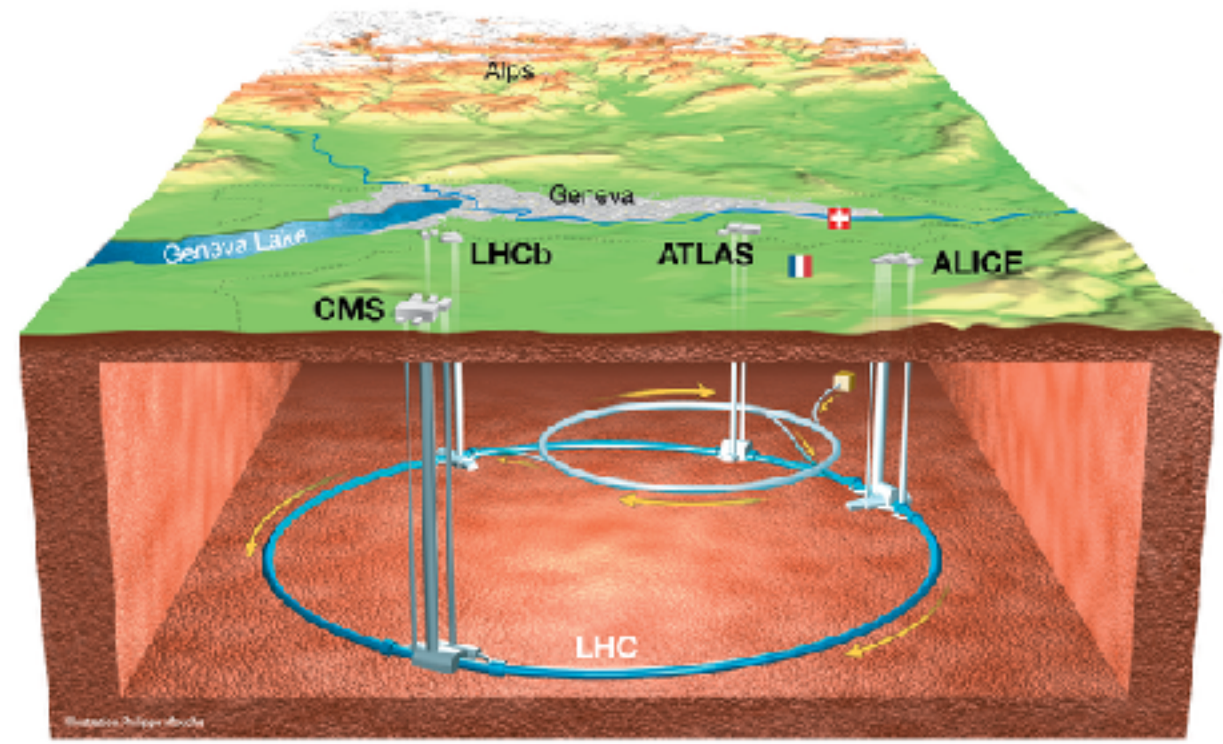
Precision computations for the LHC

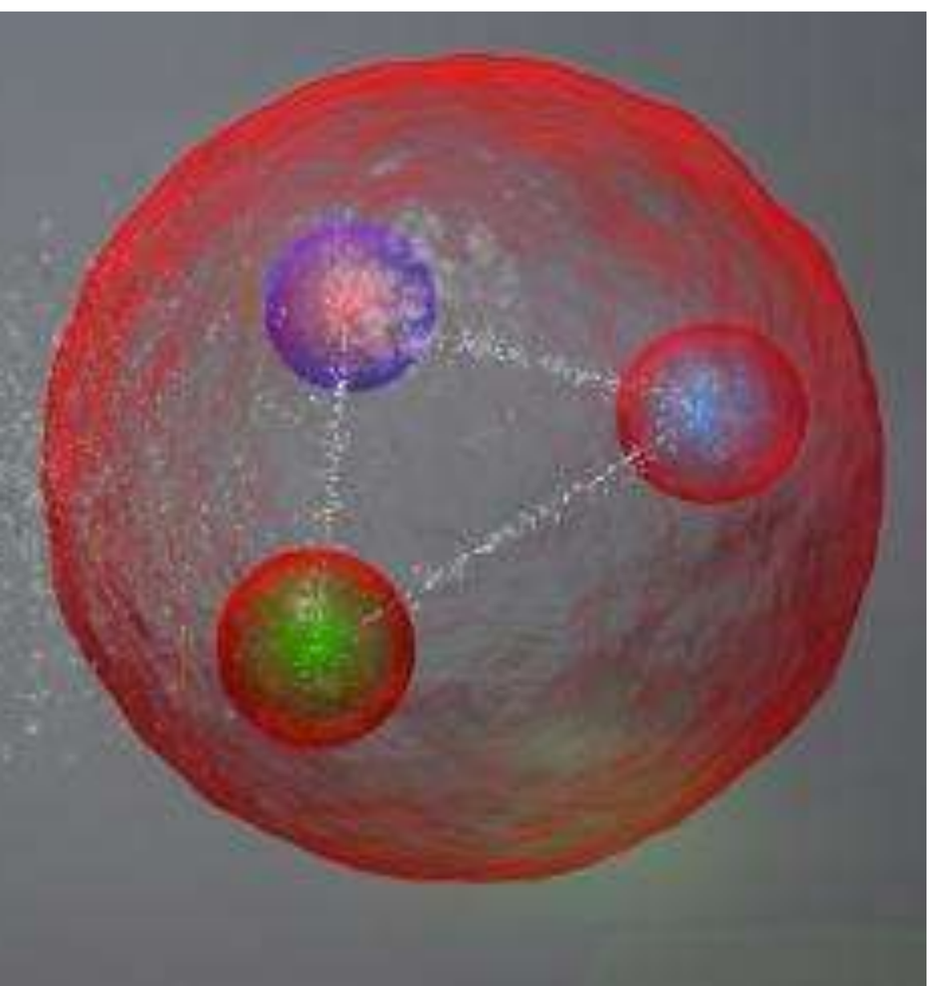
Claude Duhr

Annual Theory Meeting 2022
Durham
14 December 2022

The need for precision

- Particle physics today is driven by the LHC.
- Collides protons at high energies.
  - ➡ Gives us access to physics at new energy scales!





