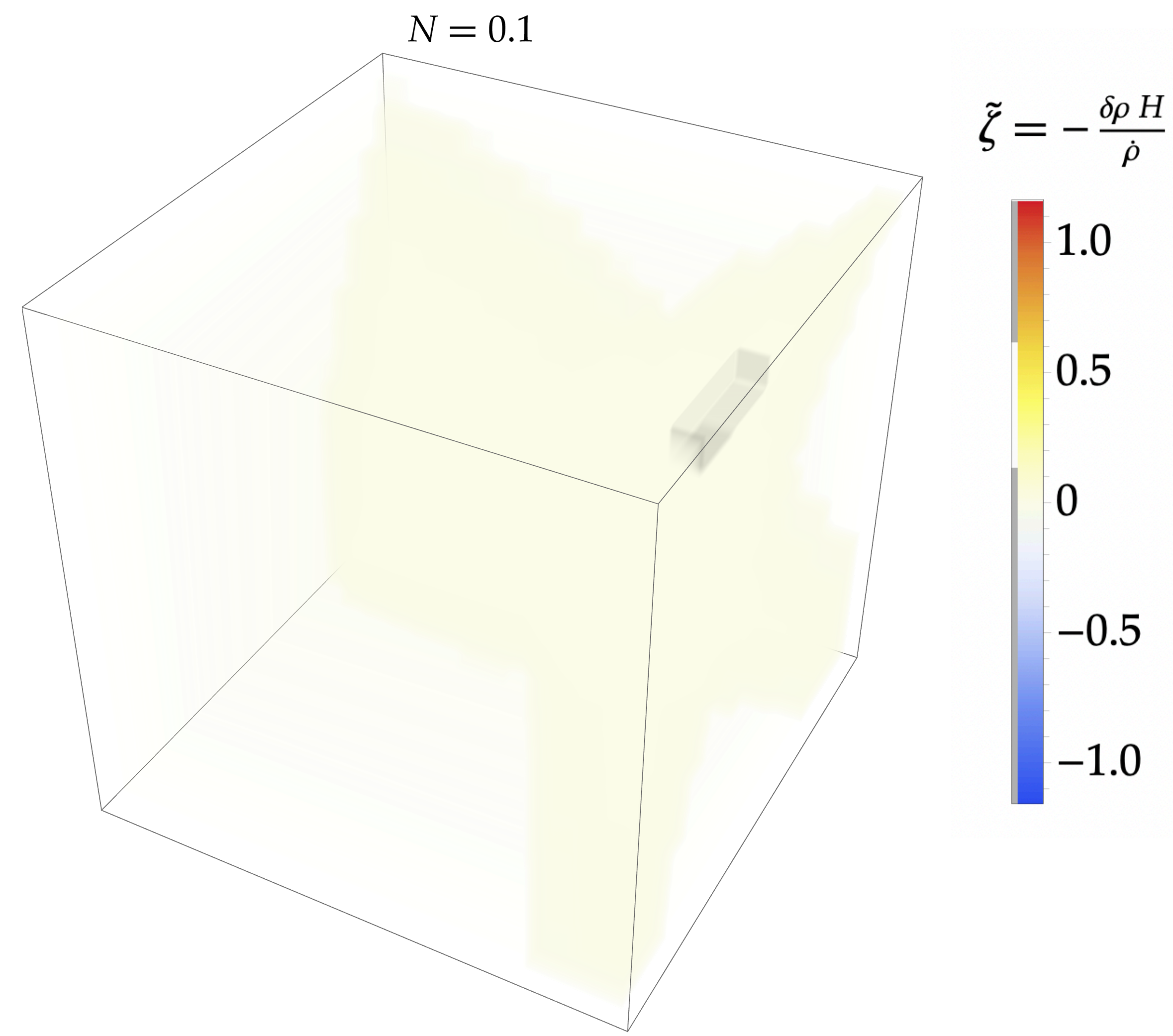

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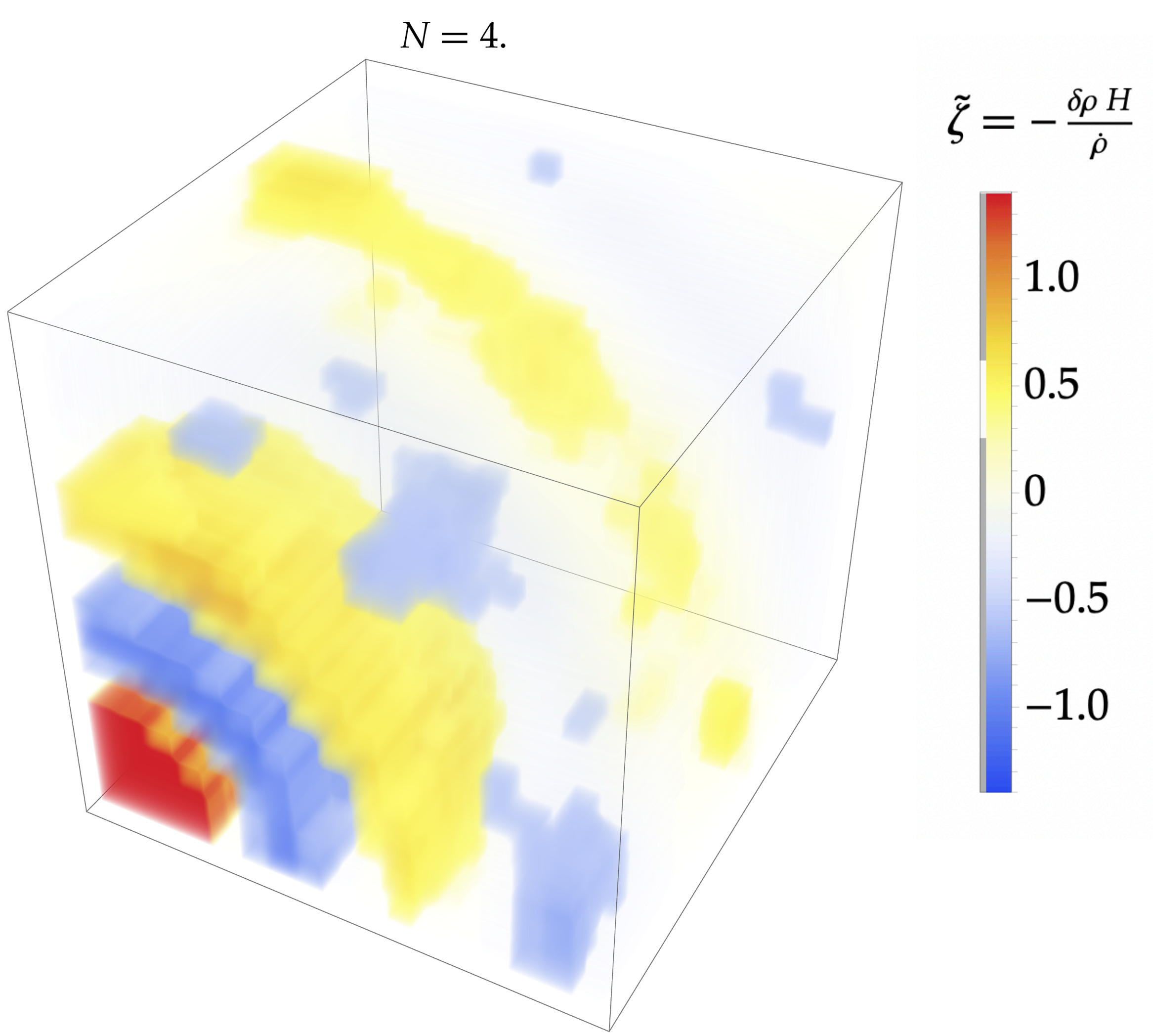


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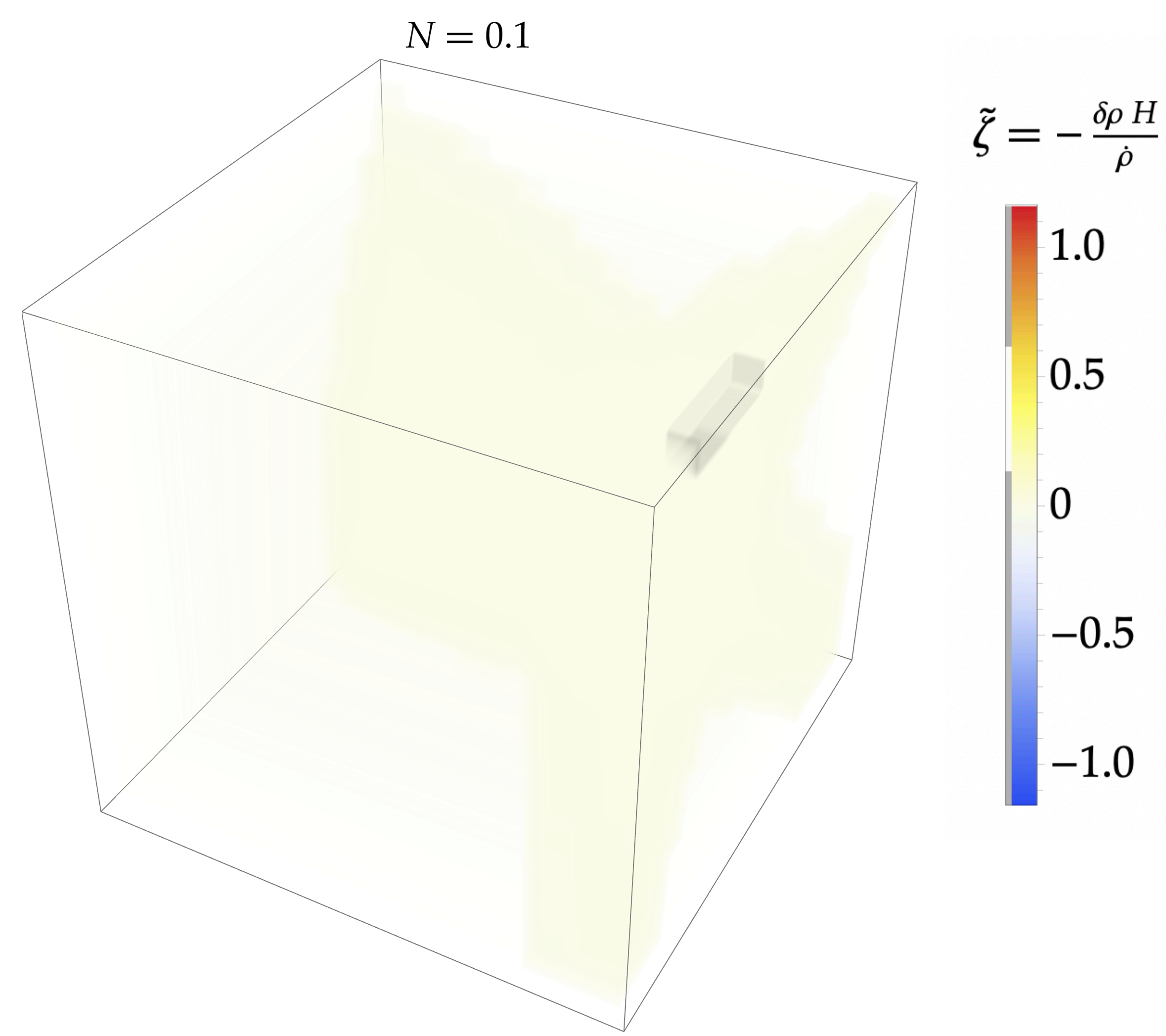

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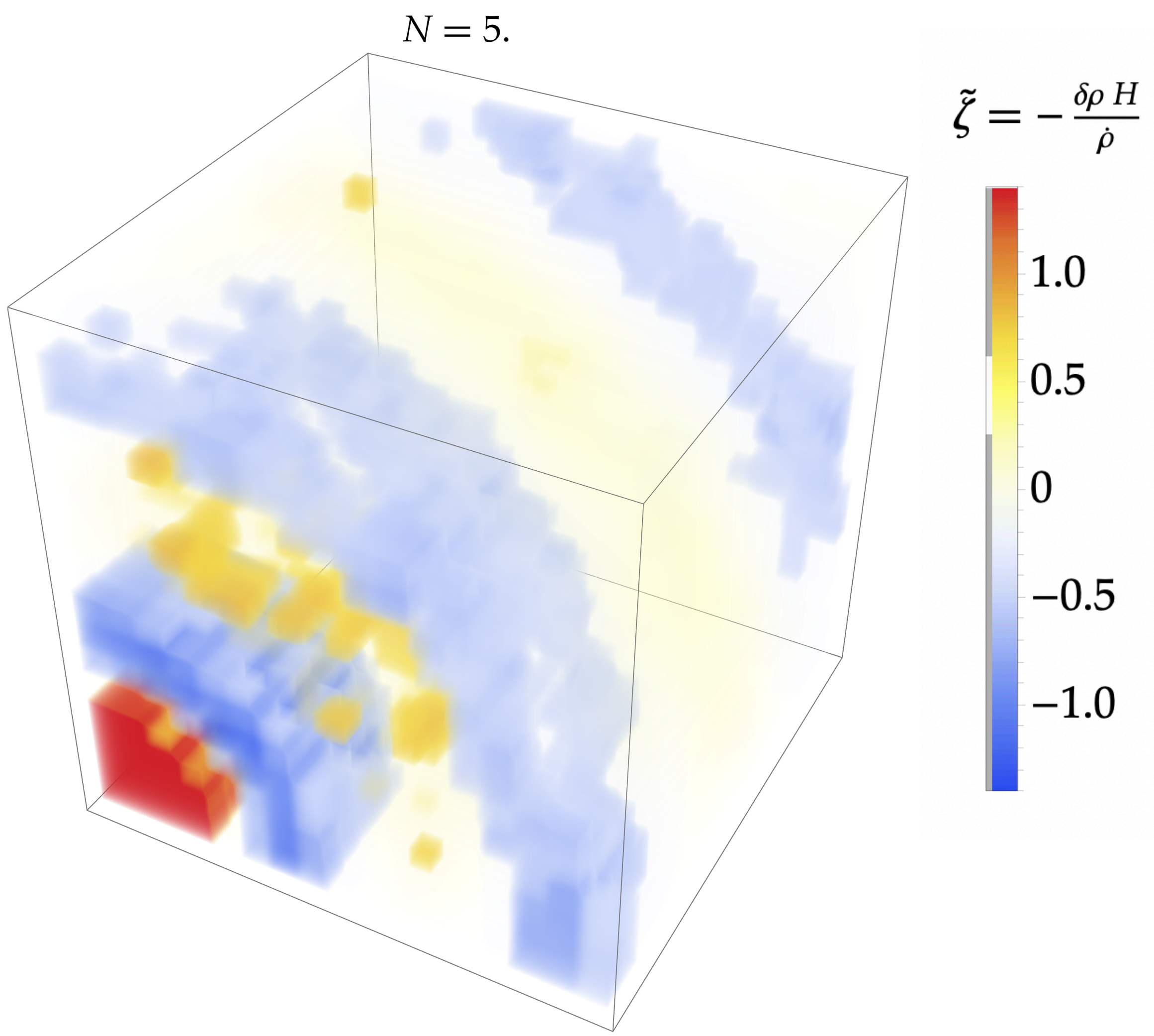


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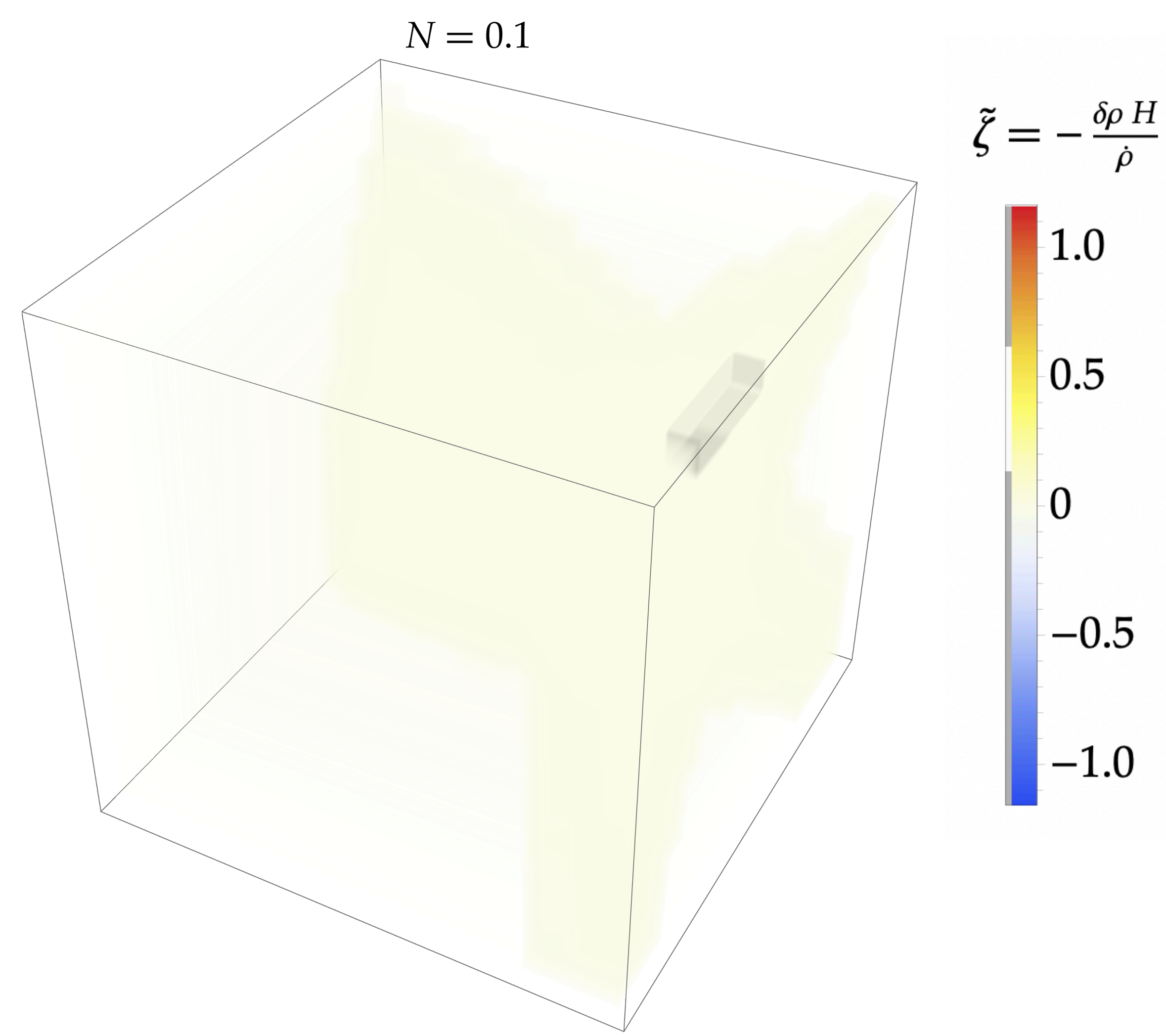

