Questions on calculation of primordial power spectrum with large spikes: the resonance model case

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arXiv: 2211.02586

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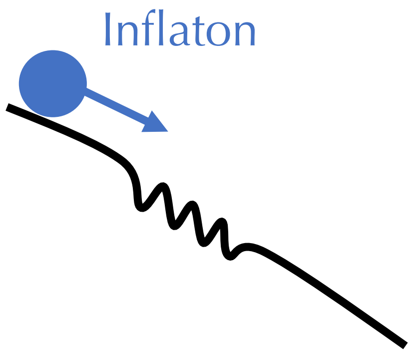
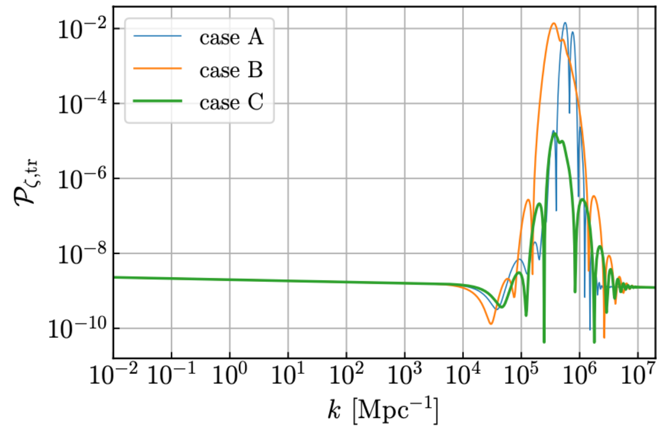
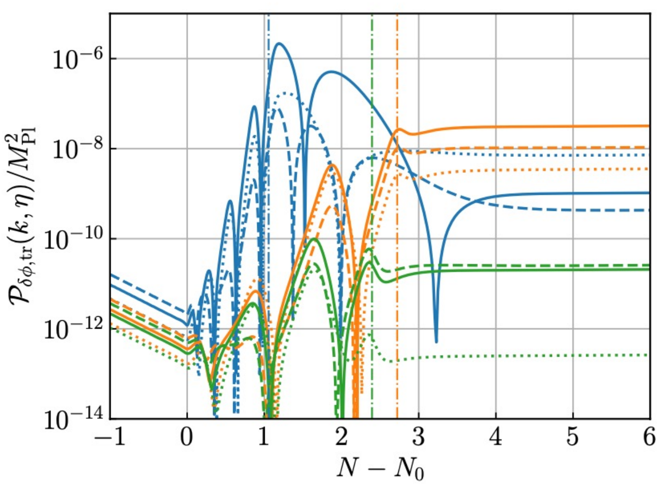
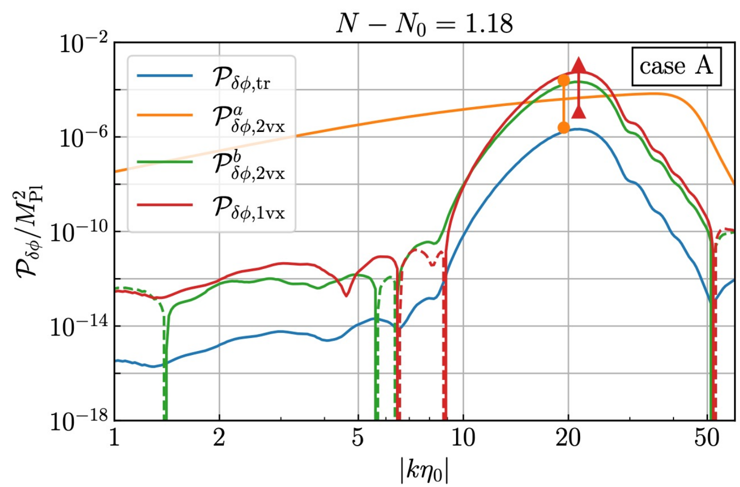
Overview

We discuss one-loop corrections to the power spectrum in inflaton potentials with oscillatory features for a sufficient PBH production.