- The LHC is a proton collider.
  - ➡ Proton = bound state of quarks and gluons.
  - ➡ LHC = Collisions of quarks and gluons.
  - ➡ How to make predictions?

QCD factorisation

- The 'master formula' for LHC observables:

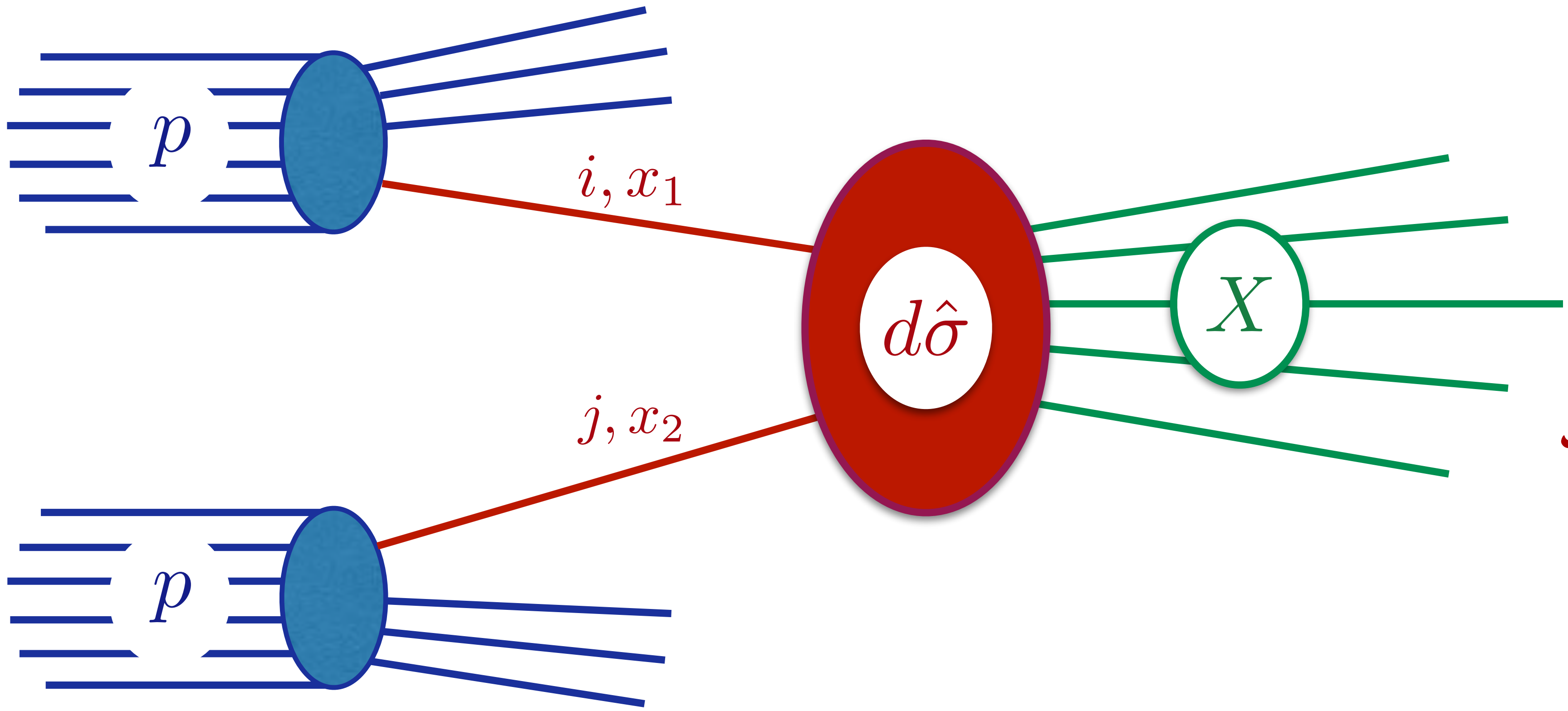
$$d\sigma(p\,p \to X) = \sum_{i,j} \int_0^1 dx_1\, dx_2\, f_i(x_1)\, f_j(x_2)\, d\hat{\sigma}(i\,j \to X)$$