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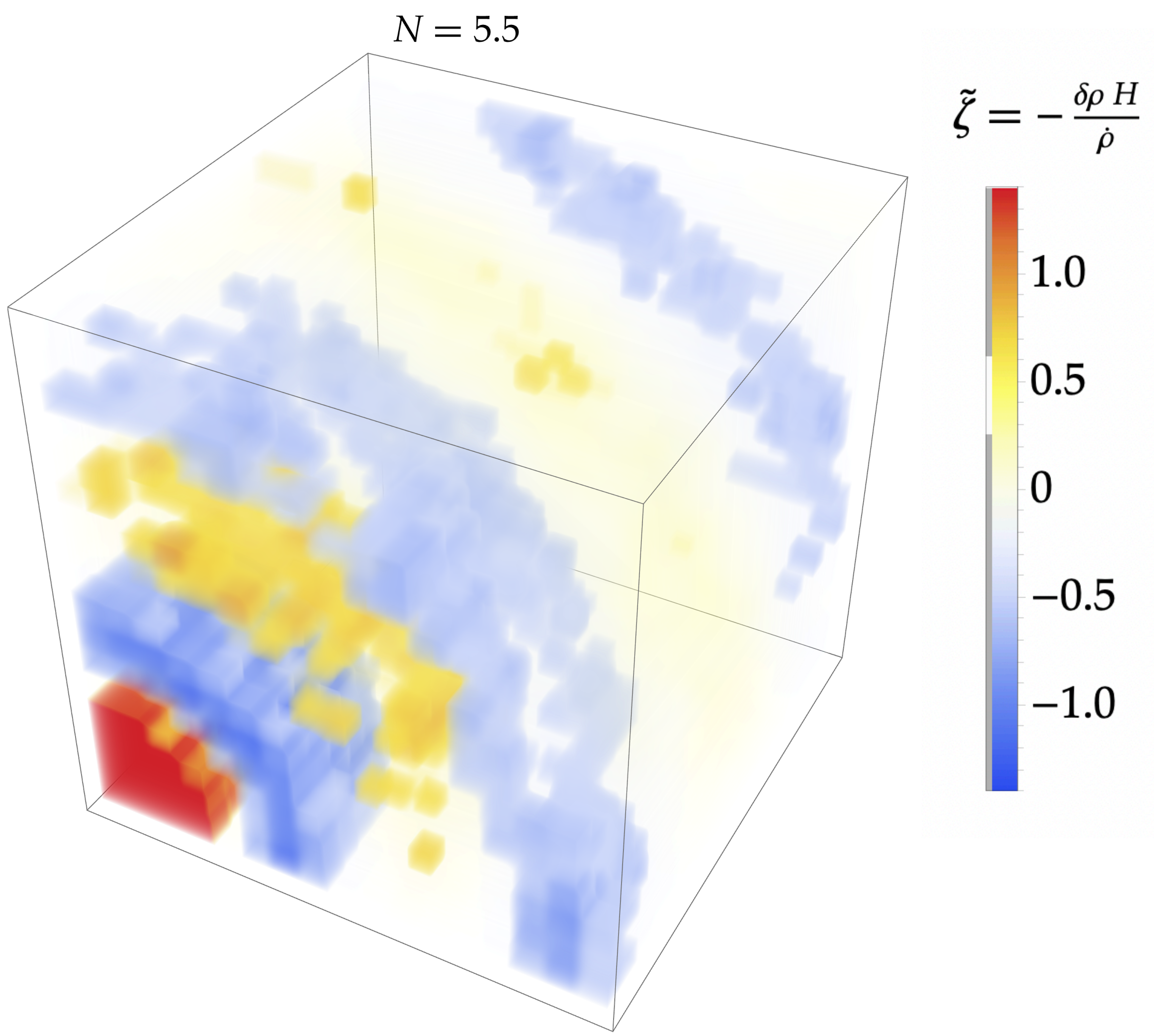


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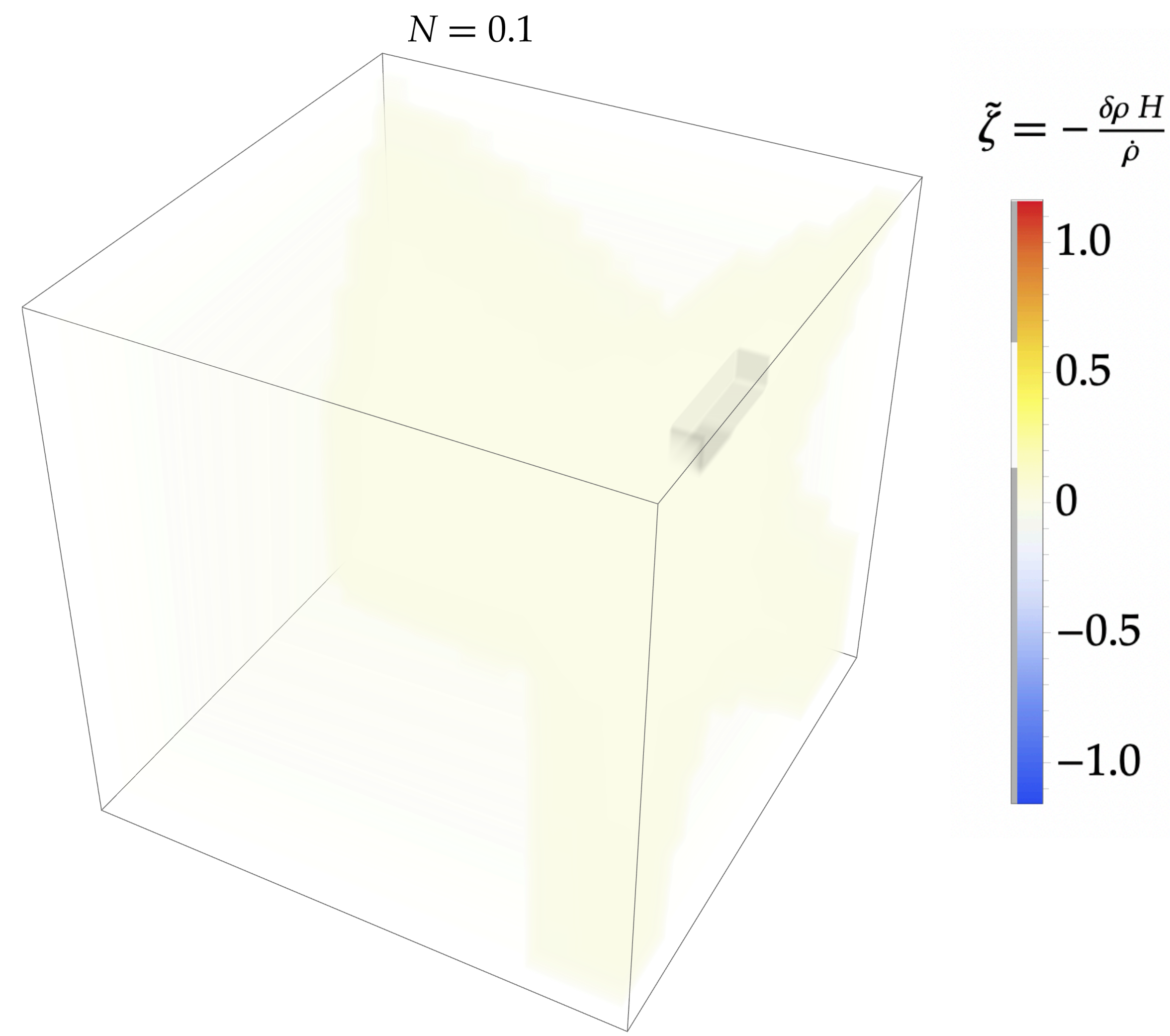

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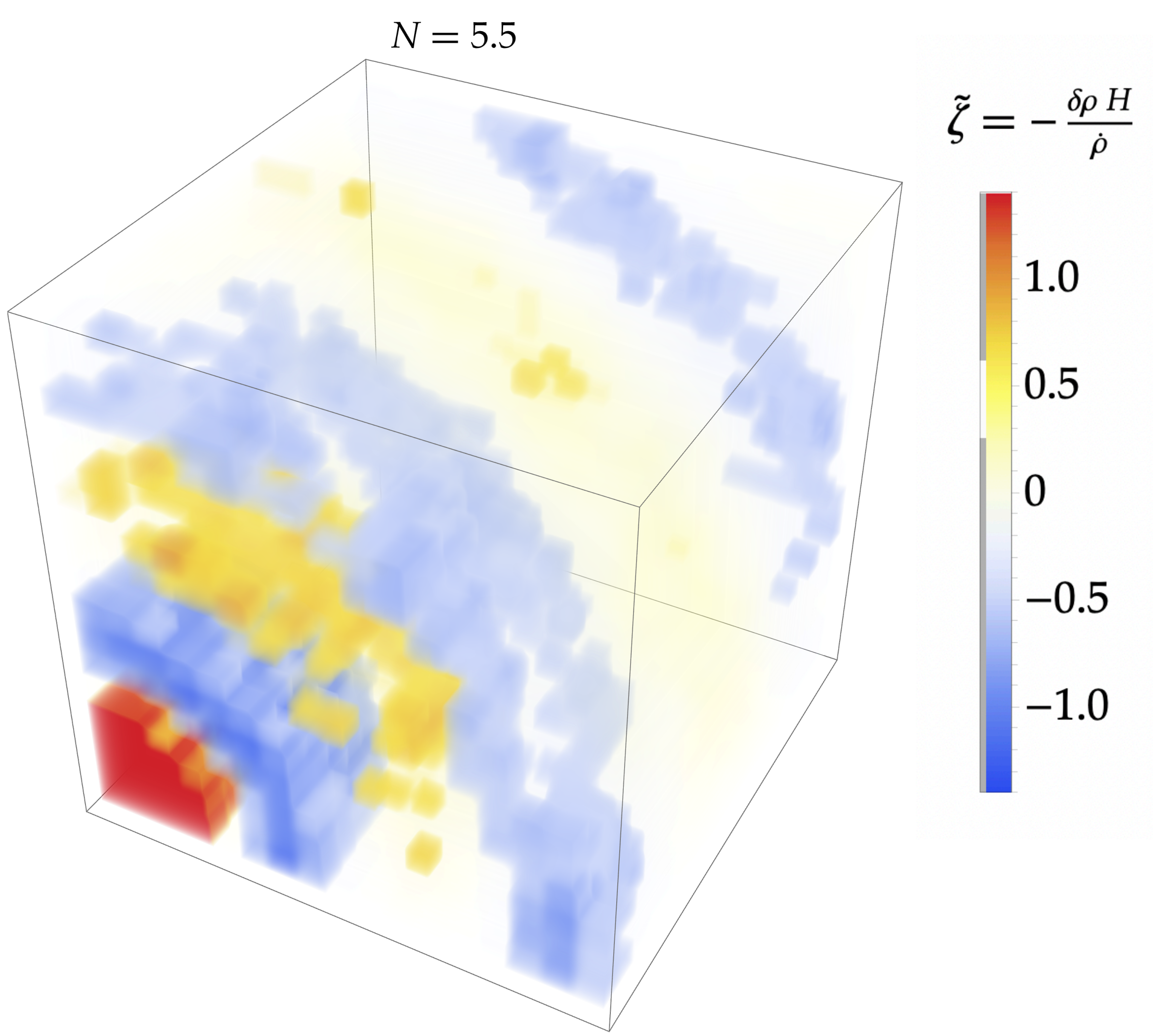


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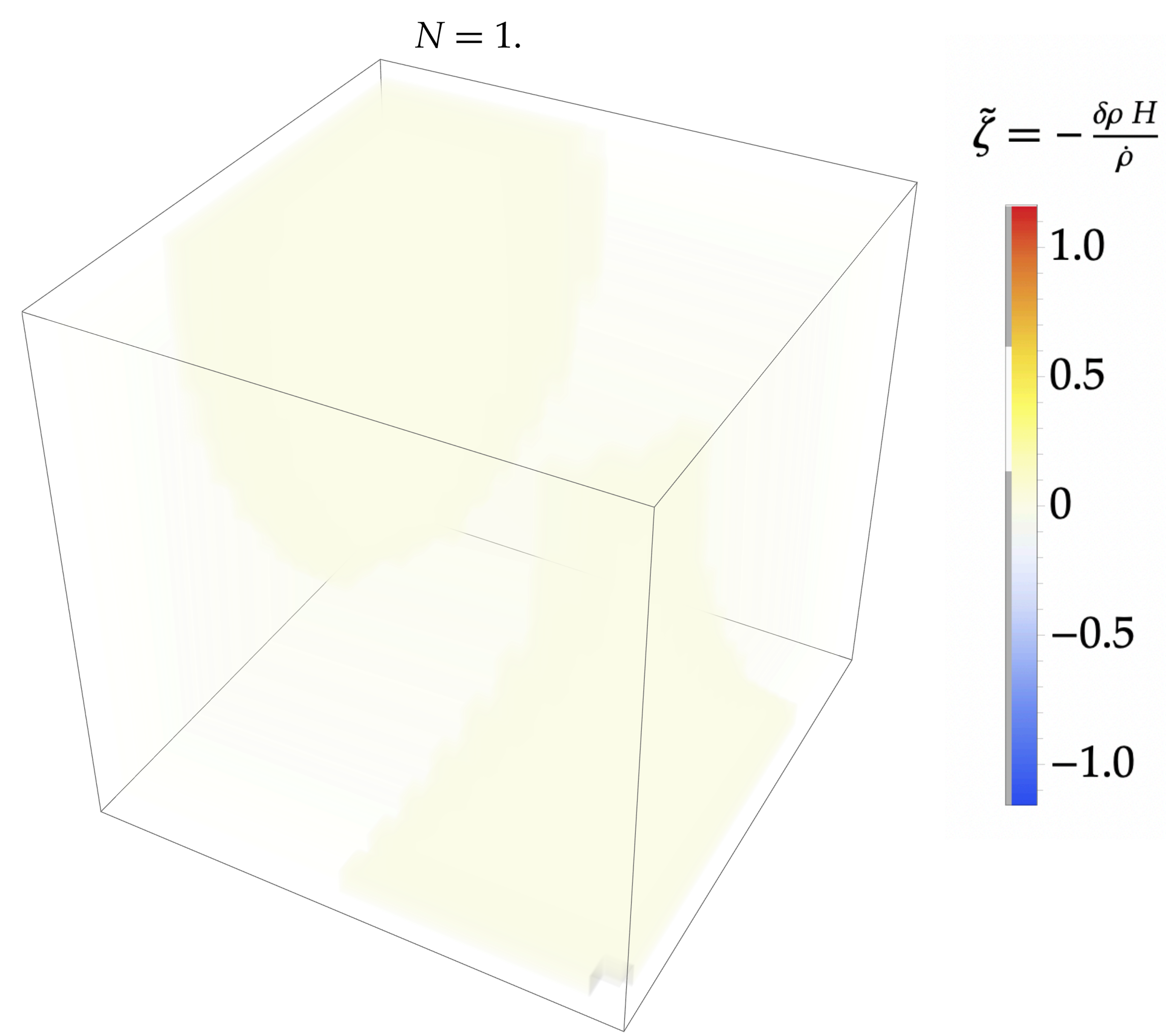

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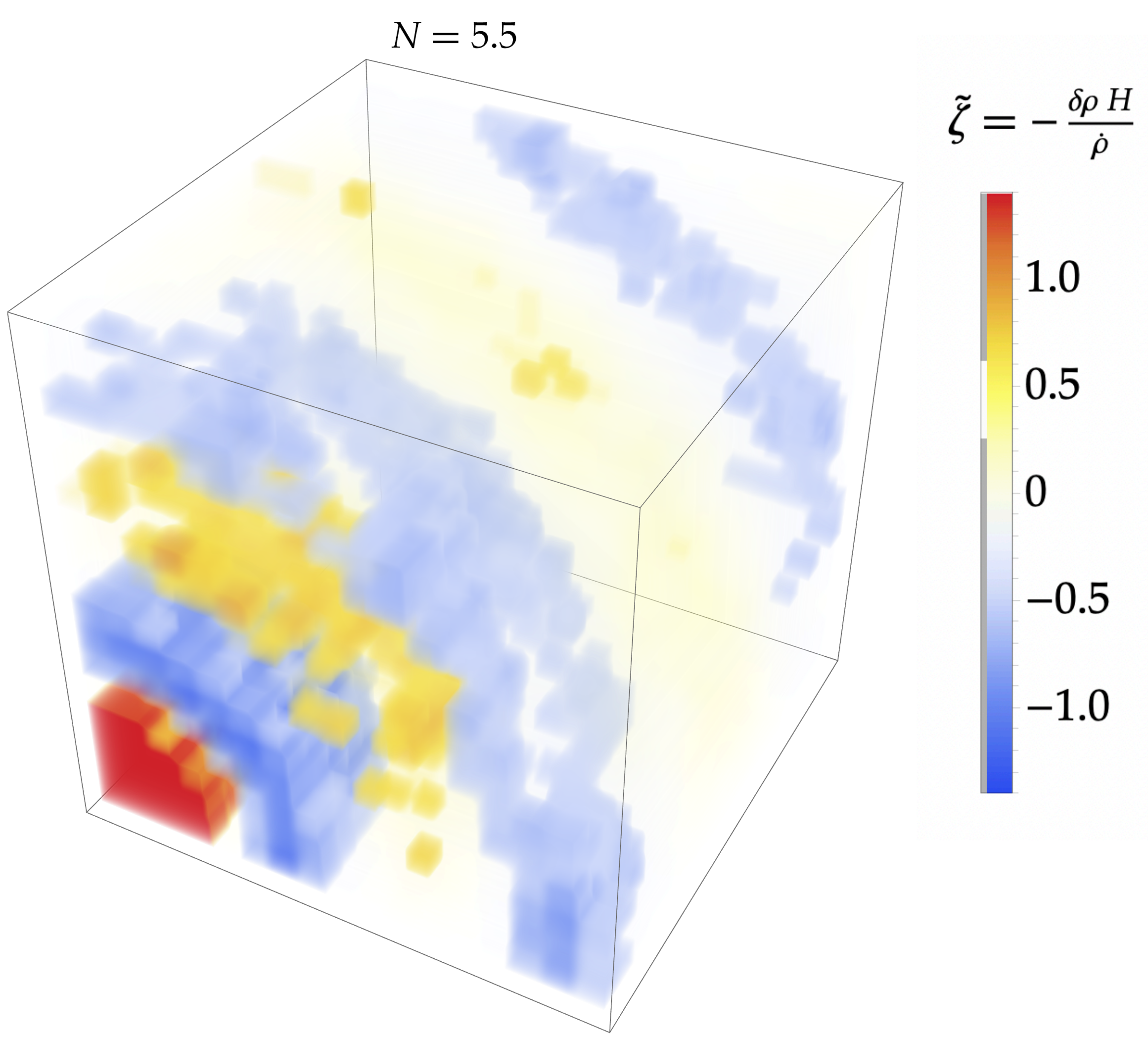