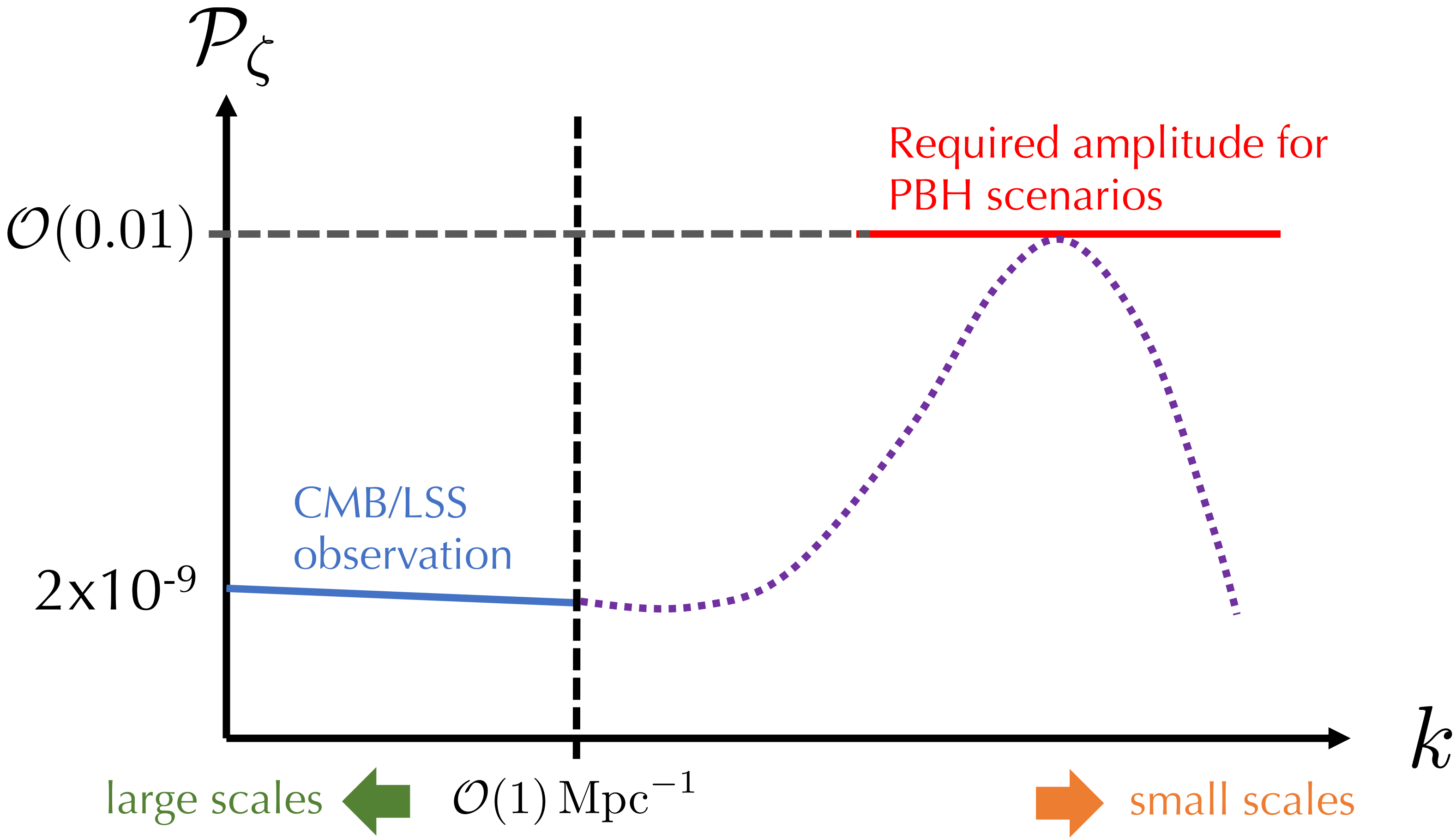




Outline

- Introduction
- Fiducial setups
- One loop power spectrum
- Analytical estimates of loop power spectrum
- Implications on PBH scenarios
- Summary

Large perturbations for PBH scenarios

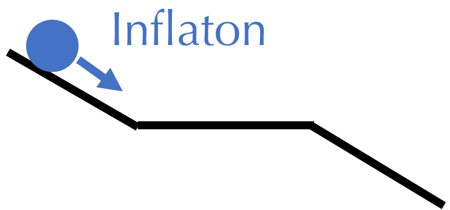


For the PBH scenarios, the enhancement of the power spectrum on small scales should be O(10^7).

Inflaton potentials for large amplification

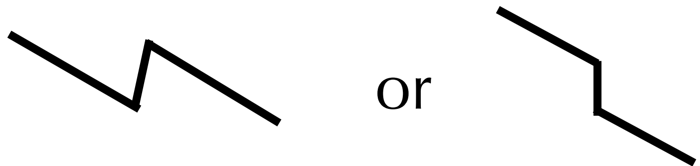
Canonical single field models for large amplification of density perturbations:

flatter region



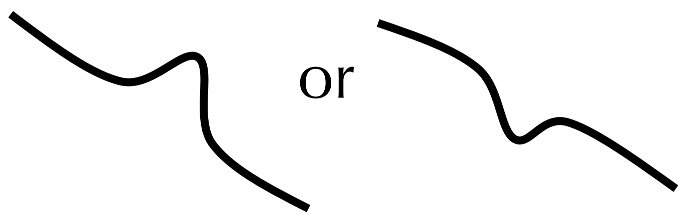
(Starobinsky 1992, Ivanov *et al*. 1994, Inoue and Yokoyama 2001, Kinney 2005)

step feature



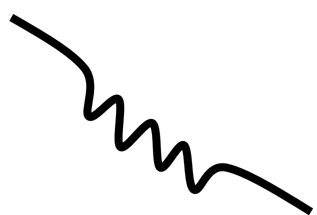
(Kefala et al. 2020, Inomata et al. 2021)

bump/dip feature



(Ozsoy et al. 2018, Mishra and Sahni 2019)

oscillatory feature



(R.G. Cai et al. 2019, Zhou et al. 2020, Peng et al. 2021)

Motivation of this work

Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$$

E.o.m. for the inflaton perturbations: (slow-roll-parameter suppressed terms neglected)

$$(V^{(n)} \equiv \partial^n V/\partial\phi^n)$$

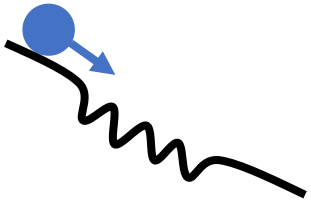
$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi = \underline{-a^2\sum_{n>2}\frac{1}{(n-1)!}V^{(n)}(\delta\phi)^{n-1}}$$

The right-hand side comes from the higher order perturbations, which are often neglected in many works.

However, it is not obvious whether the right-hand-side can be neglected especially when we consider sharp features in the potential.

To clarify this, we discuss the modification from the higher order contributions by using the in-in formalism. (Jordan 1986, Calzetta and Hu 1987, Weinberg 2005, Sloth 2006, Seery 2007, Adshead et al 2008, Senatore and Zaldarriaga 2009)

Throughout this work, we focus on the oscillatory feature model as a concrete example.