Parton Distribution Functions

non-perturbative;
describe structure of the proton

Partonic cross section

computable in perturbation theory
as collisions between quarks and gluons

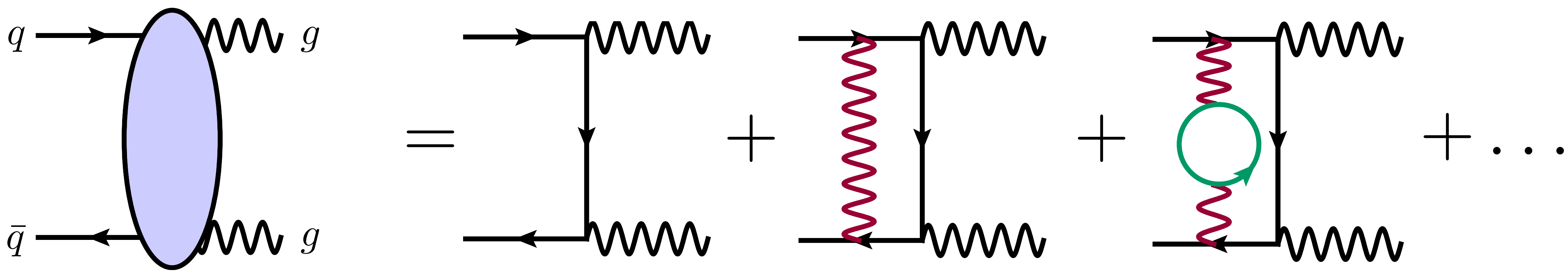


$$d\hat{\sigma} \sim \int dPS\, |\mathcal{A}|^2$$

$\mathcal{A}$ = scattering amplitude

Scattering amplitudes

- $\mathcal{A}$ computed from Feynman diagrams:

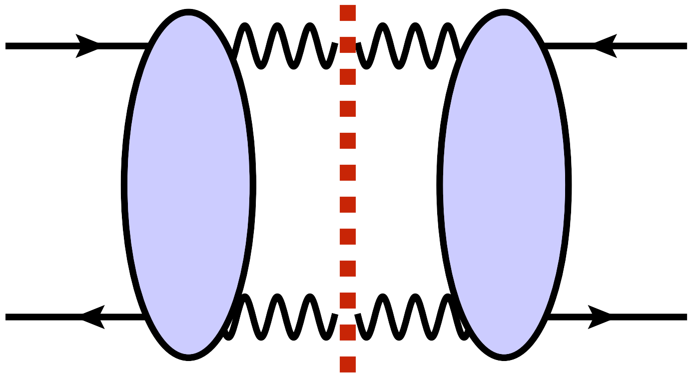


➡ Each diagram translates into an analytic formula.