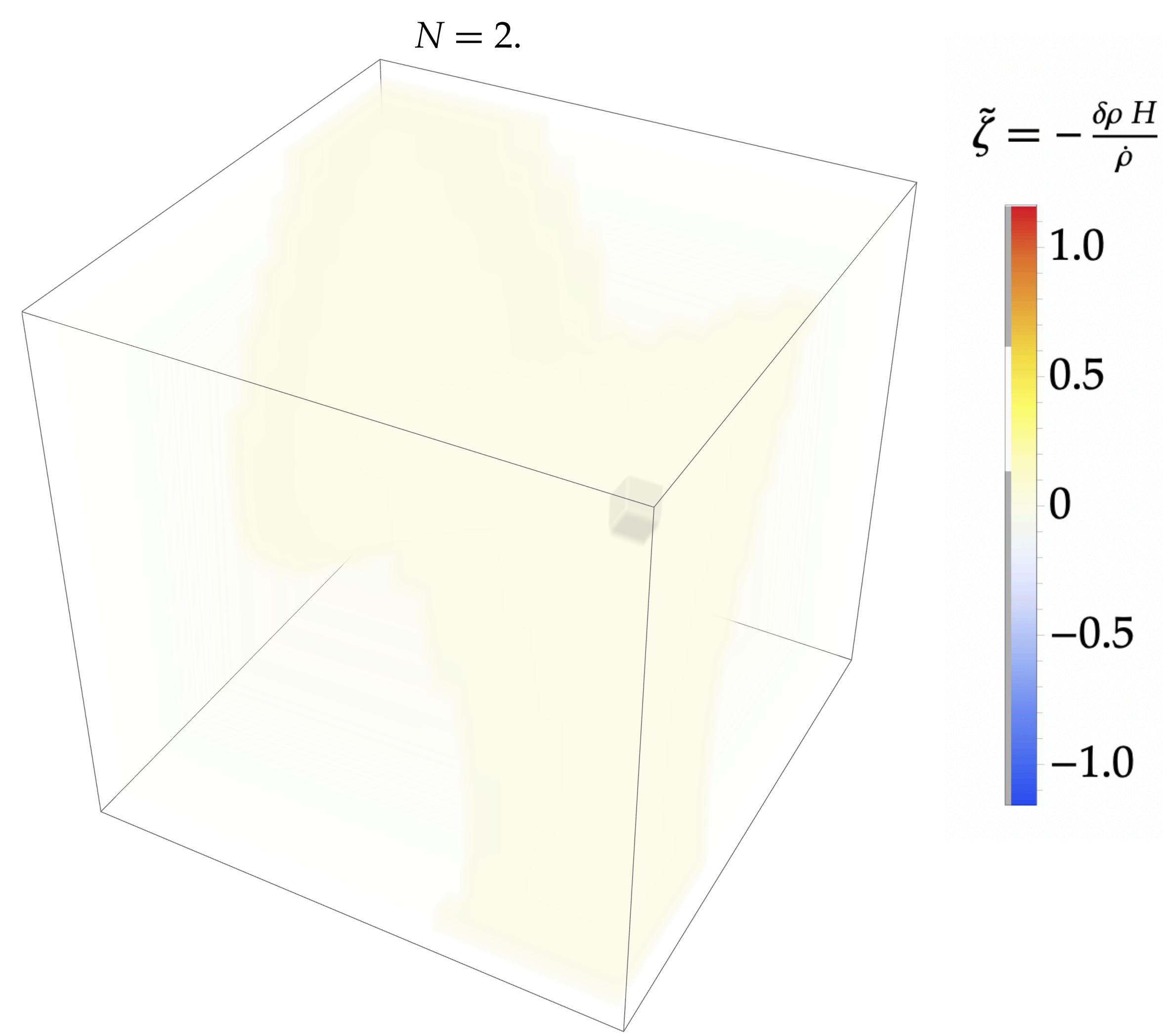

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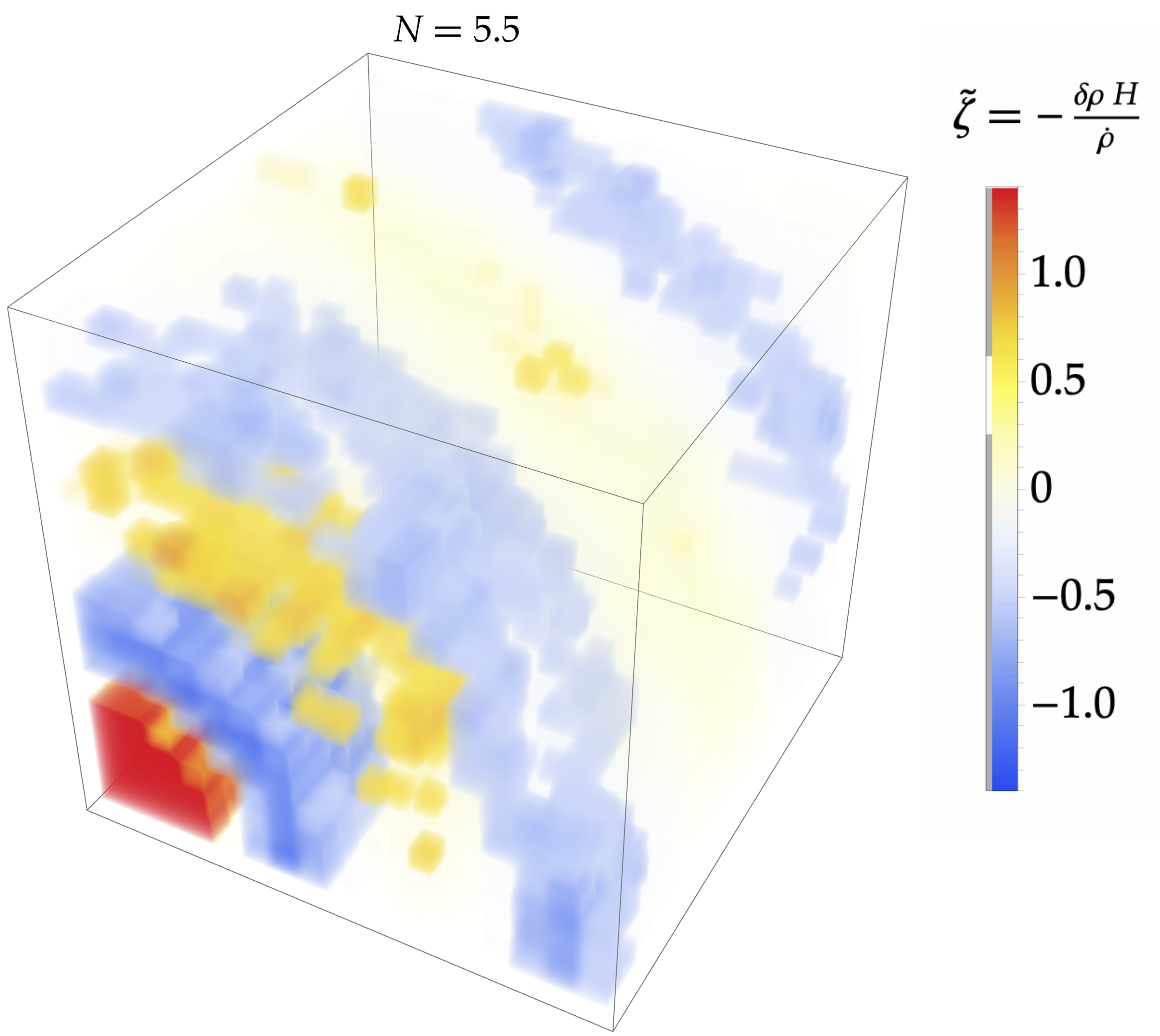
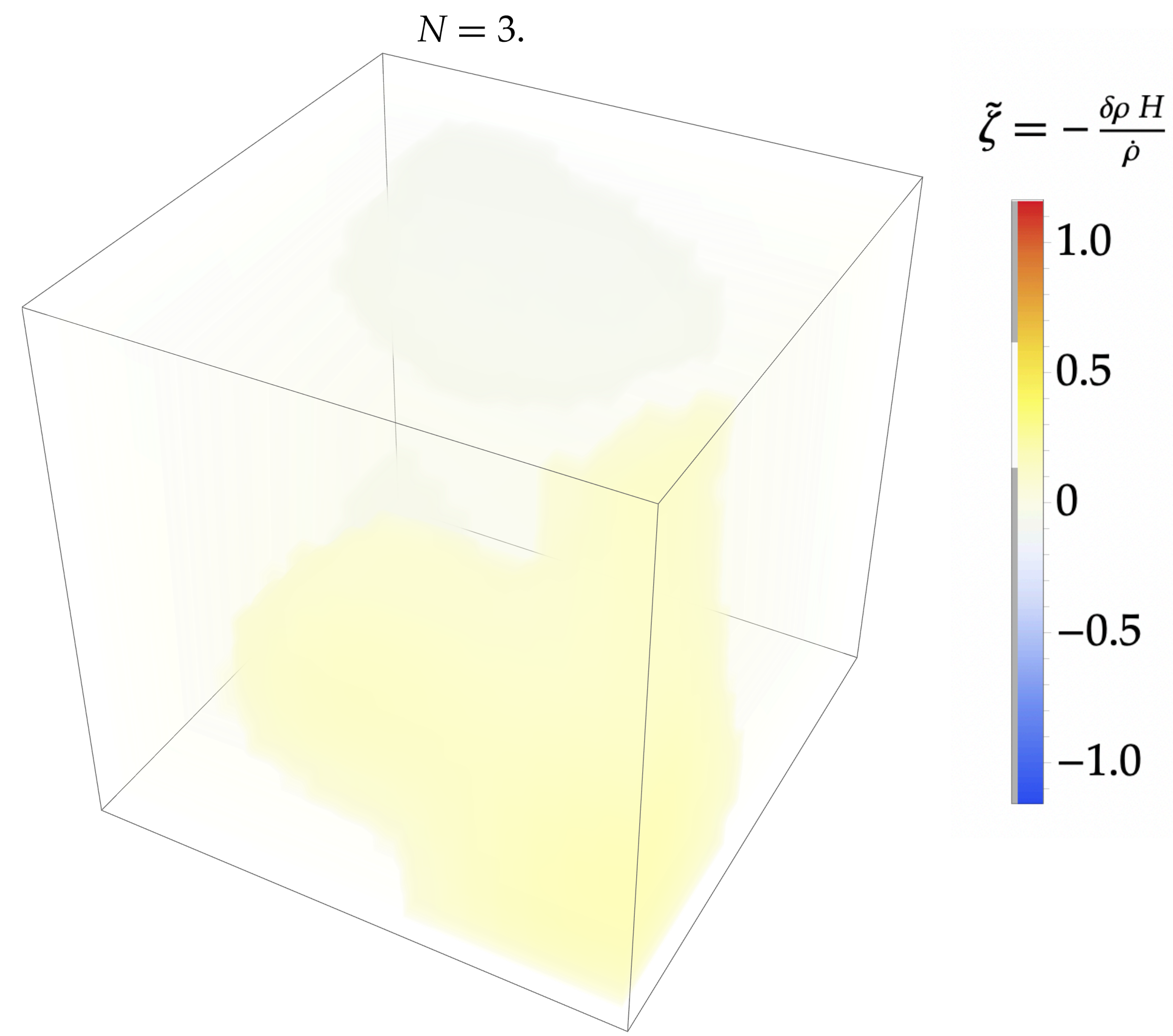