(Ultra slow roll case is discussed in: Kristiano and Yokoyama 2022, Riotto 2023, Choudhury et al 2023, Firouzjahi 2023, Motohashi and Tada 2023, Firouzhahi and Riotto 2023, Franciolini et al. 2023, Cheng et al 2023, Fumagalli 2023)

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Linear perturbation equation

Linear e.o.m.: (slow-roll-suppressed terms neglected in spatially-flat gauge)

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2 V^{(2)}(\phi)\delta\phi = 0 \qquad (V^{(n)} \equiv \partial^n V/\partial\phi^n)$$

We expand the perturbation as

$$\delta\phi(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\phi_{\mathbf{k}}(\eta)$$
$$= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[U_k(\eta)\hat{a}(\mathbf{k}) + U_k^*(\eta)\hat{a}^\dagger(-\mathbf{k})\right]$$

where $[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0,\ [\hat{a}(\mathbf{k}), \hat{a}^\dagger(-\mathbf{k}')] = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')$

Then, the linear e.o.m. can be rewritten as

$$U_k''(\eta) + 2\mathcal{H}U_k'(\eta) + k^2 U_k(\eta) + a^2 V^{(2)}(\phi) U_k(\eta) = 0$$

Tree-level power spectrum is given by

$$\langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta),$$
$$\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta) \equiv \frac{k^3}{2\pi^2}|U_k(\eta)|^2$$

Fiducial Potential with oscillatory features

$$D(\phi,\phi_0,\phi_s,\epsilon_0,\Lambda) \equiv \left(\frac{1+\tanh\left(\frac{\phi-\phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)}{2}\right)\left(\frac{1+\tanh\left(\frac{\phi_s-\phi}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)}{2}\right)$$

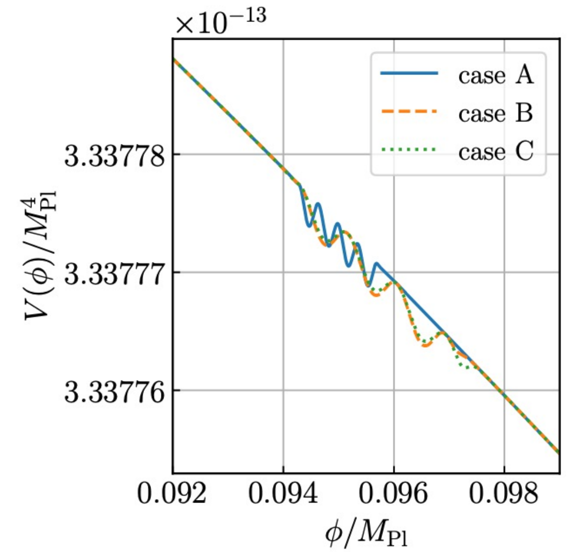
$$V(\phi) = V_0\left[\underbrace{1-\frac{1-n_s}{2}\frac{\phi^2}{2M_{\rm Pl}^2}}_{\text{base part}} + \underbrace{2c\epsilon_0\, D(\phi,\phi_0,\phi_s,\epsilon_0,\Lambda)\left(-1+\cos\left(\frac{\phi-\phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)\right)}_{\text{oscillatory feature}}\right] + V_{\rm end}(\phi)$$

base part:

This determines the large-scale power spectrum, which is tuned to be consistent with the CMB/LSS observations.

oscillatory feature:

This introduces the oscillatory feature within $\phi_0 \lesssim \phi \lesssim \phi_s$.



ϵ_0 ($\equiv -\dot{H}/H^2$): slow-roll parameter at ϕ_0, which leads to $d\phi/dN \approx \sqrt{2\,\epsilon_0}\,M_{Pl}$.

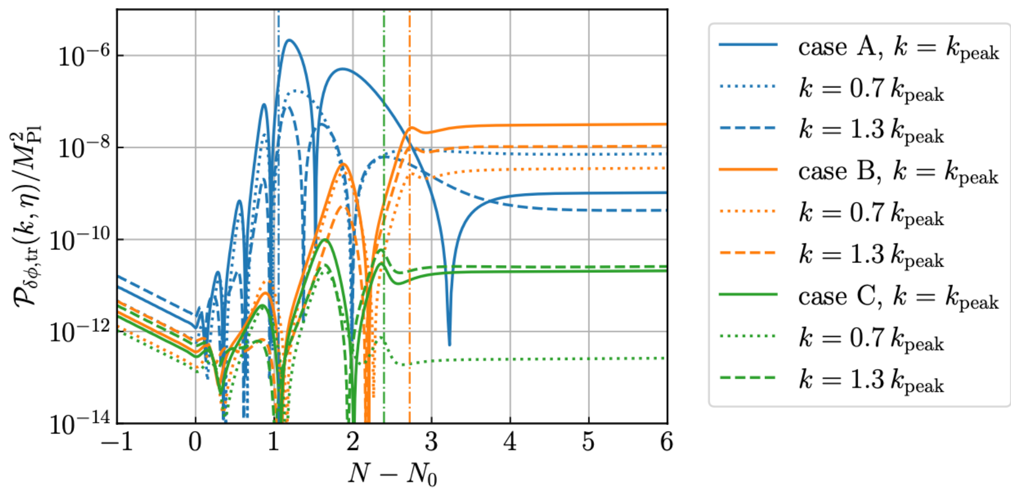
Λ : determines the oscillation timescale, $\Delta N_{osc} \sim \mathcal{O}(\Lambda)$.

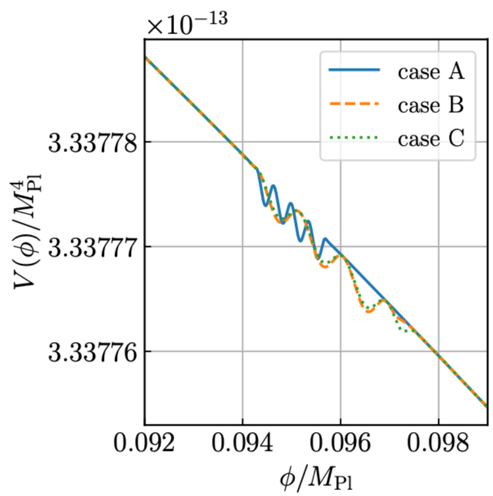
c: the amplitude of the oscillatory feature.