➡ Perturbative expansion ~ expansion in number of loops.

- Probabilities are related to the square of the amplitude:

$$\text{Proba} \sim |\mathcal{A}|^2 = \mathcal{A}\,\mathcal{A}^* =$$

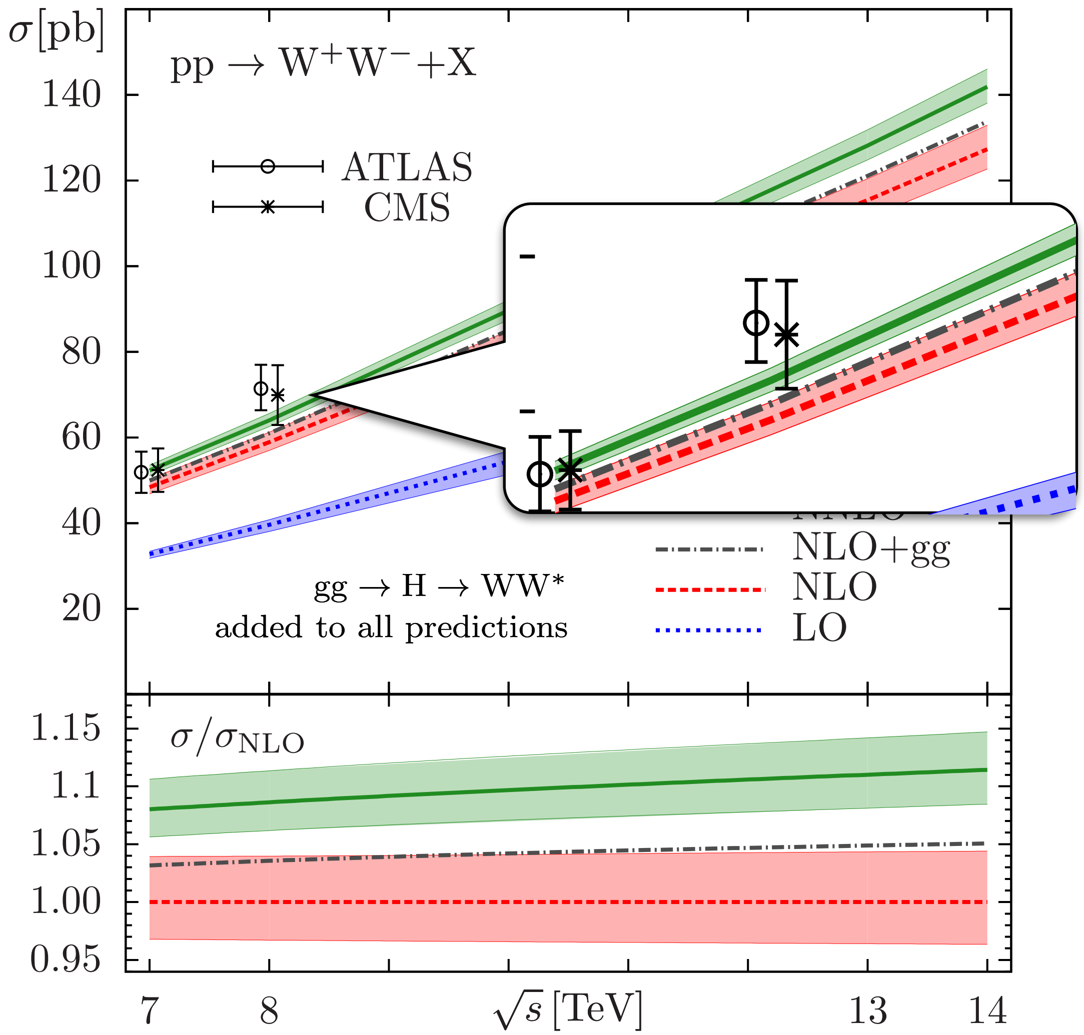


The need for precision

- In general we do not know how to compute amplitudes exactly.
  - ➡ Perturbation theory: $\alpha_s =$ coupling constant $\simeq 0.118$

$$\mathcal{A} = \underbrace{\mathcal{A}^{(0)}}_{\text{LO}} + \underbrace{\alpha_s\,\mathcal{A}^{(1)}}_{\text{NLO} \sim 10\%} + \underbrace{\alpha_s^2\,\mathcal{A}^{(2)}}_{\text{NNLO} \sim 1\%} + \dots$$