***Inflection***

$$\phi = 3.60547, \pi = -2.37409 \times 10^{-7}$$



# Result2: Previous simulation w/bias

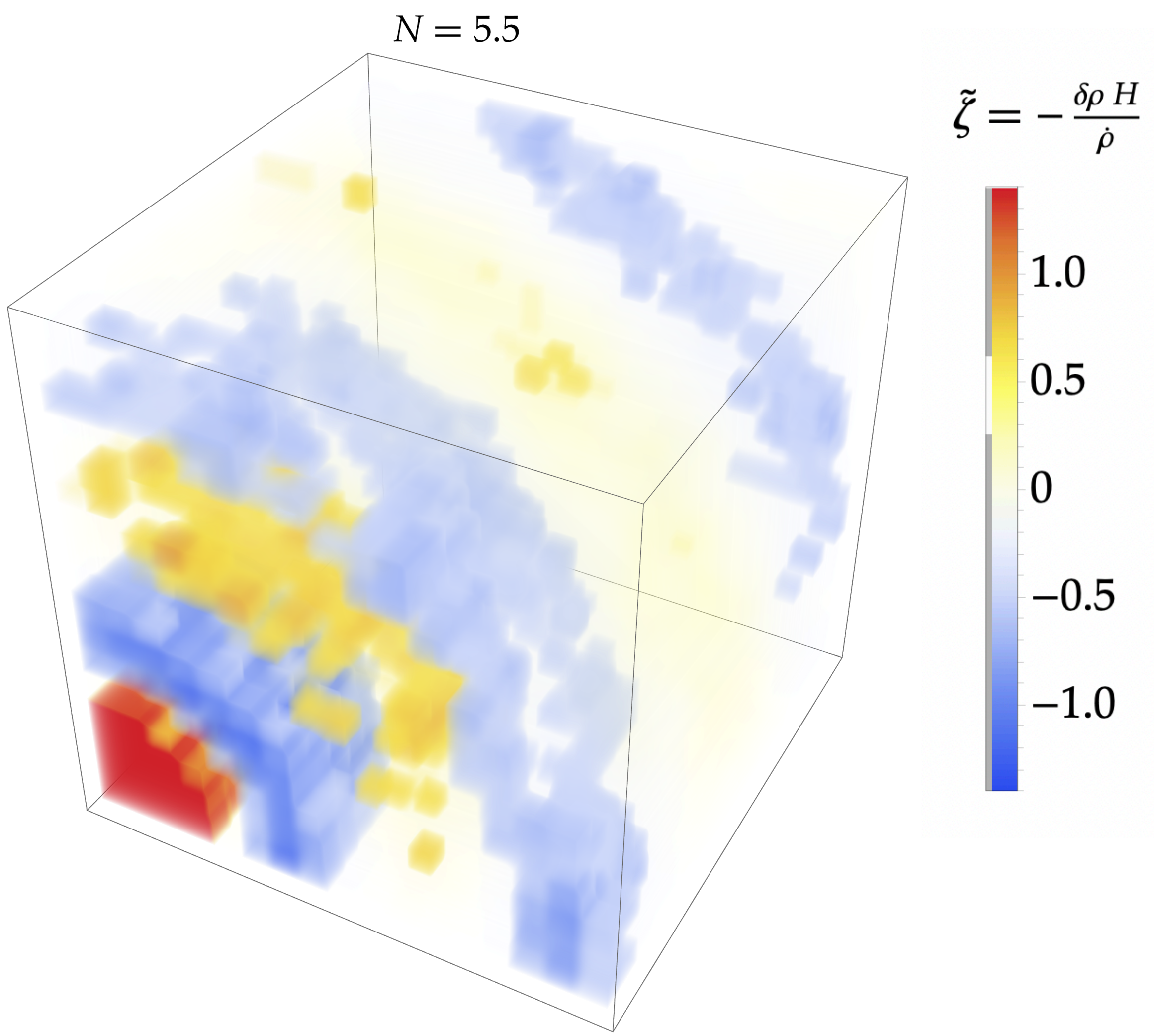
Jackson, Joseph H. P. *et al*,2022

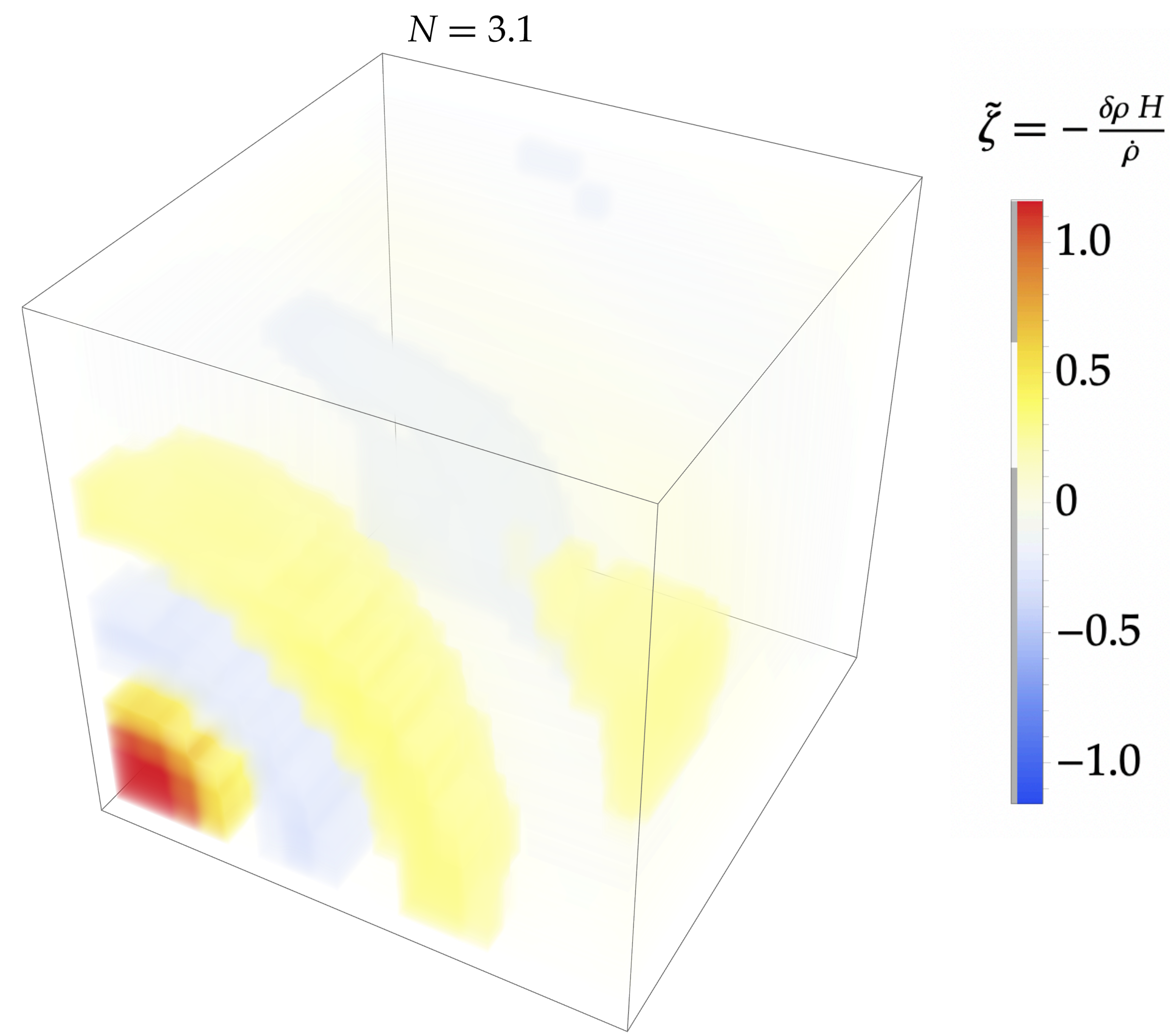
***Chaotic inflation***

$$m = 10^{-2}\,,\ \phi = 15.0,\ \pi = -\,10^{-11}$$

***Inflection***

$$\phi = 3.60547,\ \pi = -\,2.37409 \times 10^{-7}$$

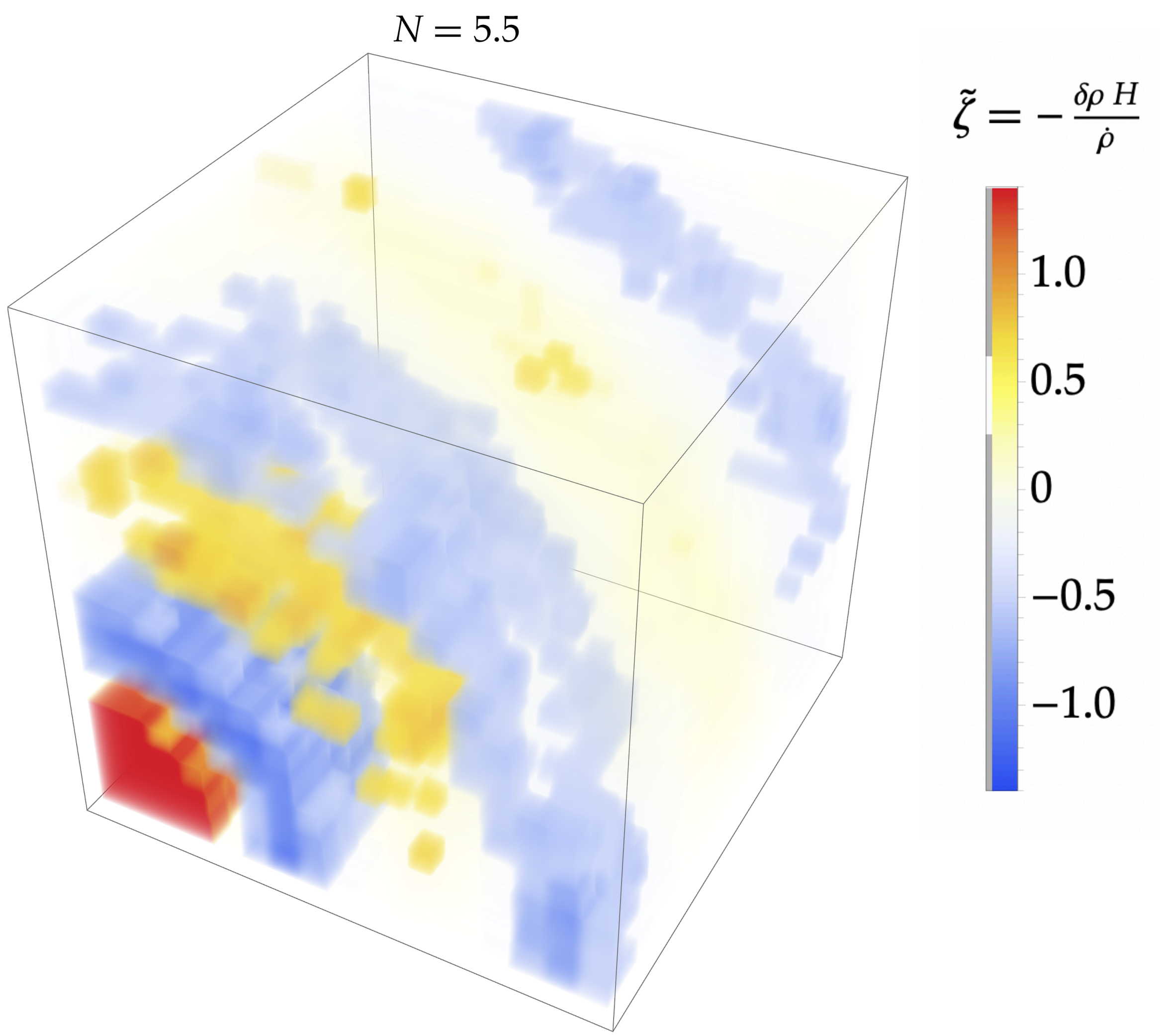


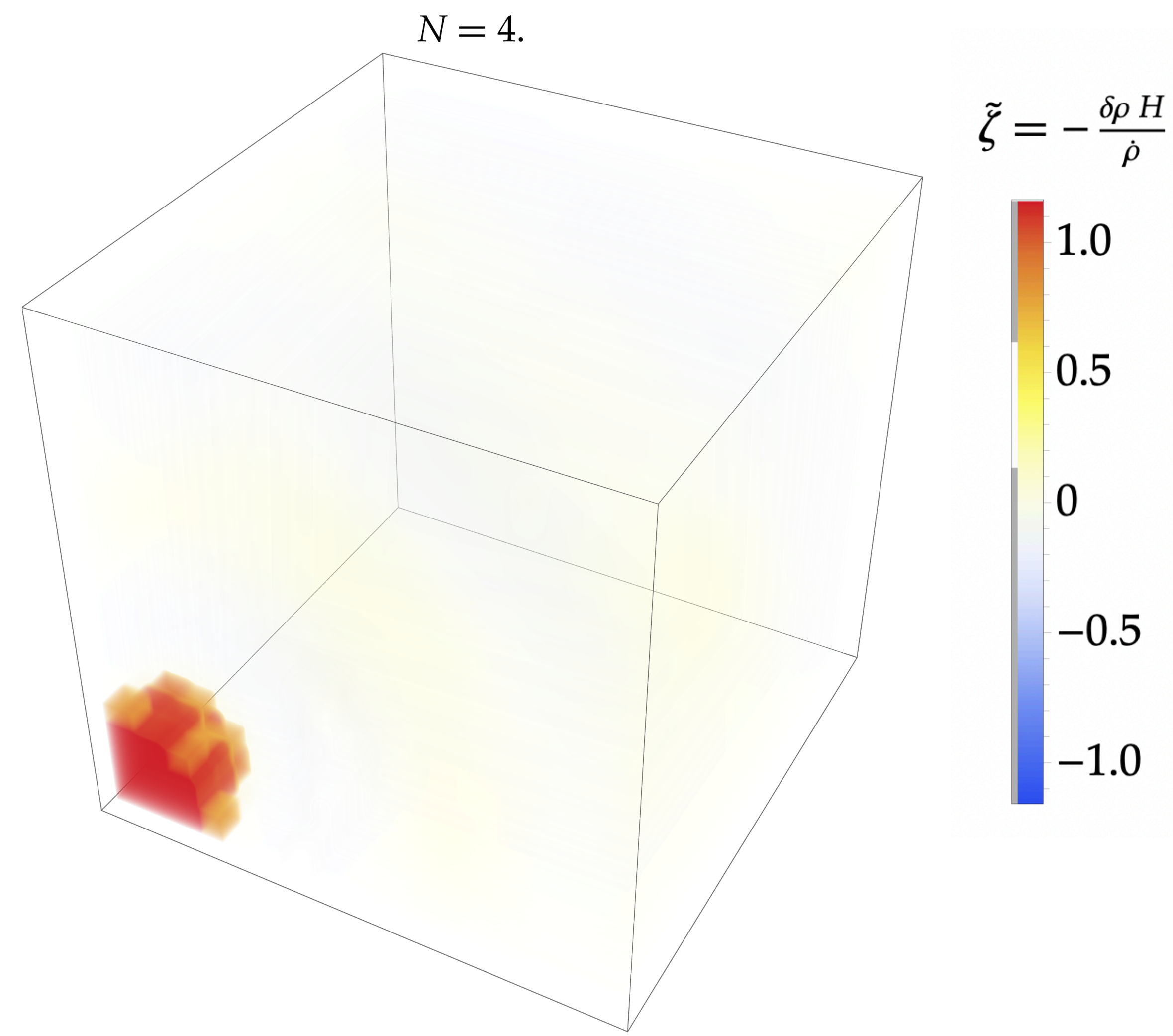



# Result2: Previous simulation w/bias

Jackson, Joseph H. P. *et al*,2022

| ***Chaotic inflation*** | ***Inflection*** |
| --- | --- |
| $m = 10^{-2}\,, \phi = 15.0, \pi = -\,10^{-11}$ | $\phi = 3.60547, \pi = -\,2.37409 \times 10^{-7}$ |



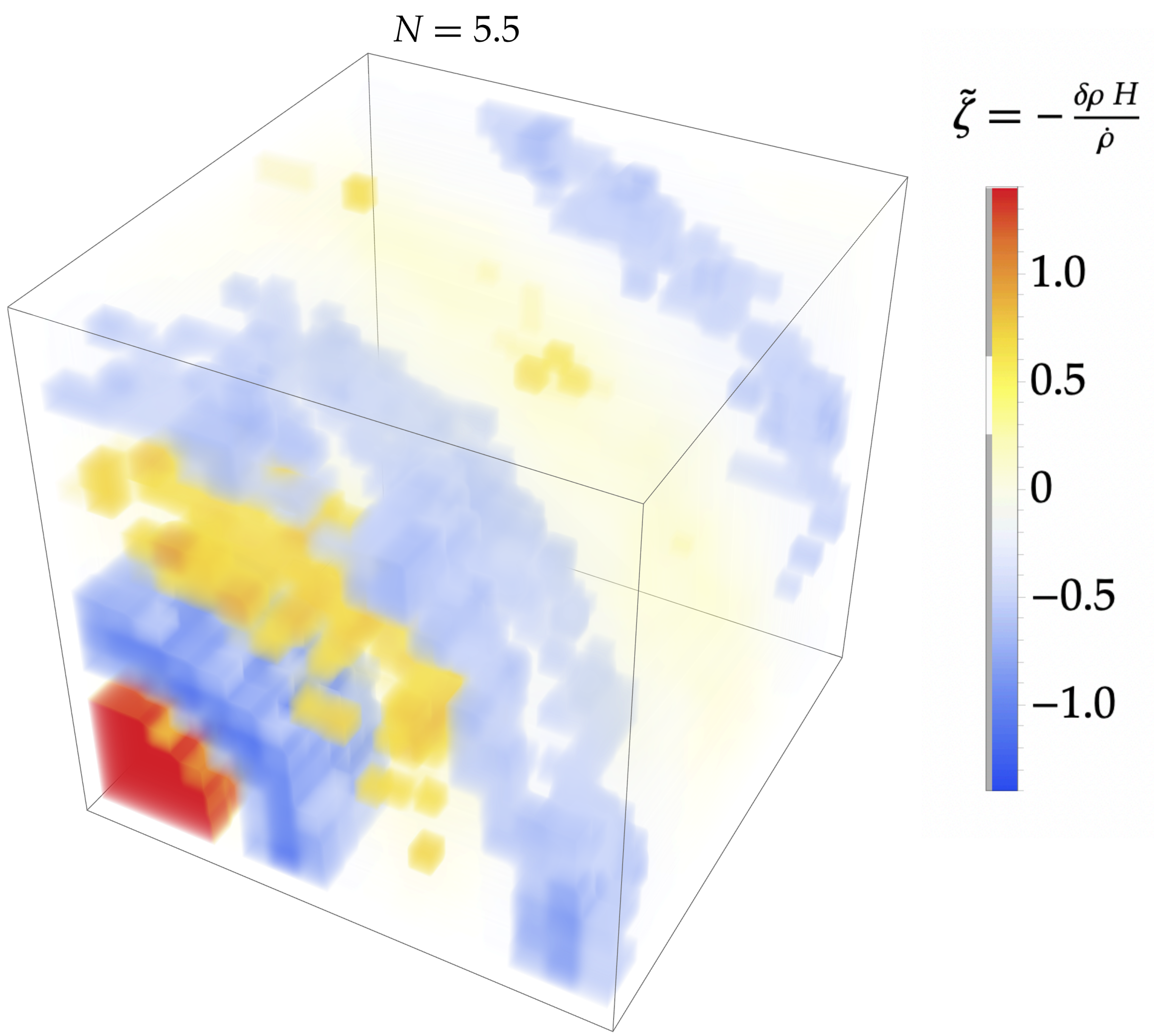



# Result2: Previous simulation w/bias

Jackson, Joseph H. P. *et al*,2022

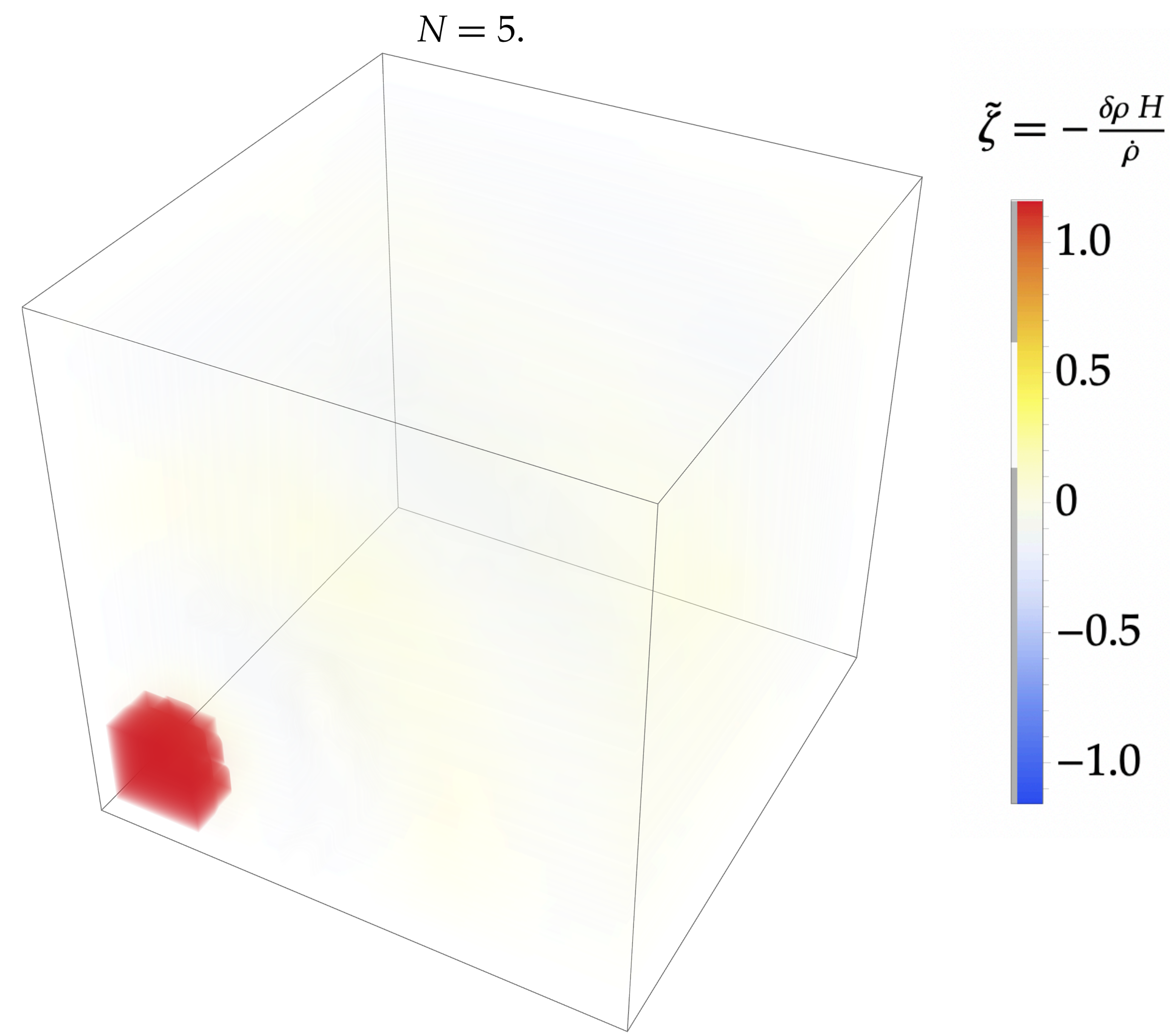
***Chaotic inflation***

$m = 10^{-2}\,, \phi = 15.0, \pi = -\,10^{-11}$




***Inflection***

$\phi = 3.60547, \pi = -\,2.37409 \times 10^{-7}$

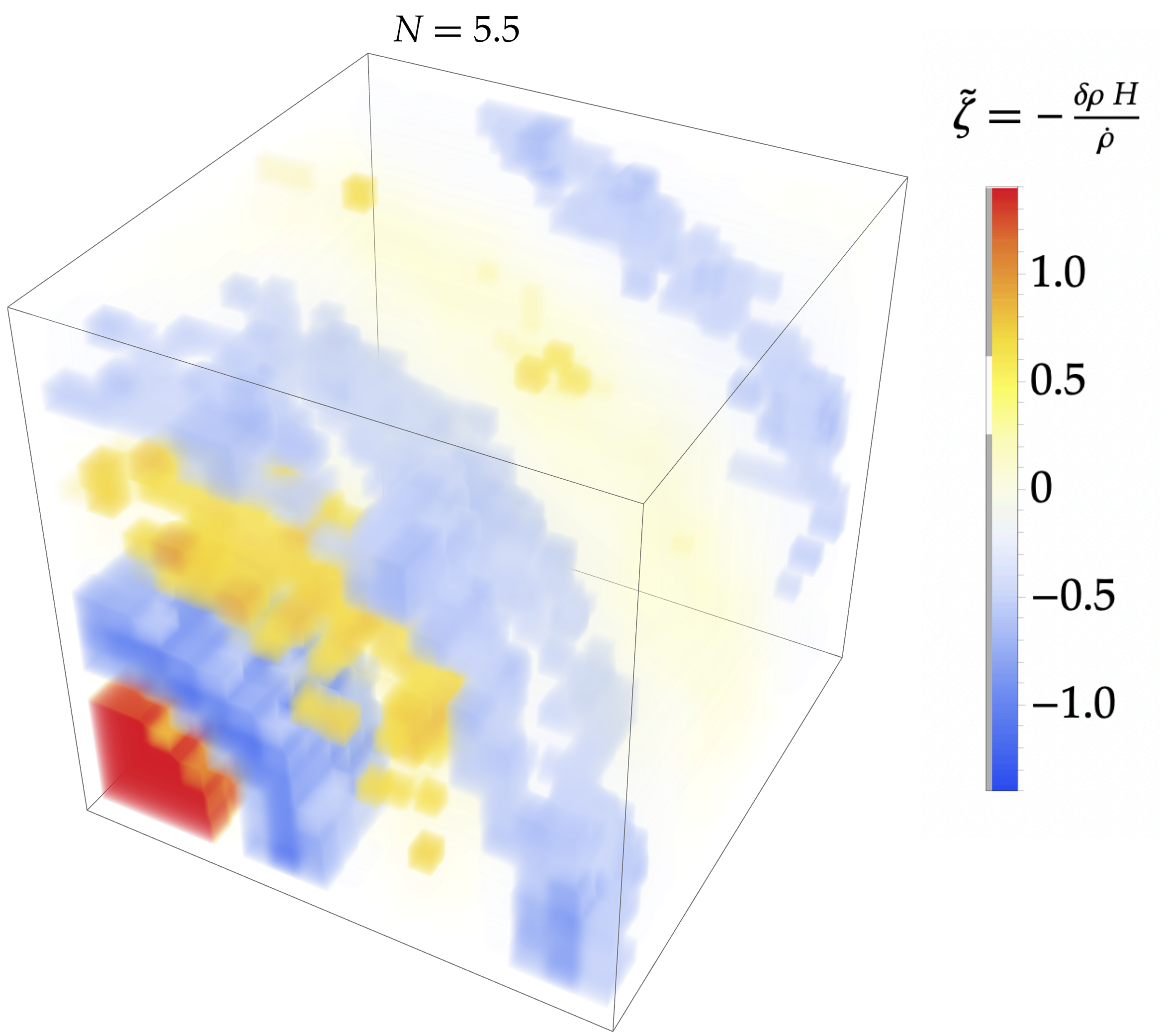


# Result2: Previous simulation w/bias

Jackson, Joseph H. P. *et al*,2022

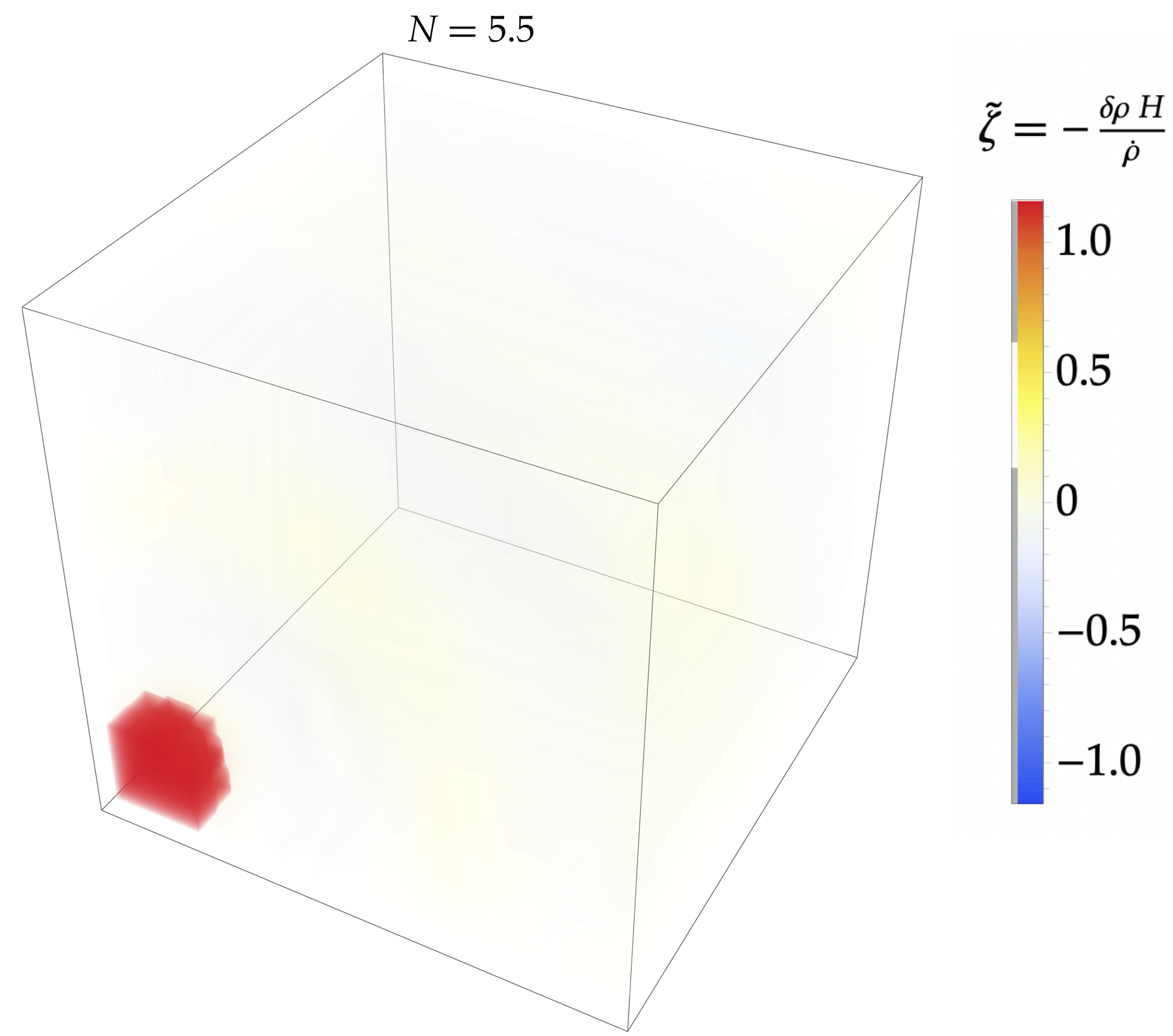
**Chaotic inflation**

$m = 10^{-2}$ , $\phi = 15.0$, $\pi = -10^{-11}$




**Inflection**

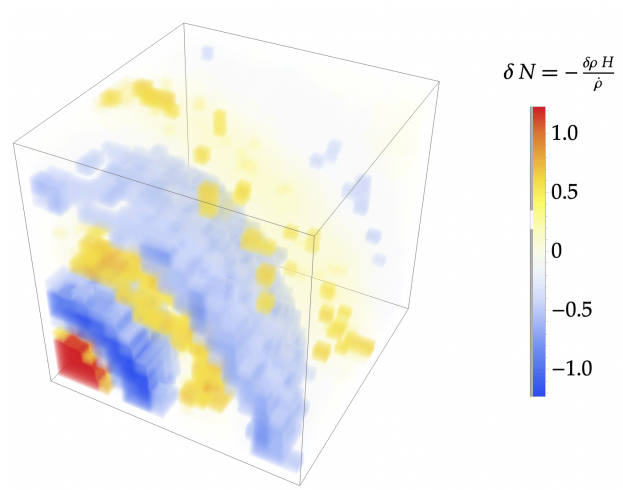
$\phi = 3.60547$, $\pi = -2.37409 \times 10^{-7}$