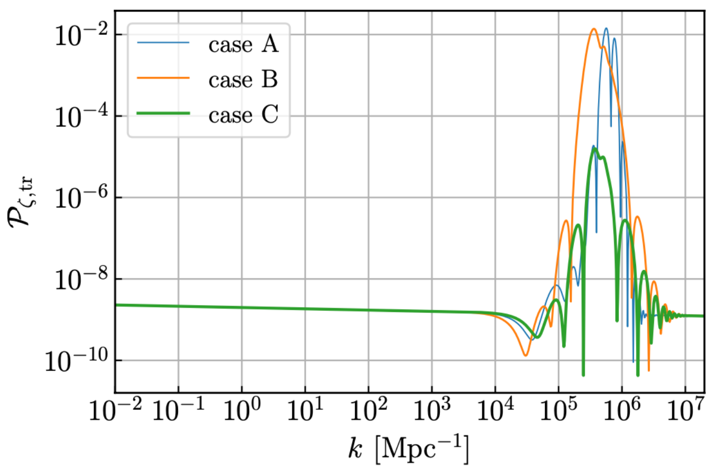
Perturbations in fiducial parameter sets

$$U_k''(\eta) + 2\mathcal{H}U_k'(\eta) + k^2 U_k(\eta) + a^2 V^{(2)}(\phi)U_k(\eta) = 0$$

$$V^{(2)}(\phi) \simeq \frac{V_0}{M_{\rm Pl}^2}(n_s - 1) - \frac{cV_0}{(\Lambda M_{\rm Pl})^2}\cos\left(\frac{\phi - \phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)$$







$\epsilon_0 = 10^{-6}, V_0/M_{\rm Pl}^4 = 3.338 \times 10^{-13}$

	c	Λ	$\frac{\phi_0-\phi_s}{\sqrt{2\epsilon_0}M_{\rm Pl}}$	$\lvert k_{\rm peak}\eta_0\rvert$
case A	0.203	0.04	1	21.4
B	0.22	0.1	2	11.4
C	0.19	0.1	2.13	10.2

Case A and B are for PBH scenarios.

The wavenumber at resonance peak is

$$k_{\rm peak} \sim \mathcal{O}(1/\Lambda)/\eta_0. \quad (\Delta N_{\rm osc} \sim \mathcal{O}(\Lambda))$$

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In-in formalism

In the in-in formalism, the two-point correlation function is given by
(Jordan 1986, Calzetta and Hu 1987, Weinberg 2005)

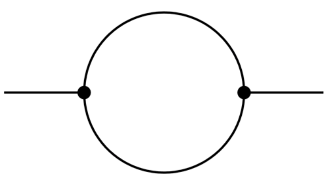
$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle = \langle 0|\left(T e^{-i\int_{-\infty}^{\eta} \mathrm{d}\eta' H_{\mathrm{int}}(\eta')}\right)^{\dagger} \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta) \left(T e^{-i\int_{-\infty}^{\eta} \mathrm{d}\eta'' H_{\mathrm{int}}(\eta'')}\right)|0\rangle$$

$$\left(H_{\mathrm{int},n} \equiv \int \mathrm{d}^3x\ a^4 \mathcal{H}_n,\ \mathcal{H}_{n(>2)} = \frac{1}{n!} V^{(n)}(\phi)\delta\phi^n\right)$$

tree

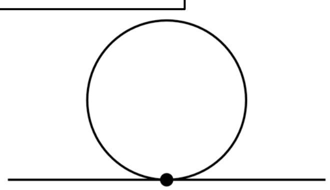
$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{\mathrm{tr}} = \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle$$

two vertices



$$\begin{aligned}\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{2\mathrm{vx}} = {}& \langle 0|\left(T\left[-i\int_{-\infty}^{\eta} \mathrm{d}\eta' H_{\mathrm{int},3}(\eta')\right]\right)^{\dagger} \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta) \left(T\left[-i\int_{-\infty}^{\eta} \mathrm{d}\eta'' H_{\mathrm{int},3}(\eta'')\right]\right)|0\rangle \\ &+ \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\left(T\left[-\frac{1}{2}\left(\int_{-\infty}^{\eta} \mathrm{d}\eta'' H_{\mathrm{int},3}(\eta'')\right)^2\right]\right)|0\rangle \\ &+ \langle 0|\left(T\left[-\frac{1}{2}\left(\int_{-\infty}^{\eta} \mathrm{d}\eta'' H_{\mathrm{int},3}(\eta'')\right)^2\right]\right)^{\dagger} \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle\end{aligned}$$

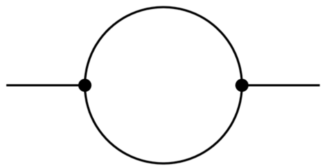
one vertex



$$\begin{aligned}\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{1\mathrm{vx}} = {}& \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\left(T\left[-i\int_{-\infty}^{\eta} \mathrm{d}\eta' H_{\mathrm{int},4}(\eta')\right]\right)|0\rangle \\ &+ \langle 0|\left(T\left[-i\int_{-\infty}^{\eta} \mathrm{d}\eta' H_{\mathrm{int},4}(\eta')\right]\right)^{\dagger} \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle\end{aligned}$$

One loop power spectrum

two vertices



We divide the two-vertex contribution into two parts:

$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{2\mathrm{vx}} \equiv (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\left[\mathcal{P}^a_{\delta\phi,2\mathrm{vx}}(k,\eta)+\mathcal{P}^b_{\delta\phi,2\mathrm{vx}}(k,\eta)\right].$$