- Precision increases with the number of terms.
  - ➡ How many terms needed?
- To reach 1%, need next-to-next-to-leading order (NNLO) precision.
  - ➡ Is this needed?

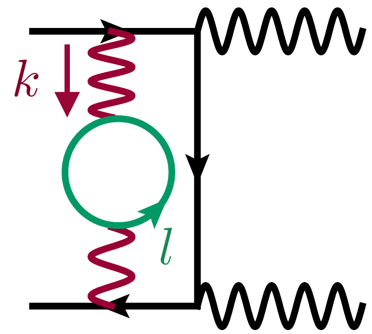


[Gehrmann, Grazzini, Kallweit, Maierhöfer, von Manteuffel, Pozzorini, Rathlev, Tancredi]

Loop integrations

- State of the art:
  - ➡ 1 loop: usually doable.
  - ➡ 2 loops: results for low multiplicity.
  - ➡ 3 loops: some $2 \to 1$.

$$\int d^4k\, d^4l$$



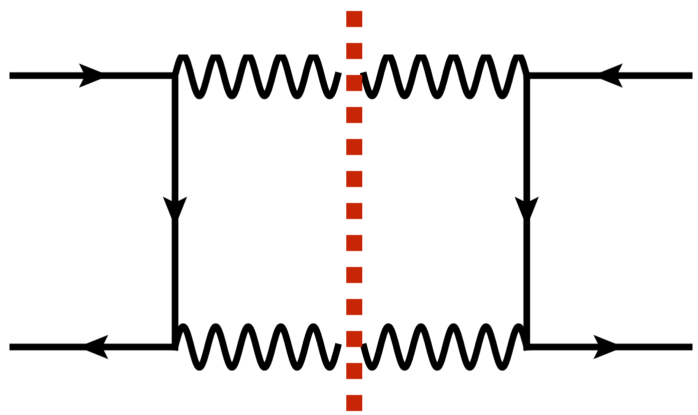
- ➡ 2x4 = 8 integrations: Cannot be so hard?
- ➡ cf. phase space for 4 jets:

  (4-1) x 3 = 9 integrations.

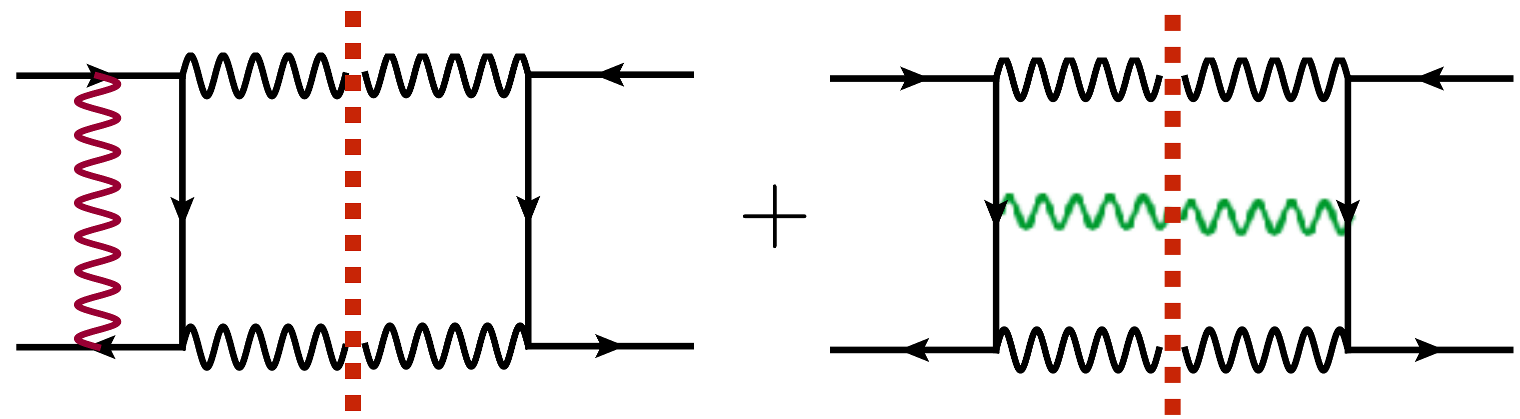
- Loop integrals are usually divergent!
  - ➡ UV divergencies: removed by renormalisation.
  - ➡ IR divergencies: cancel against real emissions.