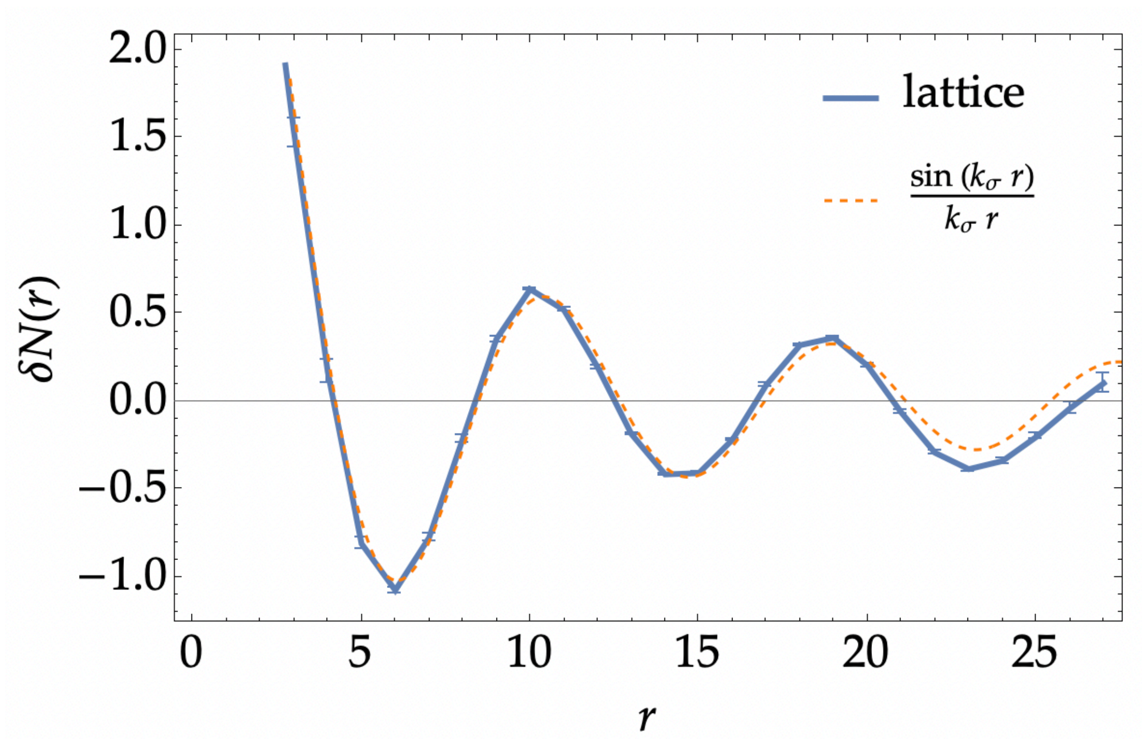


# Result2: Previous simulation w/bias

### *Chaotic inflation*

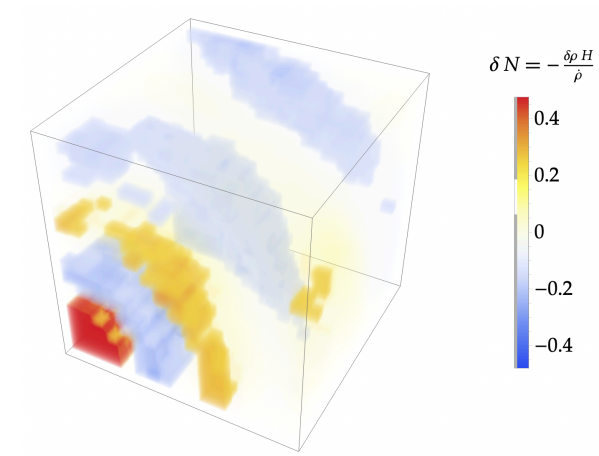
At the end of Inflation

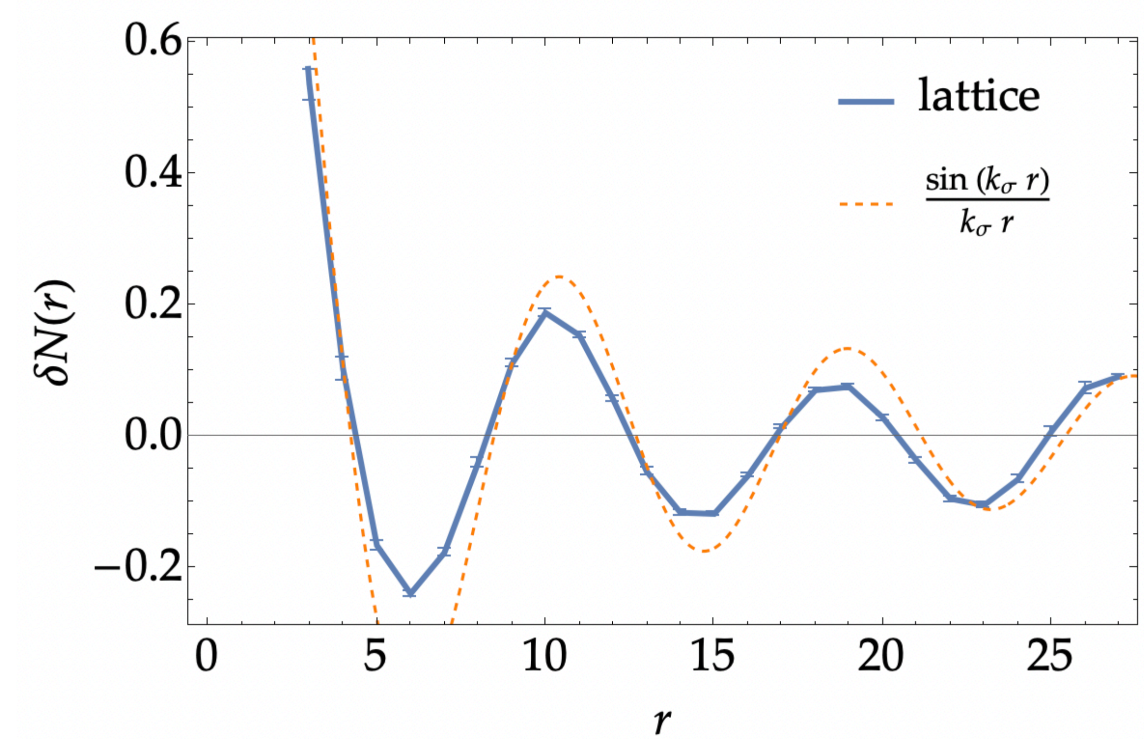



### *Inflection*

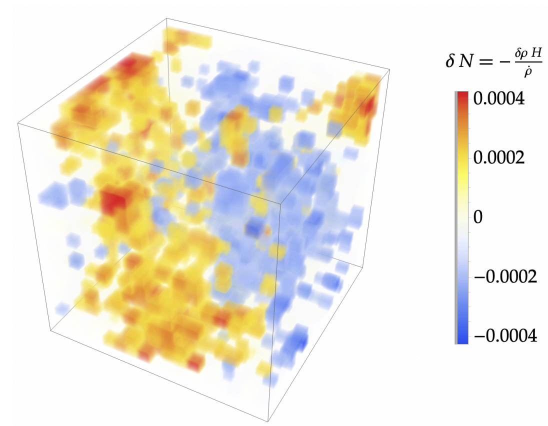
At the end of Inflation



# Summary and Future work

Profile of $\delta N$ at the end of Inflation with bias




Because collecting the statistics, repeating the simulation many times

↓

***Our goal*** **:** How accurate assumptions are in PBH formation

**https://github.com/STOchasticLAtticeSimulation**

# Reference

[1] A. A. Starobinsky, STOCHASTIC DE SITTER (INFLATIONARY) STAGE IN THE EARLY UNIVERSE, *Lect.Notes Phys.* 246 (1986) 107

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[4]Naoya Kitajima, Yuichiro Tada, Shuichiro Yokoyama, and Chul-Moon Yoo, Primordial black holes in peak theory with a non-Gaussian tail, JCAP10(2021)053

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[6]Vennin, V., Starobinsky, A.A. Correlation functions in stochastic inflation. *Eur. Phys. J. C* **75**, 413 (2015).

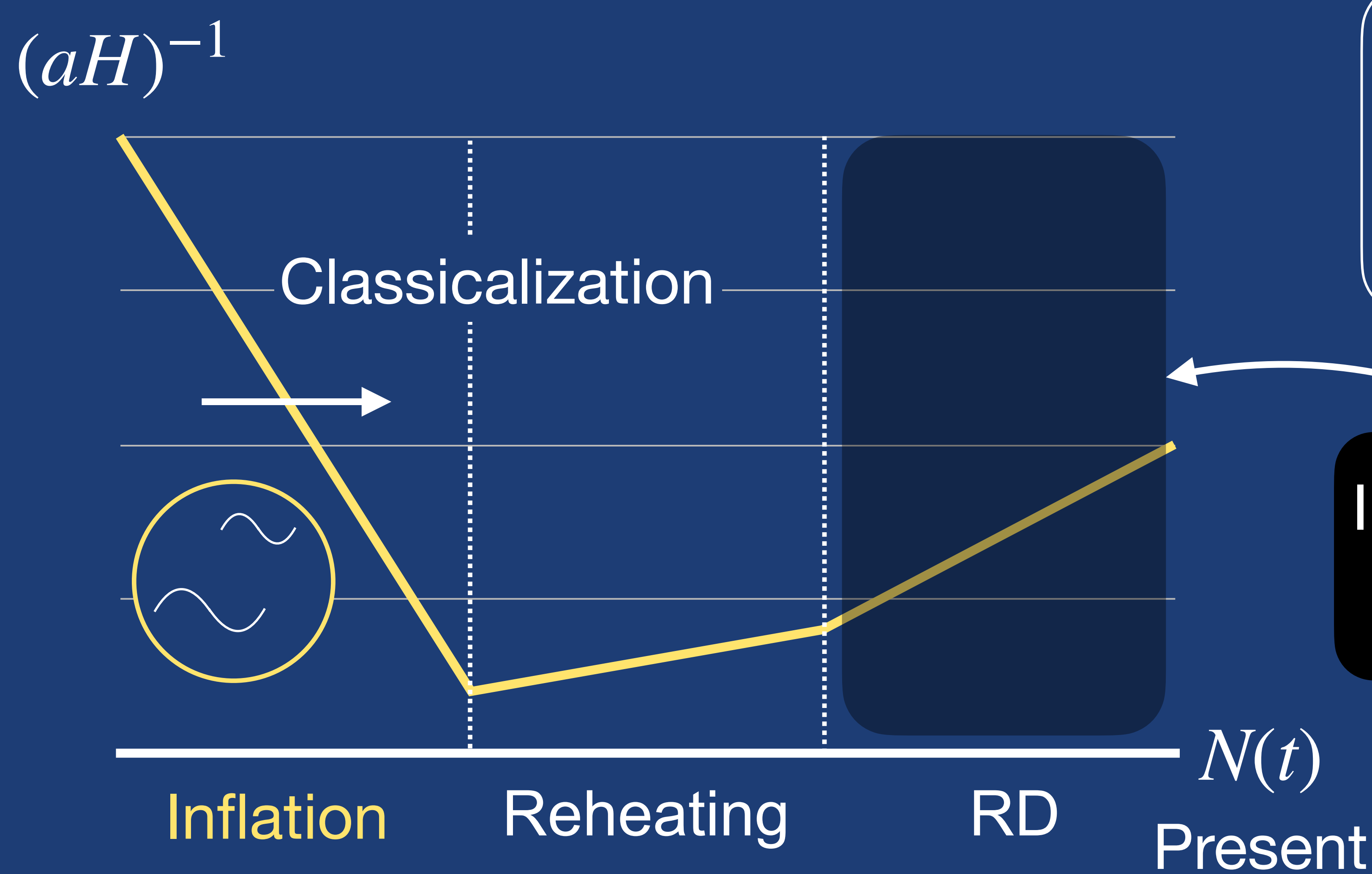
[7]M. Biagetti, G. Franciolini, A. Kehagias and A. Riotto, Primordial black holes from inflation and quantum diffusion,JCAP07(2018)032

# Appendix

# Introduction ~Inflation~

***What is cosmic Inflation?***

- solving problems of Big Bang theory
- Exponentially spatial expanding duration in early universe
- Quantum fluctuation of scalar field (Inflaton)
  → Curvature perturbation $\zeta$




After Inflation

Curvature perturbation

→ Density fluctuation

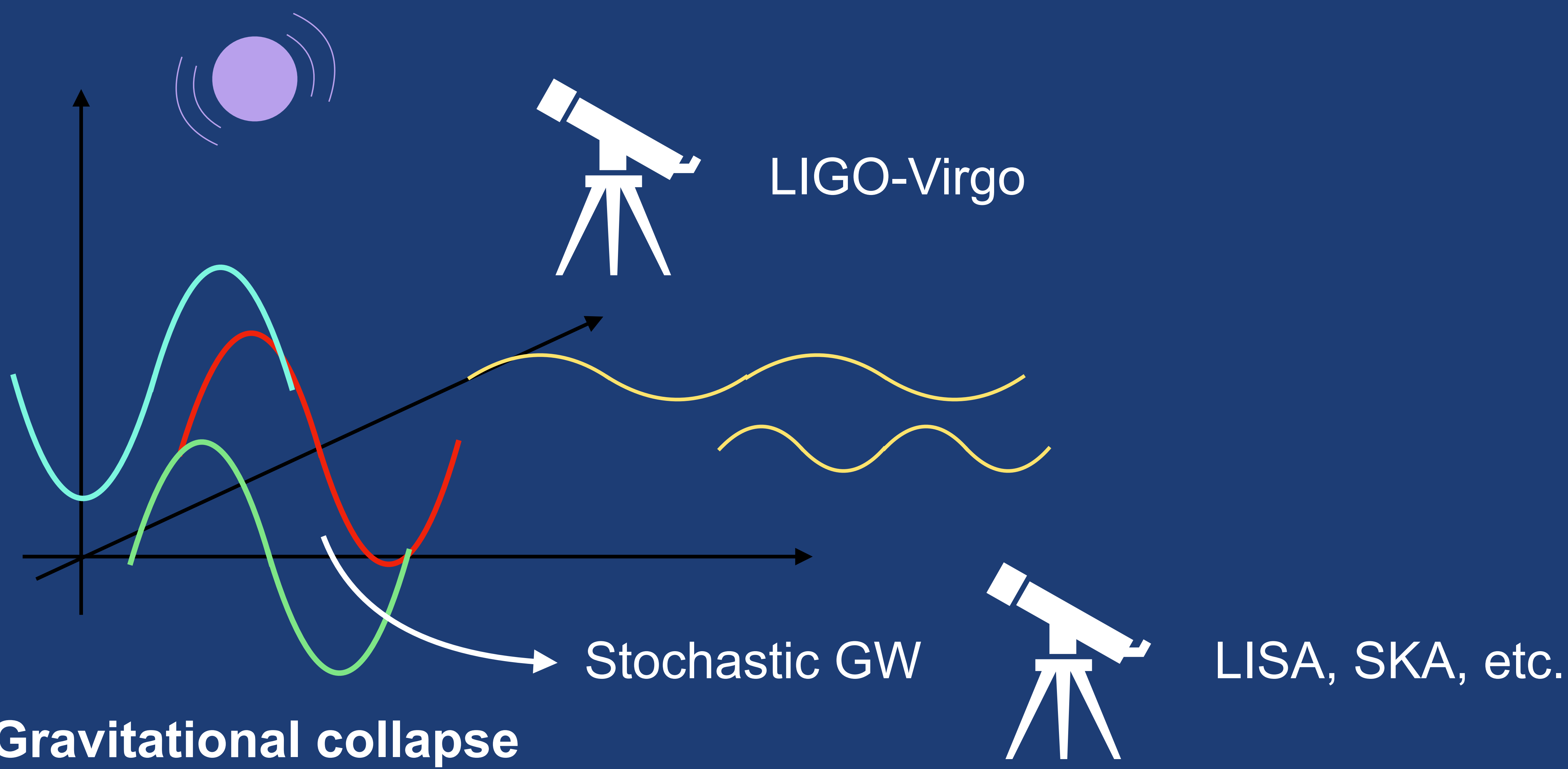
In RD era,

**Production of PBH??**

# Introduction ~Primordial Black Hole(PBH)~

**Hypothesized object produced  by large perturbation in RD**

PBH: Dark matter??



# Introduction ~Problems of PBH~

Unclear PBH formation process from initial perturbation

⟶ Conventional works: Press-Schechter theory, Peak theory, etc.

***Many assumptions***

- Press-Schechter theory
  1. Probability density function(PDF) of $\zeta$
  2. Simple threshold value
  3. Simple mass
- Peak theory
  1. **Gaussian** probability density function(PDF) of $\zeta$
  2. Already known $\mathscr{P}_\zeta$

# Motivation of our work

- **PDF of $\zeta$**
- Simply threshold value
- **Formation condition**

} How accurate are these assumptions?

**Estimation of more accurate potential in PBH formation**

***Flow***

Decision of only potential

↓

**Lattice simulation × Stochastic inflation**

↓

Data of curvature perturbations

↓

Discussion of PBH formation

# Stochastic inflation

Curvature perturbation in Inflation ⟶ Large perturbation makes PBH

- Perturbation theory

$$\begin{cases} g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \; ( \bar{g}_{\mu\nu} \gg \delta g_{\mu\nu} ) \\ \phi = \bar{\phi} + \delta\phi \end{cases}$$

Not good for $\bar{g}_{\mu\nu} \simeq \delta g_{\mu\nu}$

※ Komatsu-san's group
(Lattice simulation of Inflation, etc.)

$$\begin{cases} g_{\mu\nu}(t, \mathbf{x}) \simeq \bar{g}_{\mu\nu}(t) \\ \phi(t, \mathbf{x}) = \bar{\phi}(t, \mathbf{x}) + \delta\phi(t, \mathbf{x}) \end{cases}$$

- **Stochastic formalism**

$$\begin{cases} g_{\mu\nu} = g_{\mu\nu(IR)} + g_{\mu\nu(UV)} \\ \phi = \phi_{IR} + \hat{\phi}_{UV} \; ( \hat{\phi}_{UV} \ll \phi_{IR} ) \end{cases}$$

Good for $\bar{g}_{\mu\nu} \simeq \delta g_{\mu\nu}$

⟶ Focus on super-horizon mode

Locally, nonlinear $g_{\mu\nu}$

# Property of swinging term

Equation of motion in inflaton field $\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^2\phi + V'(\phi) = 0$

↓ Stochastic formula, Hamilton formula

$$\begin{cases} \dot{\phi}_{IR} = \pi_{IR} + \xi_\phi \\ \dot{\pi}_{IR} = -3H\pi_{IR} - V'(\phi_{IR}) + \xi_\pi \end{cases}$$

In Bunchi-Davies vacuum

$$\langle \phi_{\mathbf{k}} \rangle = \langle \pi_{\mathbf{k}} \rangle = 0 \longrightarrow \langle \xi_\phi \rangle = \langle \xi_\pi \rangle = 0$$

$$\langle \xi_\phi(t, \mathbf{x}) \xi_\phi(t', \mathbf{x}') \rangle \simeq \underbrace{\frac{H^3}{(2\pi)^2} \frac{\sin(k_c r)}{k_c r}}_{\simeq\, \theta(1 - k_c r)} \underbrace{\delta(t - t')}_{\text{White noise}}$$

**Per $t = t'$, $k_c = aH$ mode comes in IR field** AND $\phi_{\mathbf{k}}$ **is Gaussian**

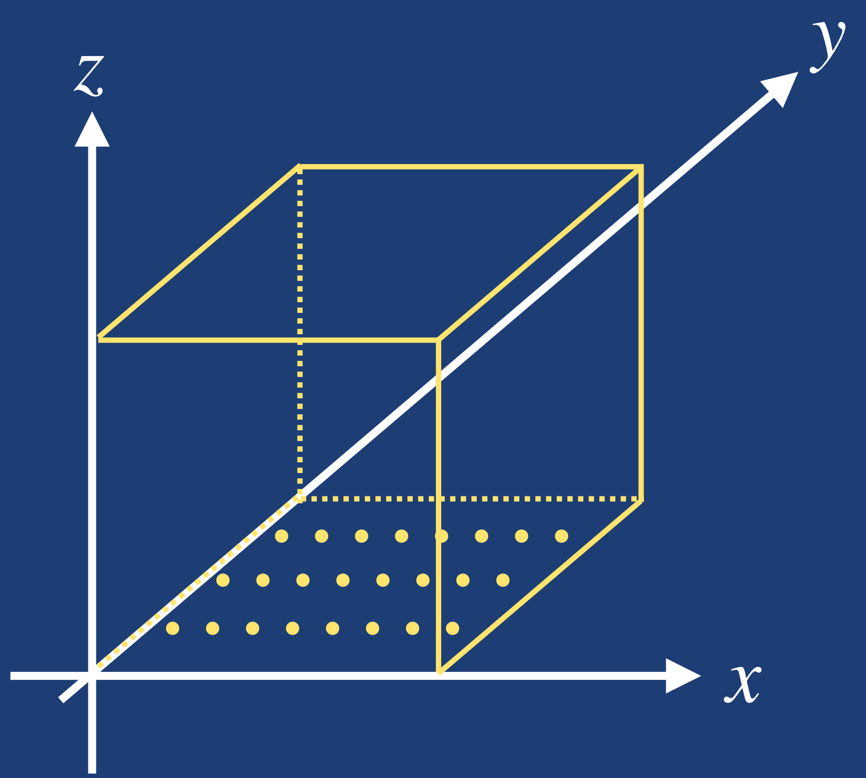
⟶ Per $k_c = aH$, $\xi_\phi$ is ***independent Gaussian noise***

# Lattice simulation and Stochastic formalism

***coarse-grained***

- Lattice simulation
- Stochastic formalism

} Good match!