Part a:

$$\mathcal{P}^a_{\delta\phi,2\mathrm{vx}}(k,\eta) = \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v}\mathrm{d}u \frac{uv}{4\pi^4} I(k,ku,kv,\eta)I^*(k,ku,kv,\eta),$$

$$I(k,ku,kv,\eta) \equiv k^3 \int_{-\infty}^{\eta}\mathrm{d}\eta'\lambda(\eta')2\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right]U_{ku}(\eta')U_{kv}(\eta').$$

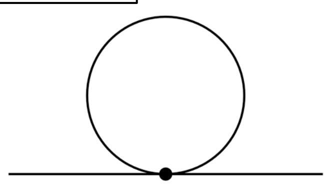
$$\left(\lambda(\eta)\equiv -a^4(\eta)\frac{V^{(3)}(\phi)}{2}\right)$$

Part b:

$$\mathcal{P}^b_{\delta\phi,2\mathrm{vx}}(k,\eta) = 8\int_0^\infty \mathrm{d}v\int_{|1-v|}^{1+v}\mathrm{d}u\frac{uv}{4\pi^4}k^6\int_{-\infty}^{\eta}\mathrm{d}\eta'\int_{-\infty}^{\eta'}\mathrm{d}\eta''\lambda(\eta')\lambda(\eta'')$$

$$\times\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right]\mathrm{Re}\left[U_k(\eta)U_k^*(\eta'')\right]\left(\mathrm{Im}\left[U_{kv}(\eta')U_{kv}^*(\eta'')\right]\mathrm{Re}\left[U_{ku}(\eta')U_{ku}^*(\eta'')\right]+(u\leftrightarrow v)\right)$$

one vertex



$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{1\mathrm{vx}} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta).$$

$$\left(\mu(\eta)\equiv -a^4(\eta)\frac{V^{(4)}(\phi)}{6}\right)$$

$$\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta) = -\frac{k^3}{\pi^2}\int_{-\infty}^{\eta}\mathrm{d}\eta'\mu(\eta')\,\mathrm{Im}[U_k(\eta)U_k^*(\eta')]\int\frac{\mathrm{d}^3p}{(2\pi)^3}6\,\mathrm{Re}[U_k(\eta)U_k^*(\eta')]U_p(\eta')U_p^*(\eta').$$

We solve the linear e.o.m. and substitute the numerical solution of $\delta\phi_k$ (that is, $U_k(\eta)$) into the above loop equations.

One loops from equation of motion

Equation of motion:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi = -a^2\sum_{n>2}\frac{1}{(n-1)!}V^{(n)}(\delta\phi)^{n-1}$$

$$\delta\phi = \delta\phi^{(1)} + \delta\phi^{(2)} + \delta\phi^{(3)}$$

Second order perturbation:

$$\delta\phi^{(2)''} + 2\mathcal{H}\delta\phi^{(2)'} - \nabla^2\delta\phi^{(2)} + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi^{(2)} = -a^2\frac{1}{2}V^{(3)}(\delta\phi^{(1)})^2$$

$$\left\langle \delta\phi^{(2)}_{\boldsymbol{k}}\delta\phi^{(2)}_{\boldsymbol{k}'}\right\rangle \to \mathcal{P}^a_{\delta\phi,2\mathrm{vx}}$$

Third order perturbation:

1. Induced by $\delta\phi^{(2)}\delta\phi^{(1)}$

$$\delta\phi^{(3)''} + 2\mathcal{H}\delta\phi^{(3)'} - \nabla^2\delta\phi^{(3)} + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi^{(3)} = -a^2\frac{V^{(3)}}{2}\left(\delta\phi^{(2)}\delta\phi^{(1)} + \delta\phi^{(1)}\delta\phi^{(2)}\right)$$

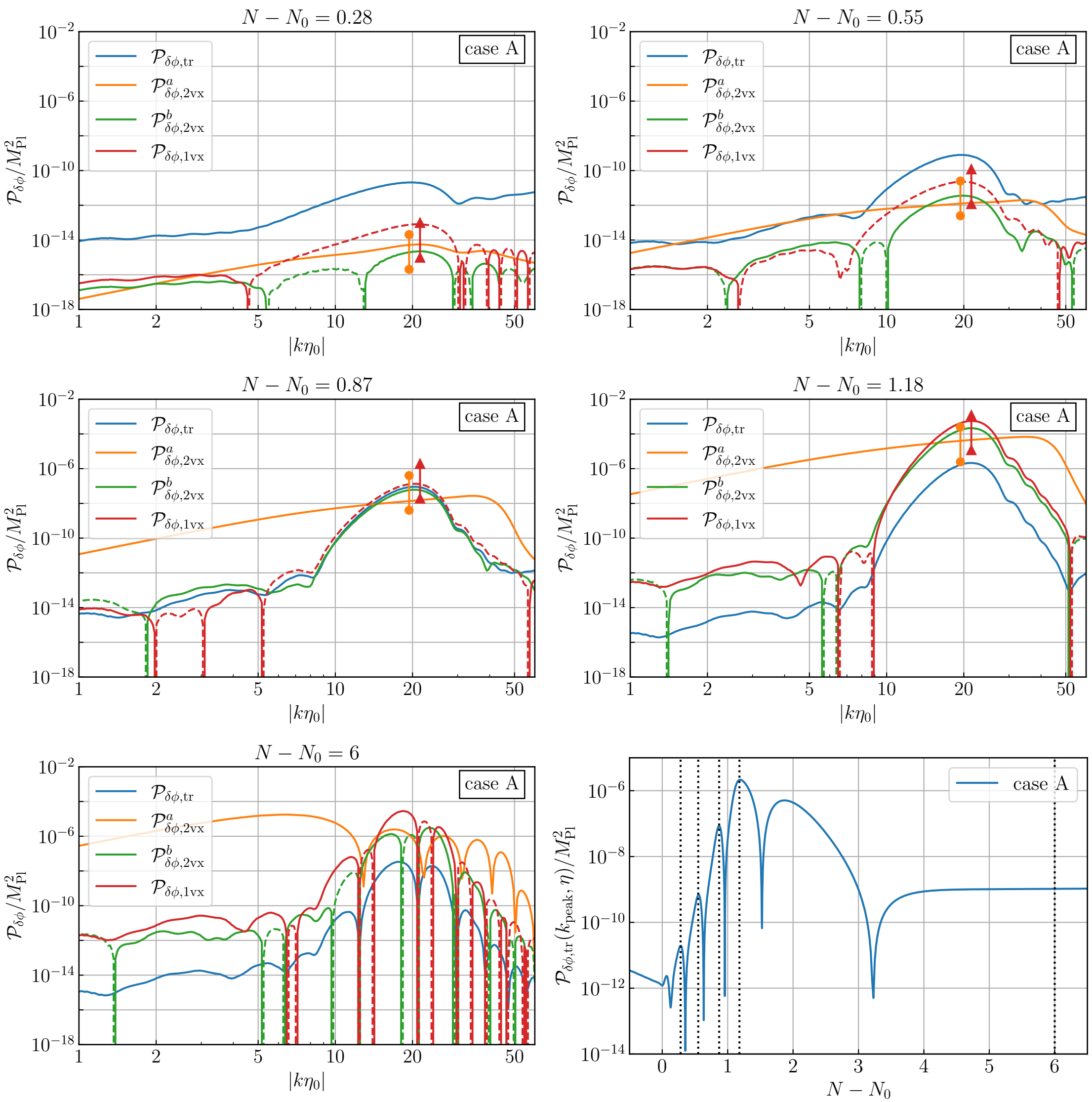
$$\left\langle \delta\phi^{(1)}_{\boldsymbol{k}}\delta\phi^{(3)}_{\boldsymbol{k}'} + \delta\phi^{(3)}_{\boldsymbol{k}}\delta\phi^{(1)}_{\boldsymbol{k}'}\right\rangle \to \mathcal{P}^b_{\delta\phi,2\mathrm{vx}}$$