Equation of inflaton

⟶ Langevin equation($dN = Hdt$)

$$
\begin{cases}
d\phi_{\mathbf{x}} = \dfrac{\pi_{\mathbf{x}}}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN + \underbrace{\dfrac{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})}{2\pi} dW_{\mathbf{x}}}_{\xi_\phi} \\
d\pi_{\mathbf{x}} = -3\pi_{\mathbf{x}} dN - \dfrac{V'(\phi_{\mathbf{x}})}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN
\end{cases}
$$

**Gaussian noise with correlation**

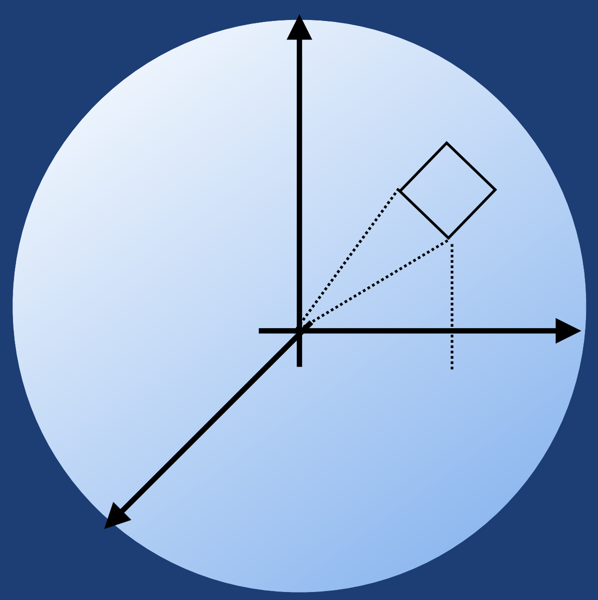
At each lattice, solving equation ⟶ Getting information of $\zeta_{\mathbf{x}}$

# Gaussian noise with correlation

Correlation function of swinging term at same time

$$\langle \xi_\phi(\mathbf{x})\xi_\phi(\mathbf{x}')\rangle \simeq \frac{H^3}{(2\pi)^2}\frac{\sin(k_c r)}{k_c r} \qquad r = |\mathbf{x}-\mathbf{y}|$$

$dW_\mathbf{x} dW_\mathbf{y}$



- Theoretical covariance matrix

$$C_{\mathbf{xy}} = \frac{\sin k_\sigma |\mathbf{x}-\mathbf{y}|}{k_\sigma |\mathbf{x}-\mathbf{y}|}$$

- Covariance matrix simulated

$$C_{\mathbf{xy}} \simeq dW_\mathbf{x} dW_\mathbf{y} = \left[\sum_i \frac{\sqrt{\Delta\Omega_i}}{2\sqrt{\pi}}[\cos(\mathbf{k}_\sigma\cdot\mathbf{x}) - \sin(\mathbf{k}_\sigma\cdot\mathbf{x})]dW_i\right]^2$$

Gaussian noise

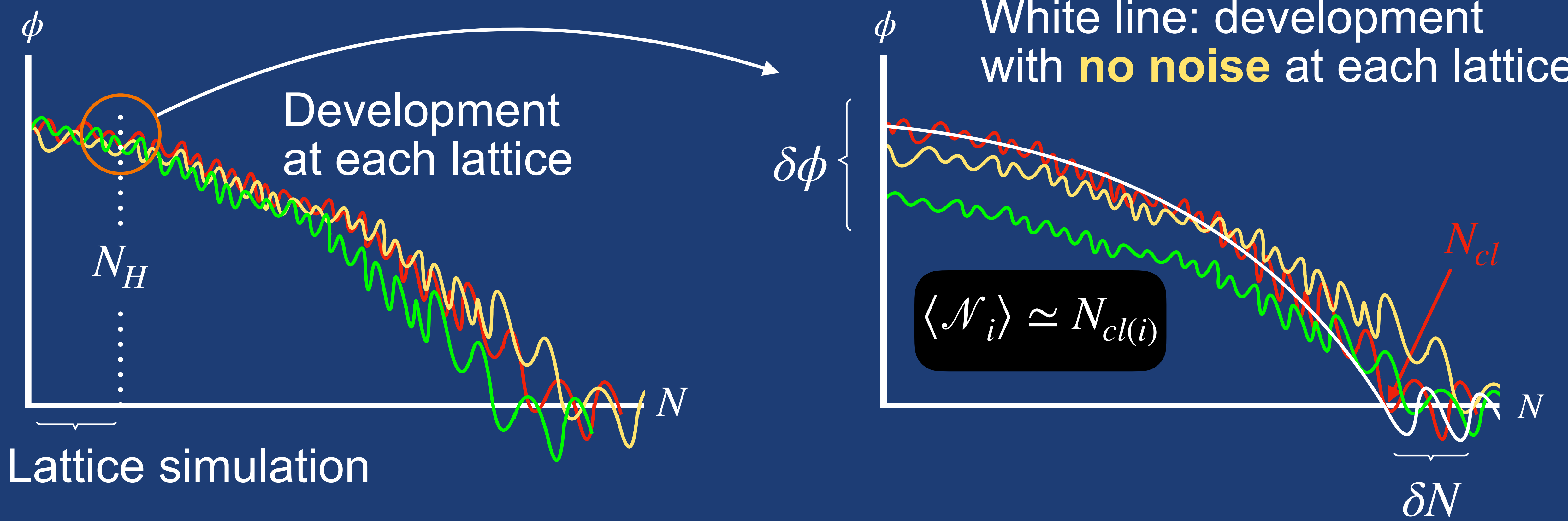
We use $dW_\mathbf{x}$ for **stochastic** perturbation

D. S. Salopek and J. R. Bond, 1991

# Relationship between $\delta N$ and $\zeta$ ($\delta N$ formula)

Vennin, V., Starobinsky, A.A., 2015

$\delta\phi \longrightarrow \delta N$




***Equation of motion with no noise***

$$\begin{cases} d\phi_{\mathbf{x}} = \dfrac{\pi_{\mathbf{x}}}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN \\[2ex] d\pi_{\mathbf{x}} = -3\pi_{\mathbf{x}} dN - \dfrac{V'(\phi_{\mathbf{x}})}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN \end{cases}$$

$$\zeta_i = \mathcal{N}_i - \langle \mathcal{N} \rangle = \delta N_i$$

***Solving until*** $\epsilon = 1$

# EoM of IR fields

EoM of scalar field $\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^2\phi + V'(\phi) = 0$

↕ Hamiltonian formula

$$\begin{cases} \dot{\pi} + 3H\pi - a^{-2}\nabla^2\phi + V'(\phi) = 0 \\ \pi = \dot{\phi} \end{cases}$$

Inflaton in k-space

$$\phi_{IR}(t, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \phi_{\boldsymbol{k}}(t)(1 - \theta(k - \epsilon a(t)H(t)))$$

$$\hat{\phi}_{UR}(t, \boldsymbol{x}) = \int \frac{d^3k}{(2\phi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\phi}_{\boldsymbol{k}}(t)\theta(k - \epsilon a(t)H(t))$$

$\theta(t)$ : Heviside step function

Focus on the picture in **IR fields**

$\phi = \phi_{IR} + \hat{\phi}_{UV}$ $(\phi_{UV} \ll \phi_{IR})$

$$\begin{cases} \pi_{IR} + \pi_{UR} = \dot{\phi}_{IR} + \dot{\phi}_{UR} \\ \dot{\pi}_{IR} + 3H\pi_{IR} - a^{-2}\nabla^2\phi_{IR} + \dot{\pi}_{UR} + 3H\pi_{UR} - a^{-2}\nabla^2\phi_{UR} + V'(\phi_{IR}) = 0 \end{cases}$$

$$\dot{\phi}_{IR} = \pi_{IR} + \epsilon a H^2 \int \frac{d^3k}{(e\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}} \delta(k - \epsilon a H)$$

$$\dot{\pi}_{IR} = -3H\pi_{IR} - V'(\phi_{IR}) + \epsilon a H^2 \int \frac{d^3k}{(2\pi)^3} \pi_{\mathbf{k}} \delta(k - \epsilon a H)$$

# Swinging term $\xi_\phi$

$$\left\langle \epsilon a H^2 \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}} \delta(k-\epsilon a H) \epsilon a' H^2 \int \frac{d^3k'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{x}'} \phi_{\mathbf{k}'} \delta(k'-\epsilon a' H) \right\rangle$$

$$k_c(t) = \epsilon a(t) H$$

$$\langle \phi_{\mathbf{k}}(t)\phi_{\mathbf{k}'}(t')\rangle = \frac{2\pi^2}{k^3}\mathscr{P}_\phi(t,k)(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')$$

$$= k_c(t)k_c(t')H^2 \int \frac{d^3k d^3k'}{(2\pi)^6} e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')} \frac{2\pi^2}{k^3}\mathscr{P}_\phi(t,k)(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\delta(k-k_c(t))\delta(k'-k_c(t'))$$

$$= k_c(t)k_c(t')H^2 \int \frac{d^3k d^3k'}{(2\pi)^6} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}'} \frac{2\pi^2}{k^3}\mathscr{P}_\phi(t,k)(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\delta(k-k_c(t))\delta(k_c(t)-k_c(t'))$$

$$\delta(k_c(t)-k_c(t')) = \delta(t-t')/k_c H$$

$$k_c(t) = k_c(t')$$

$$= k_c^2(t)H^2 \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\delta(k-k_c(t)) \frac{\delta(t-'t)}{k_c(t)H} \frac{2\pi^2\mathscr{P}_\phi(t,k)}{k^3} \quad \int e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}'}\delta(\mathbf{k}+\mathbf{k}')d^3k'$$

$$= k_c(t)H\delta(t-'t)\frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\delta(k-k_c(t))\frac{2\pi^2\mathscr{P}_\phi(t,k)}{k^3}$$

$r = |\mathbf{x}-\mathbf{x}'|, \epsilon \ll 1$

$d^3k \rightarrow k^2 dk d\cos\theta d\phi$

$e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \rightarrow e^{ikr\cos\theta}$

$k_c(t) = k$

$$= H\delta(t-'t)\frac{\sin(k_c r)}{k_c r}\mathscr{P}_\phi(t,k_c)$$

$$P_\phi(t,k_c) = \left(\frac{H}{2\pi}\right)^2 \quad \epsilon \ll 1 \longrightarrow \langle \xi_\phi(t,\mathbf{x})\xi_\phi(t',\mathbf{x}')\rangle \simeq \frac{H^3}{(2\pi)^2}\frac{\sin(k_c r)}{k_c r}\delta(t-t')$$

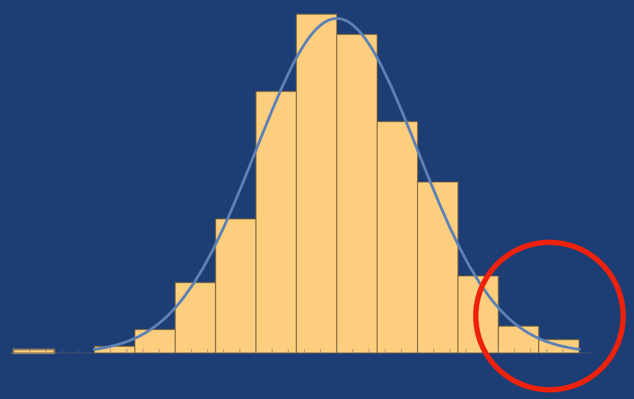
White spectrum

$$\simeq \theta(1-k_c r)$$

# Importance sampling

Discussing PBH formation
→ PBH：rare object
**We want to get data effectively**

Ex: $P(\zeta)$( PDF of $\zeta$)



***Direct sampling***

$$\begin{cases} d\phi_{\mathbf{x}} = \dfrac{\pi_{\mathbf{x}}}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN + \dfrac{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})}{2\pi} dW_{\mathbf{x}} \\ d\pi_{\mathbf{x}} = -3\pi_{\mathbf{x}} dN - \dfrac{V'(\phi_{\mathbf{x}})}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN \end{cases}$$

Number of attempts: **Increasing**

***Importance sampling***

→ Introduction of bias function

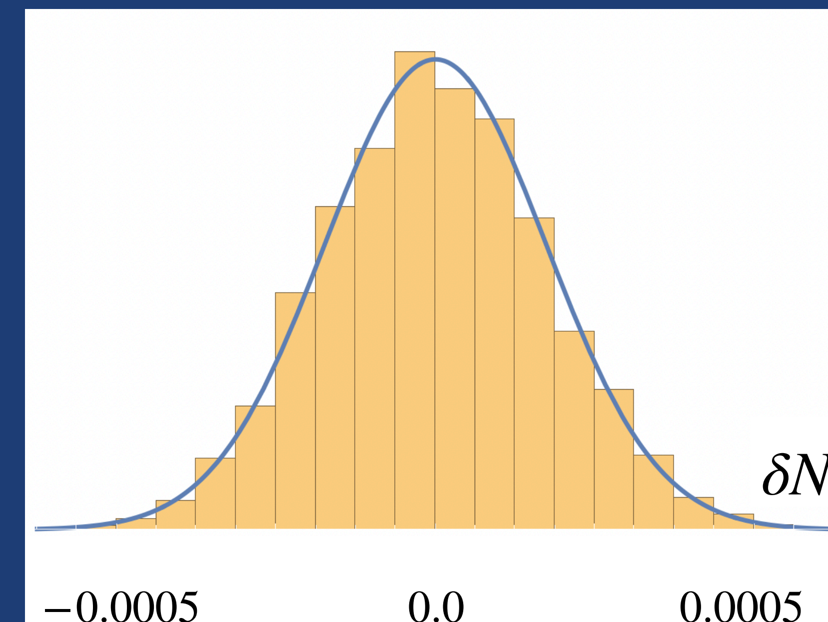
$$\begin{cases} \phi_{\mathbf{x}} = \left( \dfrac{\pi_{\mathbf{x}}}{H_{\mathbf{x}}} + \mathscr{B}_{\mathbf{x}} \right) dN + \dfrac{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})}{2\pi} dW_{\mathbf{x}} \\ d\pi_{\mathbf{x}} = -3\pi_{\mathbf{x}} dN - \dfrac{V'(\phi_{\mathbf{x}})}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN \end{cases}$$

Raising probability at tail
Adding weight instead

# Necessary of importance sampling

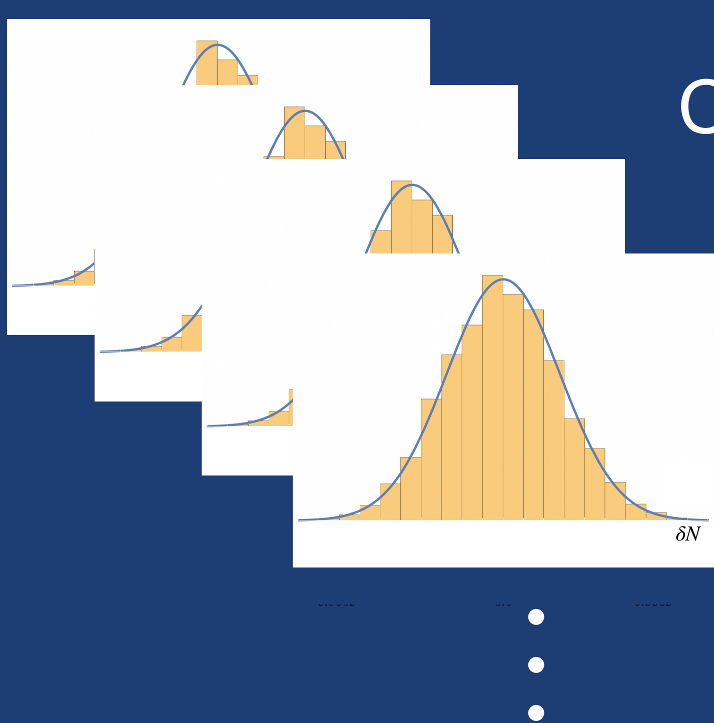
PBH formation from large perturbation

⟶ PBH: rare object




***For getting data effectively***
***through large sampling***

Direct sampling




Collect statistics

**Importance sampling**

$$\begin{cases} d\phi_{\mathrm{x}} = \dfrac{\pi_{\mathrm{x}}}{H(\phi_{\mathrm{x}}, \pi_{\mathrm{x}})} dN + \dfrac{H(\phi_{\mathrm{x}}, \pi_{\mathrm{x}})}{2\pi} dW_{\mathrm{x}} \\ d\pi_{\mathrm{x}} = -3\pi_{\mathrm{x}} dN - \dfrac{V'(\phi_{\mathrm{x}})}{H(\phi_{\mathrm{x}}, \pi_{\mathrm{x}})} dN \end{cases}$$

\+ **bias function**

↓

Raising Probability at tail

Good!

# General Importance sampling

***Langevin equation***

$$\frac{dx}{dt} = [D(t,x) + \mathscr{B}(t,x)] + S(t,x)\xi$$

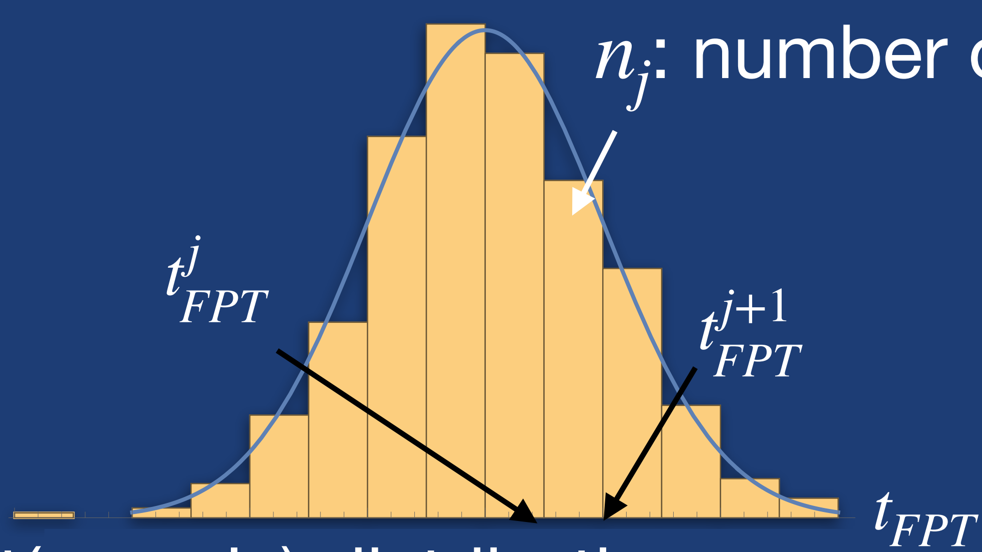
$$\longrightarrow \quad x_{m+1} - x_m = [D(t_m, x_m) + \mathscr{B}(t_m, x_m)]\Delta t_m + S(t_m, x_m)\xi_m\sqrt{\Delta t_m}$$

the deterministic drift   Bias term   amplitude of stochastic diffusion

$\xi$: random white Gaussian noise w/ $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$

***Statistical weight***

$$w_i^{(j)}(\boldsymbol{X}) = \frac{p_{T,i}^{(j)}(\boldsymbol{X}\,|\,x_0)}{p_{S,i}^{(j)}(\boldsymbol{X}\,|\,x_0)}$$




- starting from some initial value $x_0$

  $M$ : number of runs

  $p_{T(S)}(\boldsymbol{X}\,|\,x_0)$ : the probability of target(sample) distribution

  $\boldsymbol{X} = (x_1, x_2, \ldots, x_M)$

[5]Jackson, Joseph H. P. *et al*, 2022

# The PDF of stochastic inflation

*The target PDF using importance sampling*

$$w_i^{(j)}(\boldsymbol{X}) = \frac{p_{T,i}^{(j)}(\boldsymbol{X}\,|\,x_0)}{p_{S,i}^{(j)}(\boldsymbol{X}\,|\,x_0)} \longrightarrow \hat{P}\left(t_{FPT}^{(j)}\right) = \frac{\sum_{i=0}^{n_j} w_j^{(i)}}{(t_{FPT}^{j+1} - t_{FPT}^{j})n_{total}}$$

Hypothesis: PDF of weight $P(w)$ is a ***lognormal distribution***

$$\langle w_j \rangle = \exp\left( \langle \ln w_j \rangle + \frac{\sigma_{\ln w_j}^2}{2} \right) \longrightarrow \hat{P}(\mathcal{N}_j) = \frac{n_j \langle \hat{w}_j \rangle}{n_{total}(\mathcal{N}_{j+1} - \mathcal{N}_j)}$$

*How does the EoM of the inflaton be described*
***with Importance sampling?***

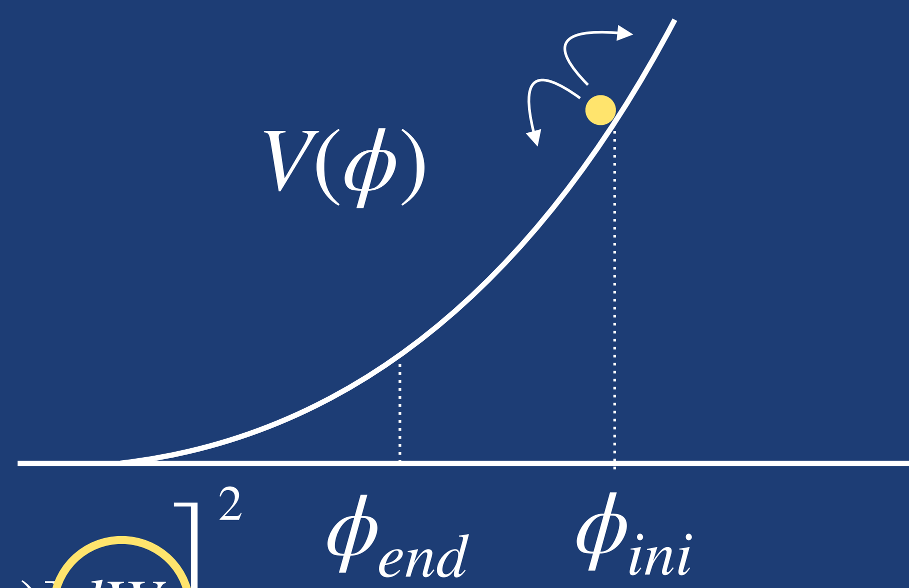
[5]Jackson, Joseph H. P. *et al*, 2022

# How does the EoM of the inflaton be described?

***Langevin equation*** $x_{m+1} - x_m = [\underline{D(t_m, x_m) + \mathscr{B}(t_m, x_m)}]\Delta t_m + \underline{S(t_m, x_m)}\xi_m\sqrt{\Delta t_m}$

***EoM of the Inflaton***

$$\begin{cases} d\phi_{\mathbf{x}} = \dfrac{\pi_{\mathbf{x}}}{\underline{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})}} dN + \dfrac{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})}{\underline{2\pi}} dW_{\mathbf{x}} \\ d\pi_{\mathbf{x}} = -3\pi_{\mathbf{x}} dN - \dfrac{V'(\phi_{\mathbf{x}})}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} dN \end{cases}$$




$$C_{\mathbf{xy}} \simeq dW_{\mathbf{x}} dW_{\mathbf{y}} = \left[ \sum_i \frac{\sqrt{\Delta\Omega_i}}{2\sqrt{\pi}} [\cos(\mathbf{k}_\sigma \cdot \mathbf{x}) - \sin(\mathbf{k}_\sigma \cdot \mathbf{x})] dW_i \right]^2$$

Independent random Gaussian with bias function

$$\begin{cases} dW_i \longrightarrow dW_i + \mathscr{B} \text{ ?} \\ \dfrac{\pi_{\mathbf{x}}}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} \longrightarrow \dfrac{\pi_{\mathbf{x}}}{H(\phi_{\mathbf{x}}, \pi_{\mathbf{x}})} + \mathscr{B} \text{ ?} \end{cases}$$

It is important to **choose properly the bias function**

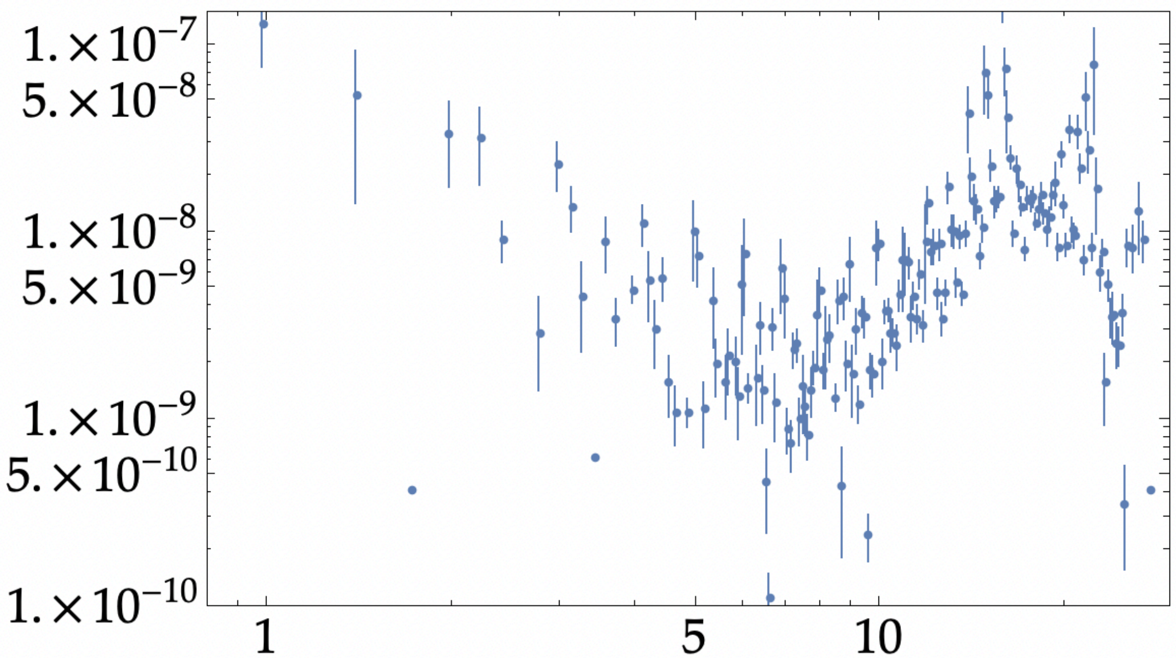
We want to focus on ***the symmetric*** of the region where curvature perturbations are large

# Powerspectrum

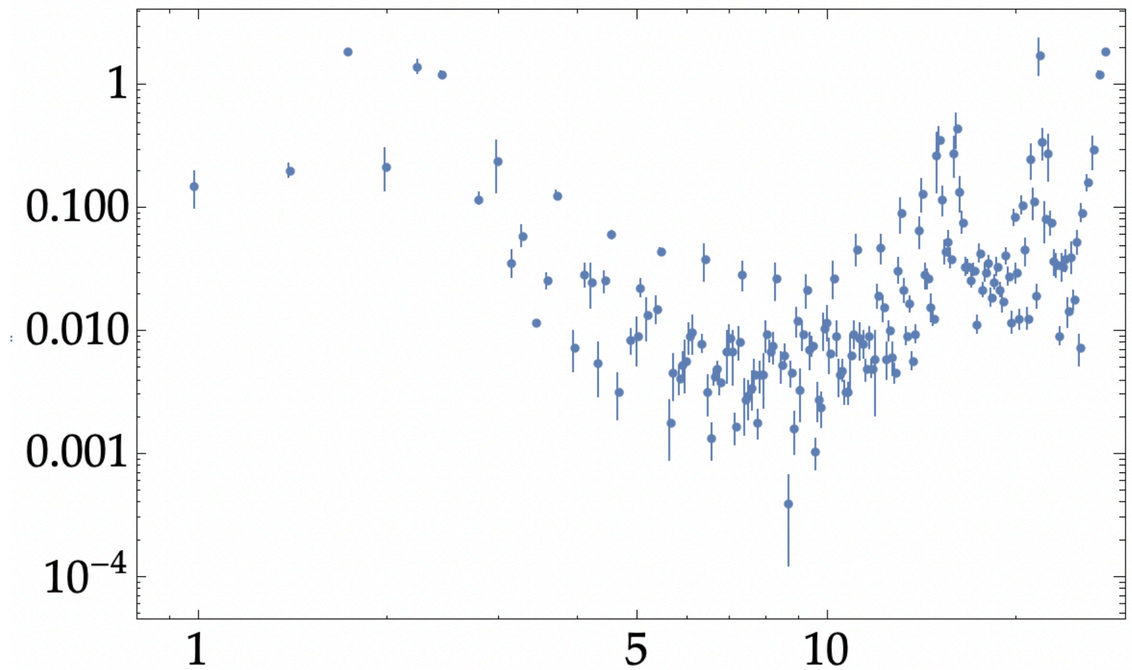
***Chaotic inflation***

$$m = 10^{-5}\,, \phi = 15.0, \pi = -10^{-11}$$

No biased




Biased

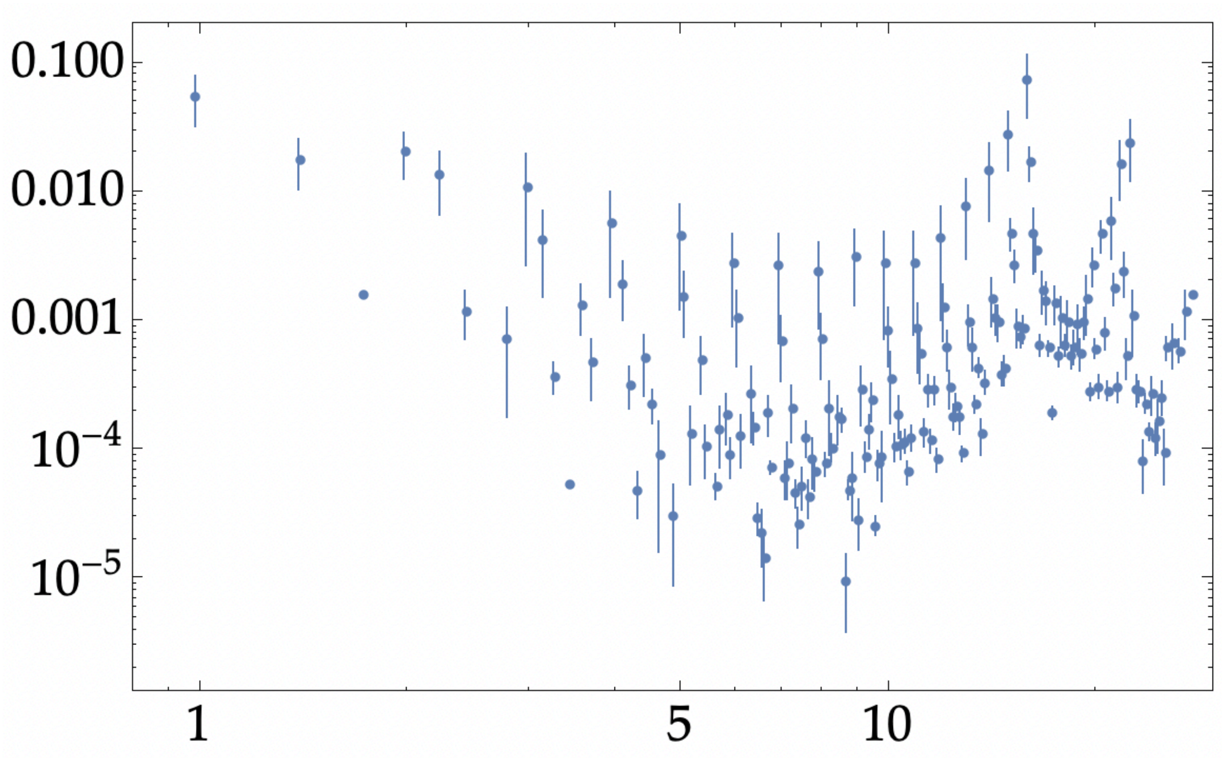


# Powerspectrum

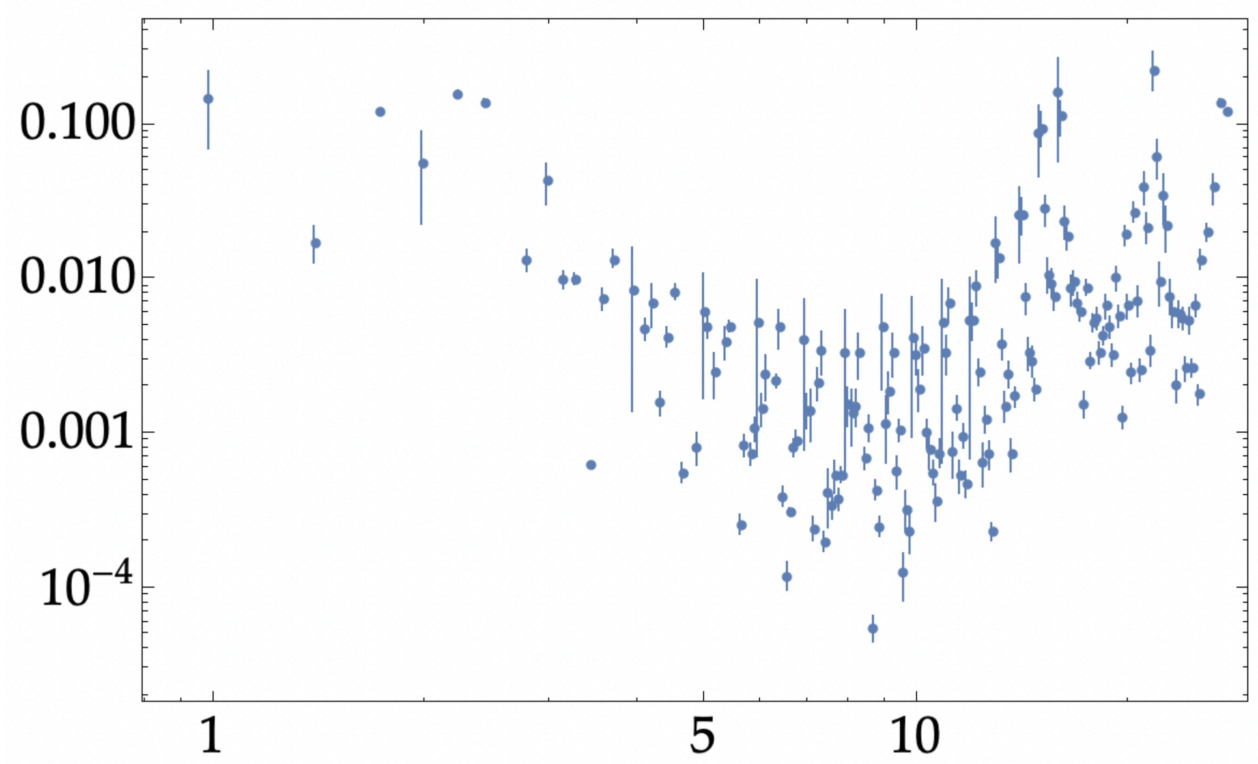
***Inflection***

$$\phi = 3.60547, \pi = -2.37409 \times 10^{-7}$$

No biased



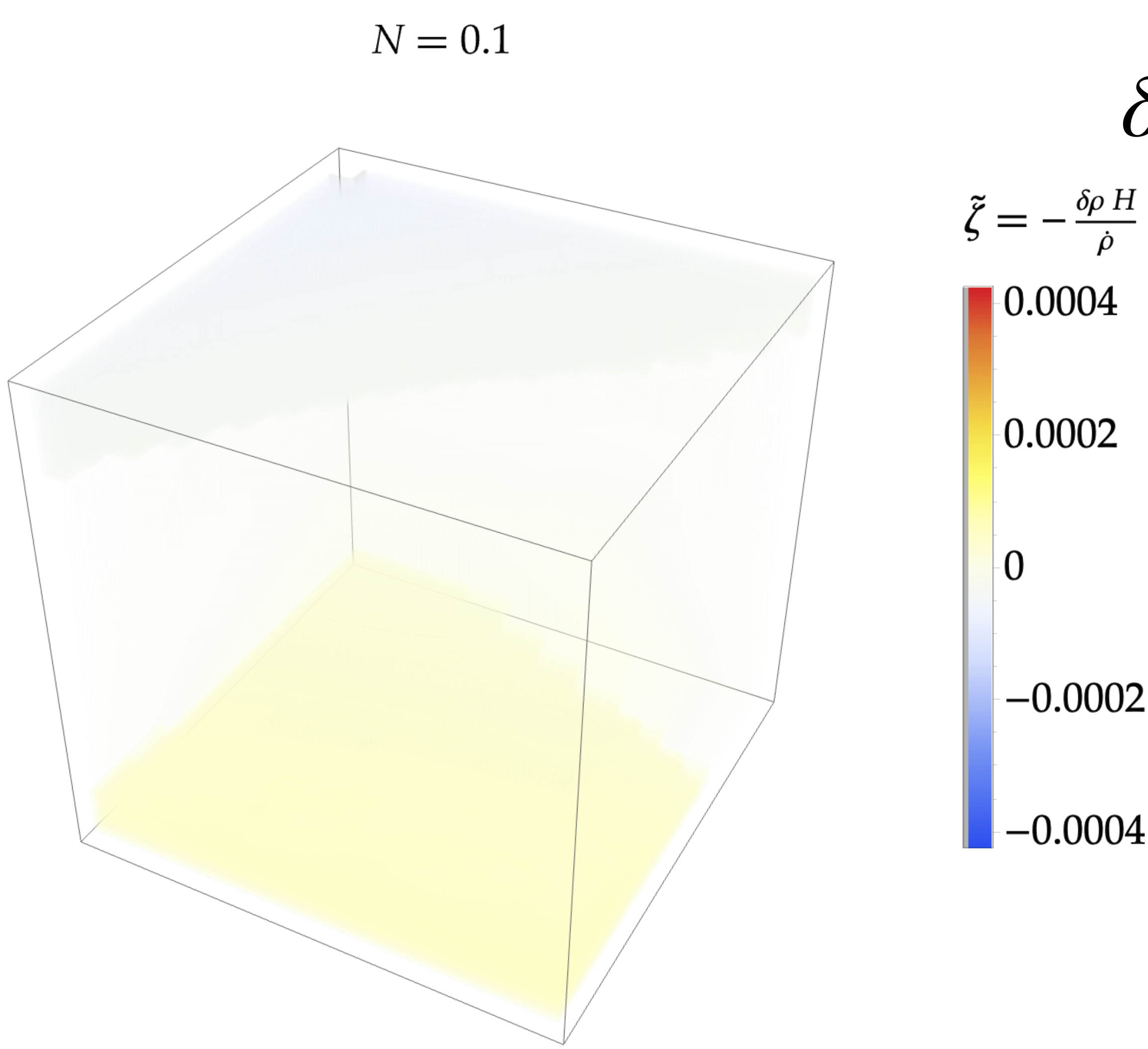

Biased



# Result1: Stochastic lattice simulation

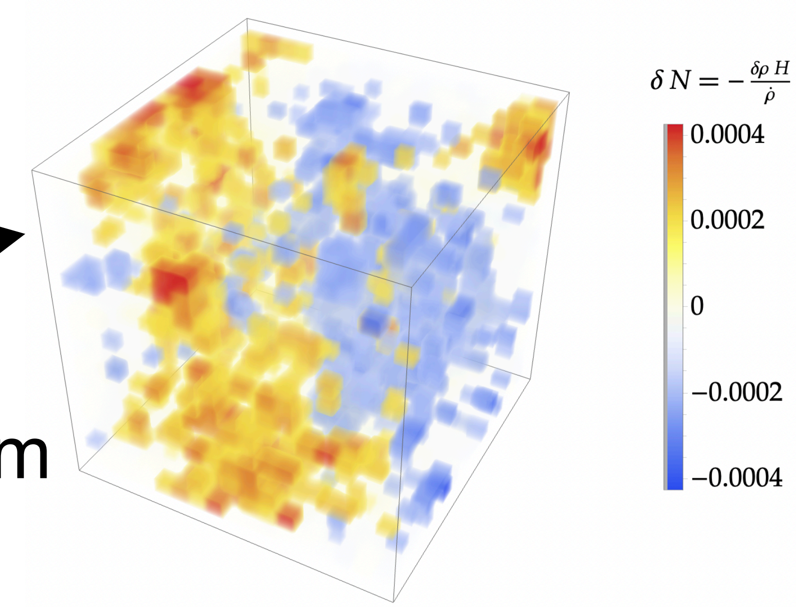
***Chaotic inflation***

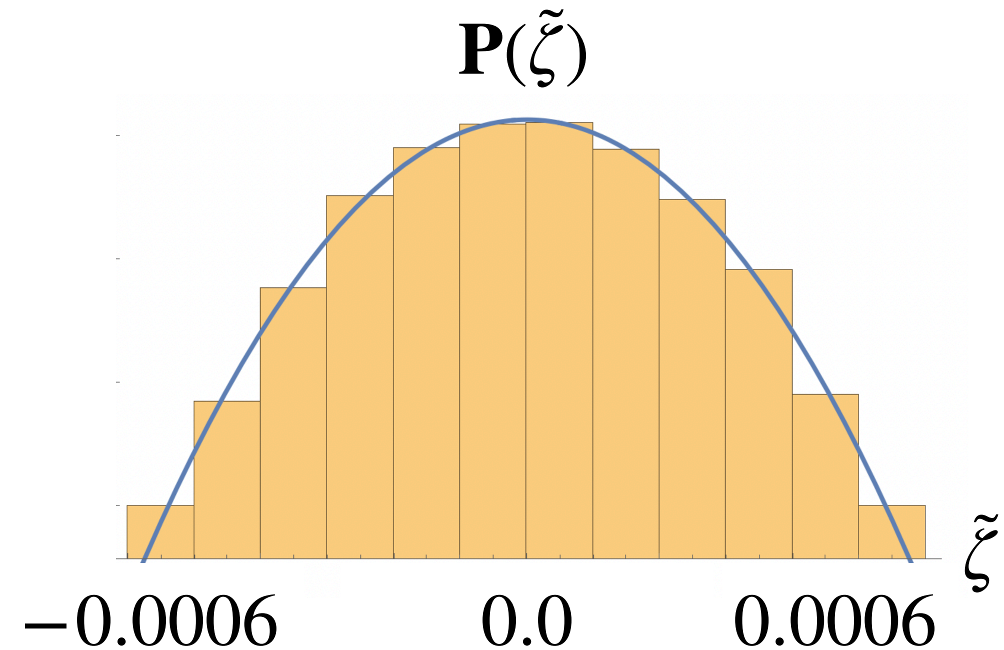
$$m = 10^{-5}\,, \phi = 15.0, \pi = -\,10^{-11}$$