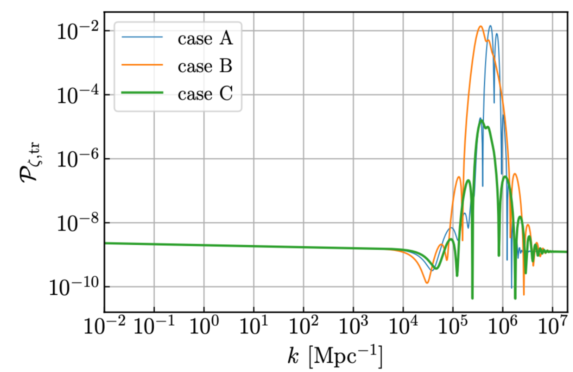
2. Induced by $\left(\delta\phi^{(1)}\right)^3$

$$\delta\phi^{(3)''} + 2\mathcal{H}\delta\phi^{(3)'} - \nabla^2\delta\phi^{(3)} + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi^{(3)} = -a^2\frac{1}{6}V^{(4)}(\delta\phi^{(1)})^3$$

$$\left\langle \delta\phi^{(1)}_{\boldsymbol{k}}\delta\phi^{(3)}_{\boldsymbol{k}'} + \delta\phi^{(3)}_{\boldsymbol{k}}\delta\phi^{(1)}_{\boldsymbol{k}'}\right\rangle \to \mathcal{P}_{\delta\phi,1\mathrm{vx}}$$

Numerical results



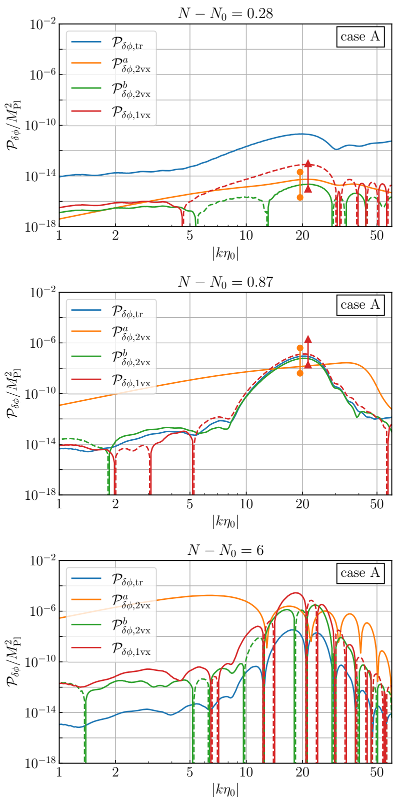


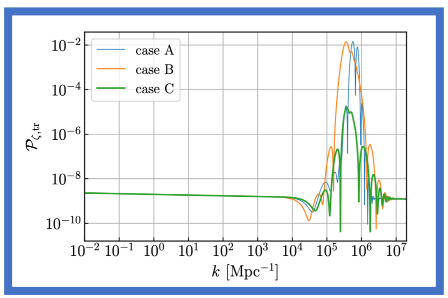
The dots are analytical estimates and dashed lines denote negative values.

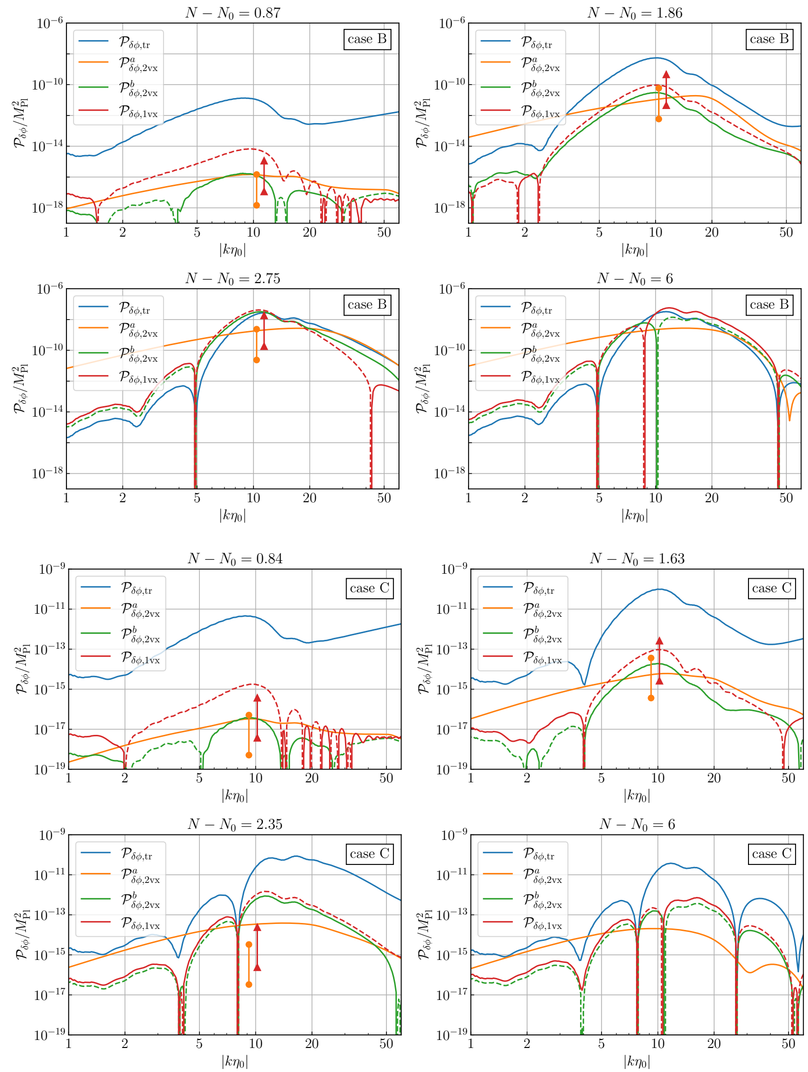
Loop power spectra dominate over the tree spectrum at some point.

Break down of perturbation theory

Numerical results in other cases







Only case C avoids the domination of the one-loop power spectra.

Order estimates

Equation of motion:

$$\underline{\delta\phi''} + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi = -a^2\sum_{n>2}\frac{1}{(n-1)!}V^{(n)}(\delta\phi)^{n-1}$$

$$\sim\left(\frac{a(\eta)}{a_0}\right)^2 k_{\rm peak}^2\delta\phi$$

modification due to $V^{(3)}$:

$$\delta\phi^{(2)} \sim \frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^2 \Rightarrow \left(\delta\phi^{(2)}\right)^2 \sim \left(\frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\right)^2\left(\delta\phi^{(1)}\right)^4$$

$$\Rightarrow \mathcal{P}_{\delta\phi,{\rm loop}} \sim \left(\frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\right)^2 \mathcal{P}_{\delta\phi,{\rm tr}}^2 \quad (\sim \mathcal{P}^a_{\delta\phi,{\rm 2vx}})$$

(same for $\mathcal{P}^b_{\delta\phi,2vx}$)

modification due to $V^{(4)}$:

$$\delta\phi^{(3)} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^3 \Rightarrow \delta\phi^{(3)}\delta\phi^{(1)} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^4$$

$$\Rightarrow \mathcal{P}_{\delta\phi,{\rm loop}} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\mathcal{P}_{\delta\phi,{\rm tr}}^2 \quad (\sim \mathcal{P}_{\delta\phi,{\rm 1vx}})$$

Implications on PBH scenarios

The linear perturbation theory is valid only when the loop power spectrum does not dominate over the tree power spectrum during the resonance.

The conditions for the negligible one loops are: (η_e: the global maximum time)

$$\frac{\mathcal{P}_{\delta\phi,\mathrm{2vx}}(k_{\mathrm{peak}},\eta_e)}{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_e)} \sim \frac{c^2}{\Lambda^4}\mathcal{P}_{\zeta,\mathrm{tr}}(k_{\mathrm{peak}}) < 1,$$

$$\left|\frac{\mathcal{P}_{\delta\phi,\mathrm{1vx}}(k_{\mathrm{peak}},\eta_e)}{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_e)}\right| \sim \frac{c}{\Lambda^4}\mathcal{P}_{\zeta,\mathrm{tr}}(k_{\mathrm{peak}}) < 1$$

The resonance strength is determined by the amplitude (c) and the frequency ($1/\Lambda$) of the oscillatory features in the potential.

$$V(\phi) = V_0\left[1 - \frac{1-n_s}{2}\frac{\phi^2}{2M_{\mathrm{Pl}}^2} + 2c\epsilon_0\, D(\phi,\phi_0,\phi_s,\epsilon_0,\Lambda)\left(-1+\cos\left(\frac{\phi-\phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\mathrm{Pl}}}\right)\right)\right] + V_{\mathrm{end}}(\phi)$$

Actually, to realize the large enhancement, c^2/Λ cannot be much smaller than $\mathcal{O}(1)$.
So, the larger Λ can lead to the smaller loops with the tree power spectrum fixed.

However, the resonance models typically have $\Lambda \ll 1$ to realize the resonance with the rapid oscillation of the slow-roll parameter within less than one e-fold.

Case B realizes the PBH scenarios with $\Lambda = 0.1$, but the loop dominates the tree power spectrum.