$\delta N$ formalism

At the end of Inflation



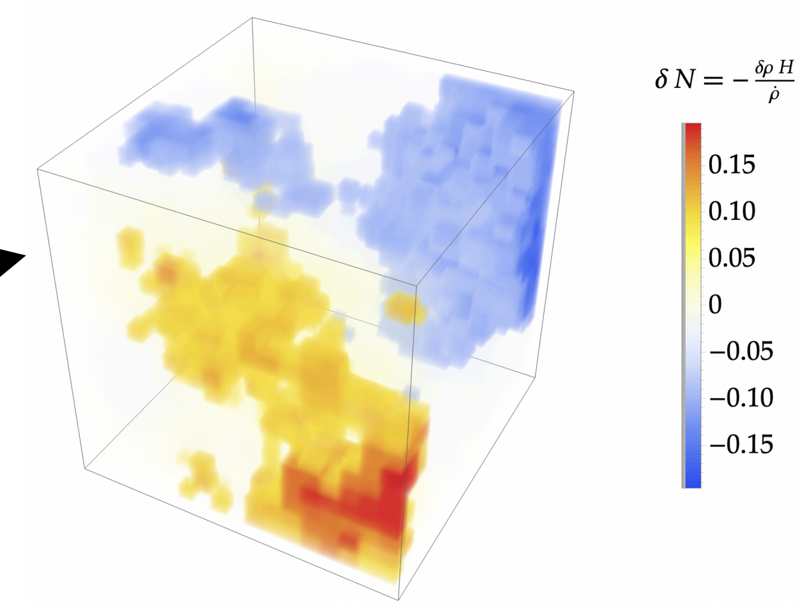



# Result1: Stochastic lattice simulation

***Inflection***

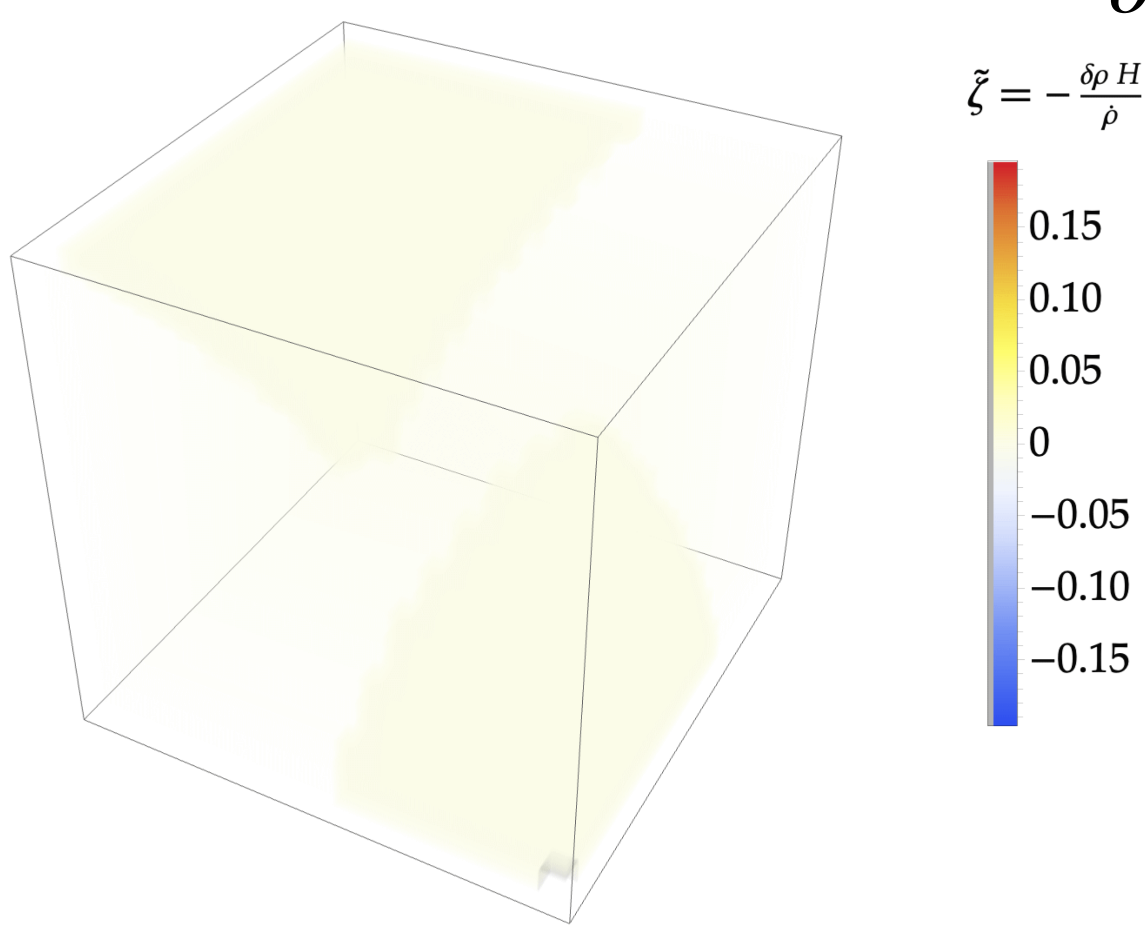
$$\phi = 3.60547, \pi = -2.37409 \times 10^{-7}$$

At the end of Inflation

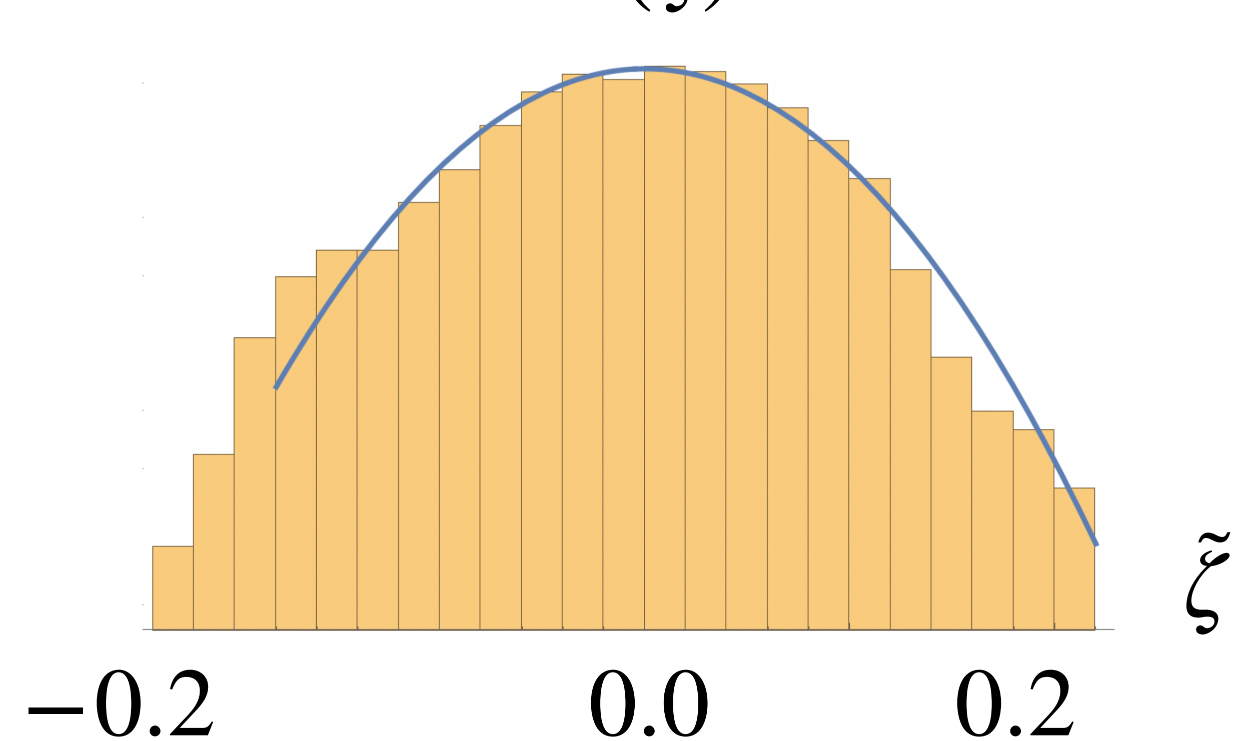



$\delta N$ formalism

$N = 0.1$




$\mathbf{P}(\tilde\zeta)$

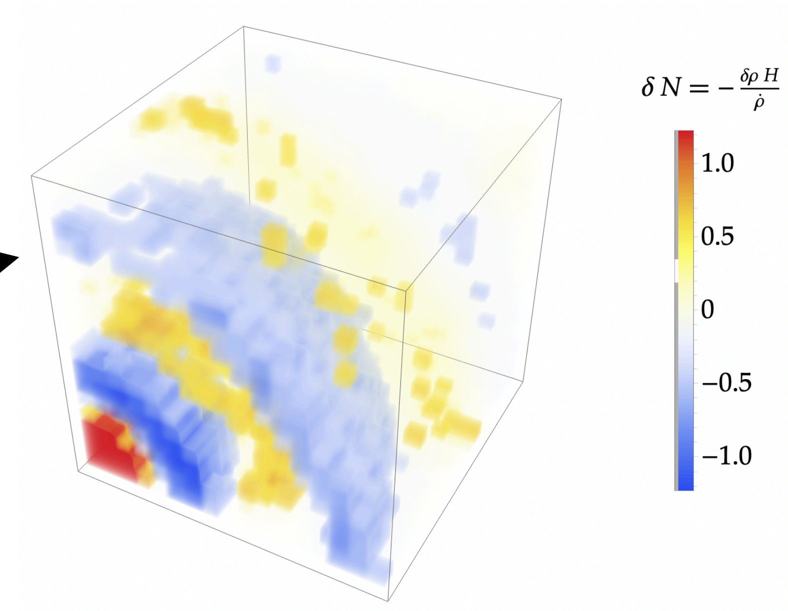


# Result2: Previous simulation w/bias

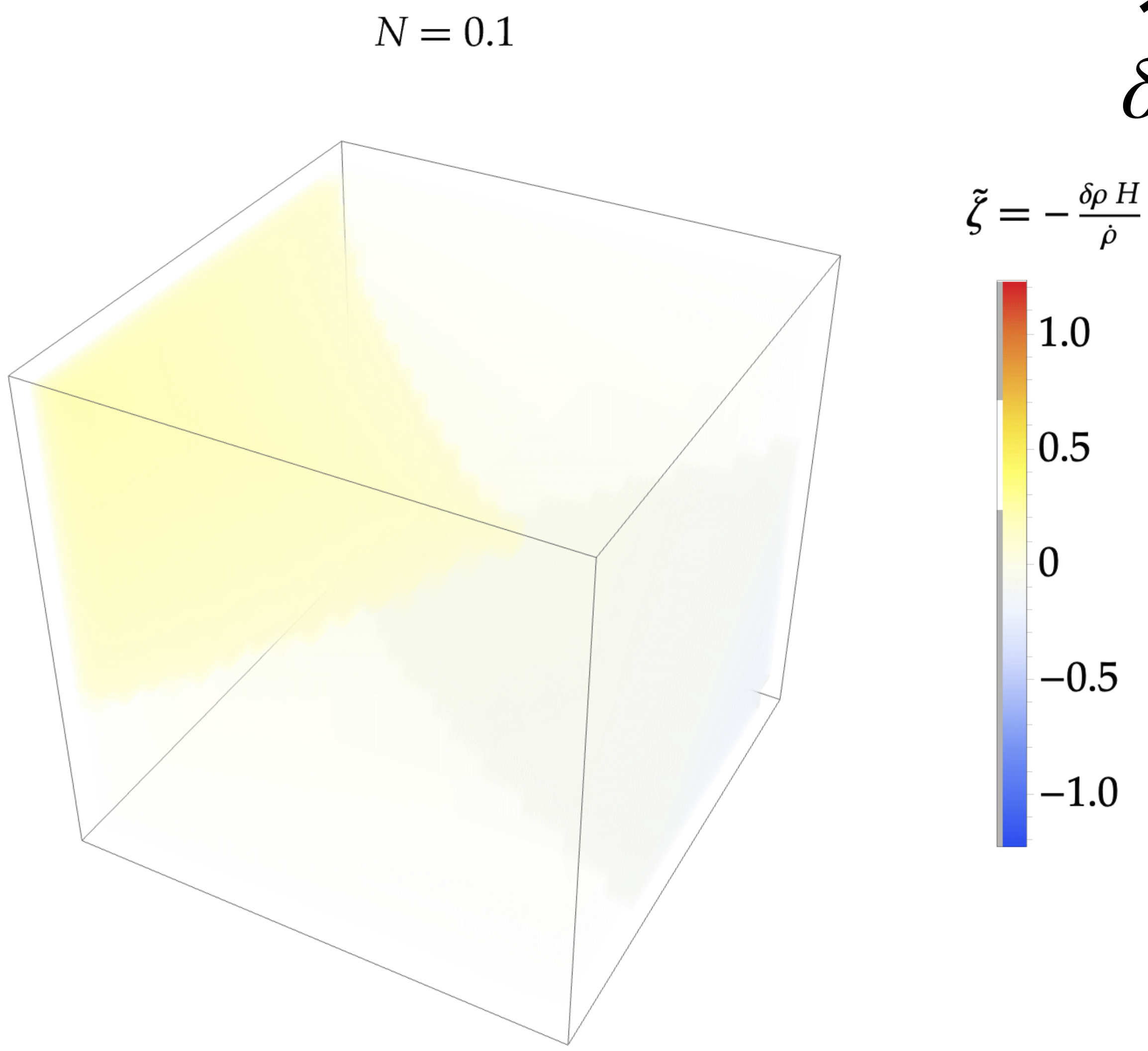
At the end of Inflation

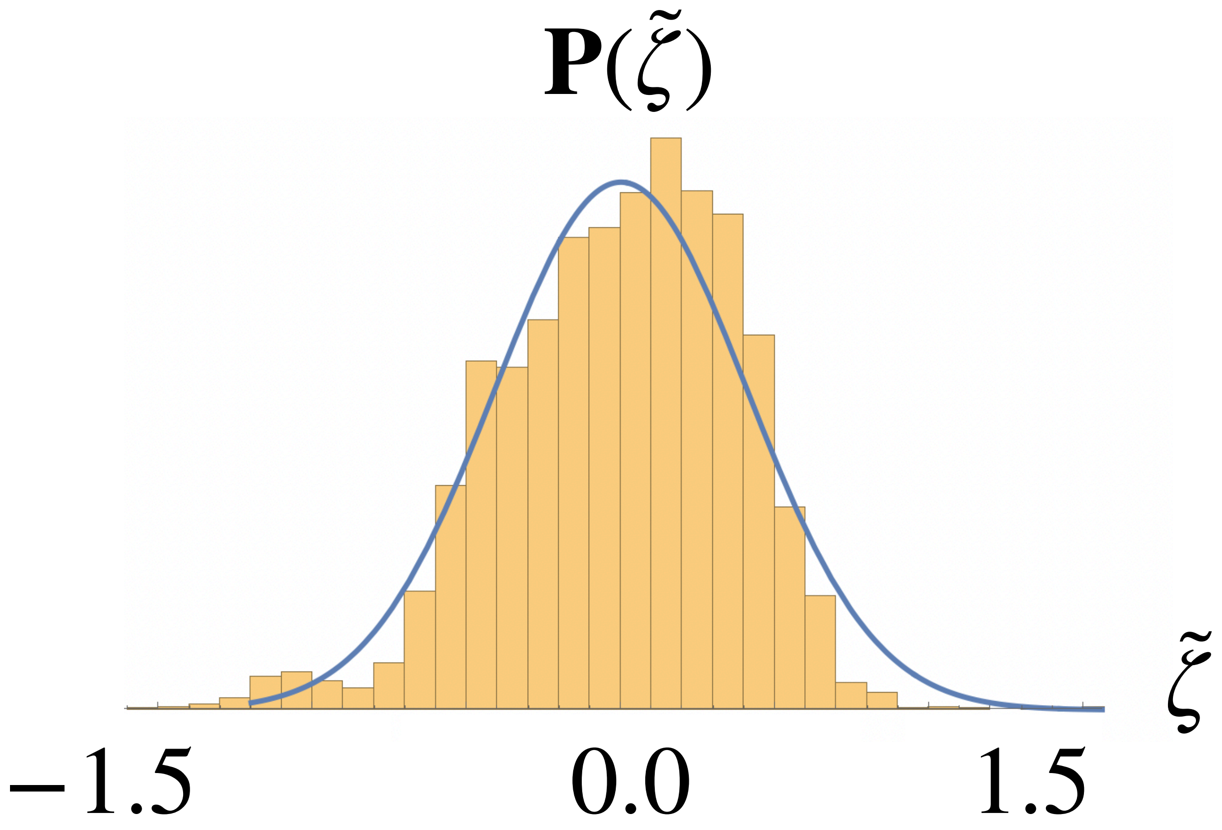
***Chaotic inflation***

$$m = 10^{-5}\,, \phi = 15.0, \pi = -10^{-11}$$



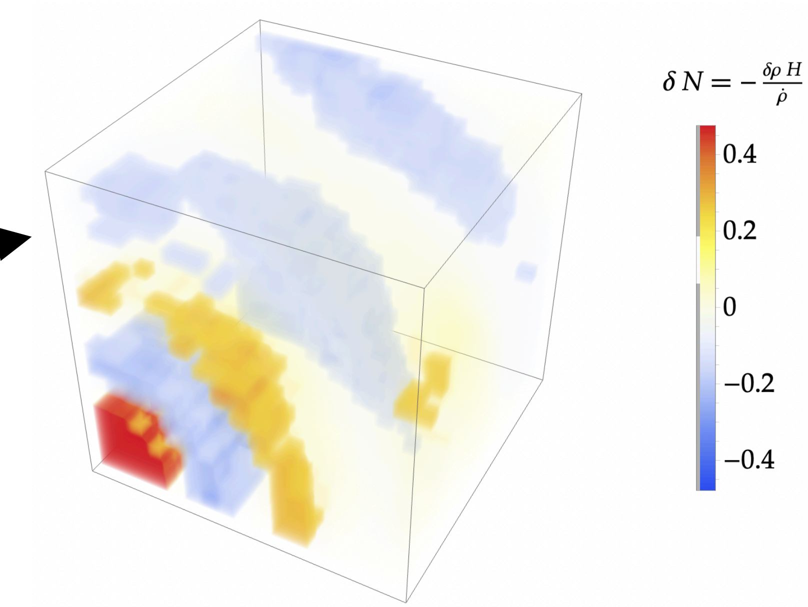

$\delta N$ formalism



# Result2: Previous simulation w/bias

At the end of Inflation




***Inflection***

$$\phi = 3.60547, \pi = -2.37409 \times 10^{-7}$$

$\delta N$ formalism

$N = 0.1$