⬇

<u>The typical resonance models for the PBH scenarios give nonnegligible loop contributions.</u>

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Summary

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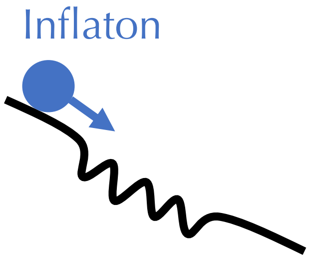
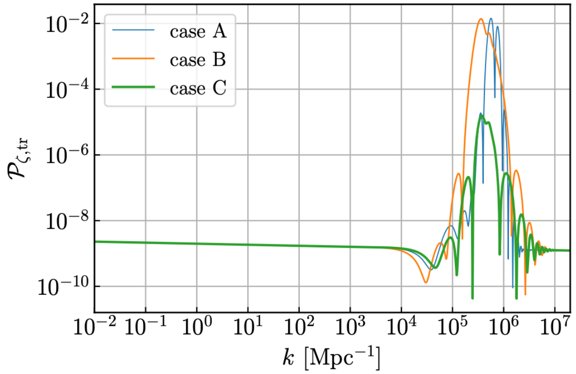
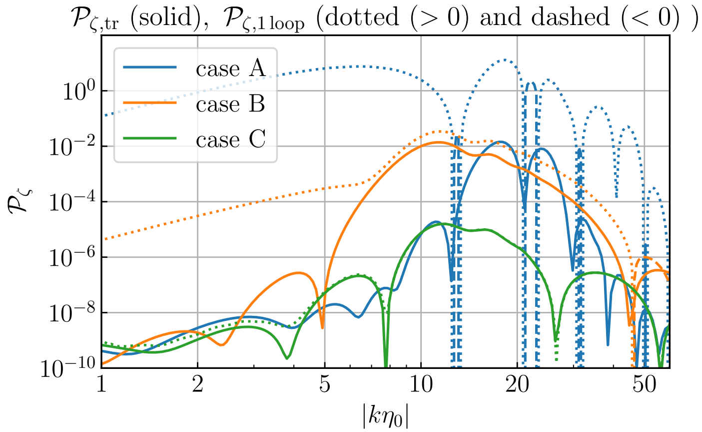






The domination of the one-loop over tree power spectrum indicates the break down of the perturbation theory.

If we consider the typical oscillatory feature models, the $\mathcal{O}(10^7)$ enhancement in in $\mathcal{P}_\zeta$ for the PBH scenarios leads to the nonnegligible loop power spectrum.

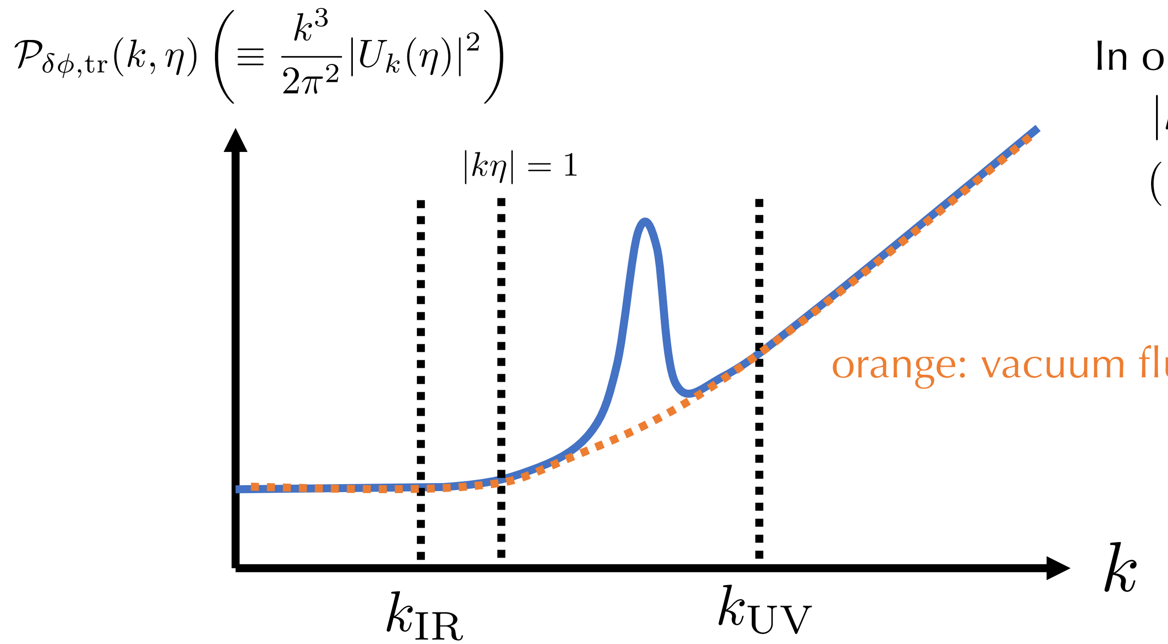
Our result indicates that we need a new computational method to discuss whether the oscillatory feature models can realize the PBH scenarios.

Backup

Loop integrals

Example (one vertex): $\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta) = -\frac{k^3}{\pi^2}\int_{-\infty}^{\eta} \mathrm{d}\eta' \mu(\eta')\,\mathrm{Im}[U_k(\eta)U_k^*(\eta')] \int \frac{\mathrm{d}^3 p}{(2\pi)^3} 6\,\mathrm{Re}[U_k(\eta)U_k^*(\eta')] U_p(\eta') U_p^*(\eta')$.

To focus on the loop corrections from the amplified perturbations (exited states), we introduce the wavenumber cutoff scales for the loop integral.



In our fiducial setups, we take
$|k_{\mathrm{IR}}\eta_0| = 0.1, |k_{\mathrm{UV}}\eta_0| = 60$
$(|k_{\mathrm{peak}}\eta_0| \lesssim 20)$

This procedure is based on the assumption that the UV/IR divergences are already renormalized by the potential parameters and the finite loop corrections from the vacuum fluctuations are much smaller than those from the exited states.