Anatomy of higher orders

- Leading order (LO):



- Next-to-LO (NLO):

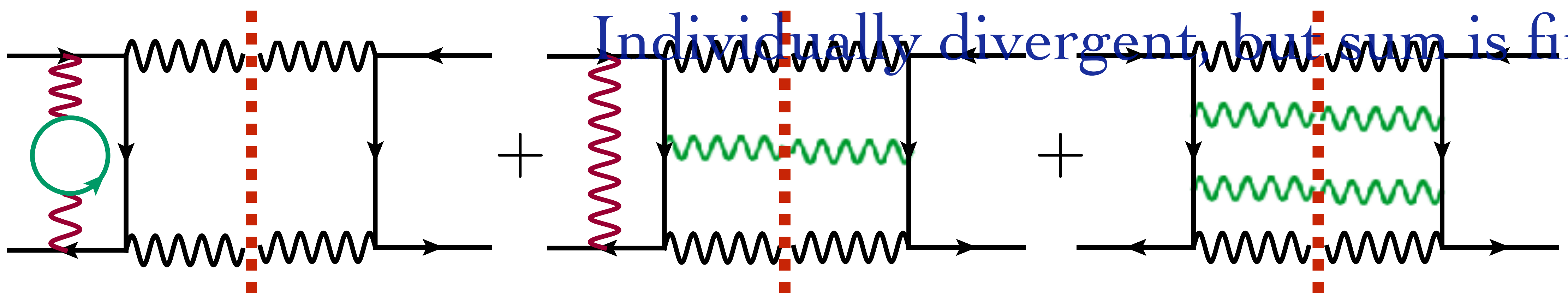


Virtual

Real

- Next-to-next-to-LO (NNLO):

Individually divergent, but sum is finite.



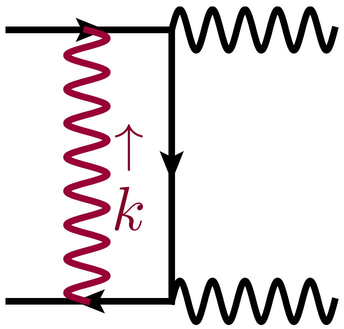
We know how to combine reals and virtuals at NLO and NNLO.

[NNLO: Anastasiou, Melnikov, Petriello; Catani, de Florian, Grazzini; Gehrmann, Gehrmann-de Ridder, Glover; Czakon; Czakon, Fiedler, Mitov; Caola, Melnikov, Schulze; Caola, Melnikov, Röntsch; Gaunt, Stahlhofen, Tackmann, Walsh; Boughezal, Focke, Giele, Liu, Petriello; Cacciari, Dreyer, Karlberg, Salam, Zanderighi; G.Bevilacqua, A.Kardos, G.Somogyi, Z.Trocsanyi, Z.Tulipant; L.Magnea, L.Maina, G.Pelliccioli, C.Signorile-Signorile, P.Torrielli, S.Uccirati, …]

State of the art

- Loop integrations:

$$\int d^4k$$



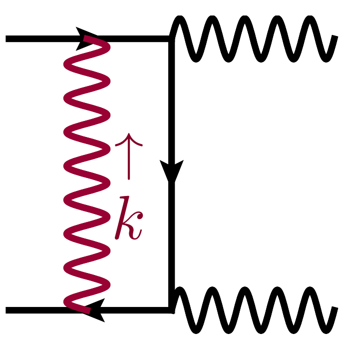
  - ➡ 1 loop: usually doable.
  - ➡ 2 loops: typically $2 \to 2$ and $2 \to 3$ with massless particles.
  - ➡ 3 loops: $2 \to 1$ and first $2 \to 2$ with massless particles.

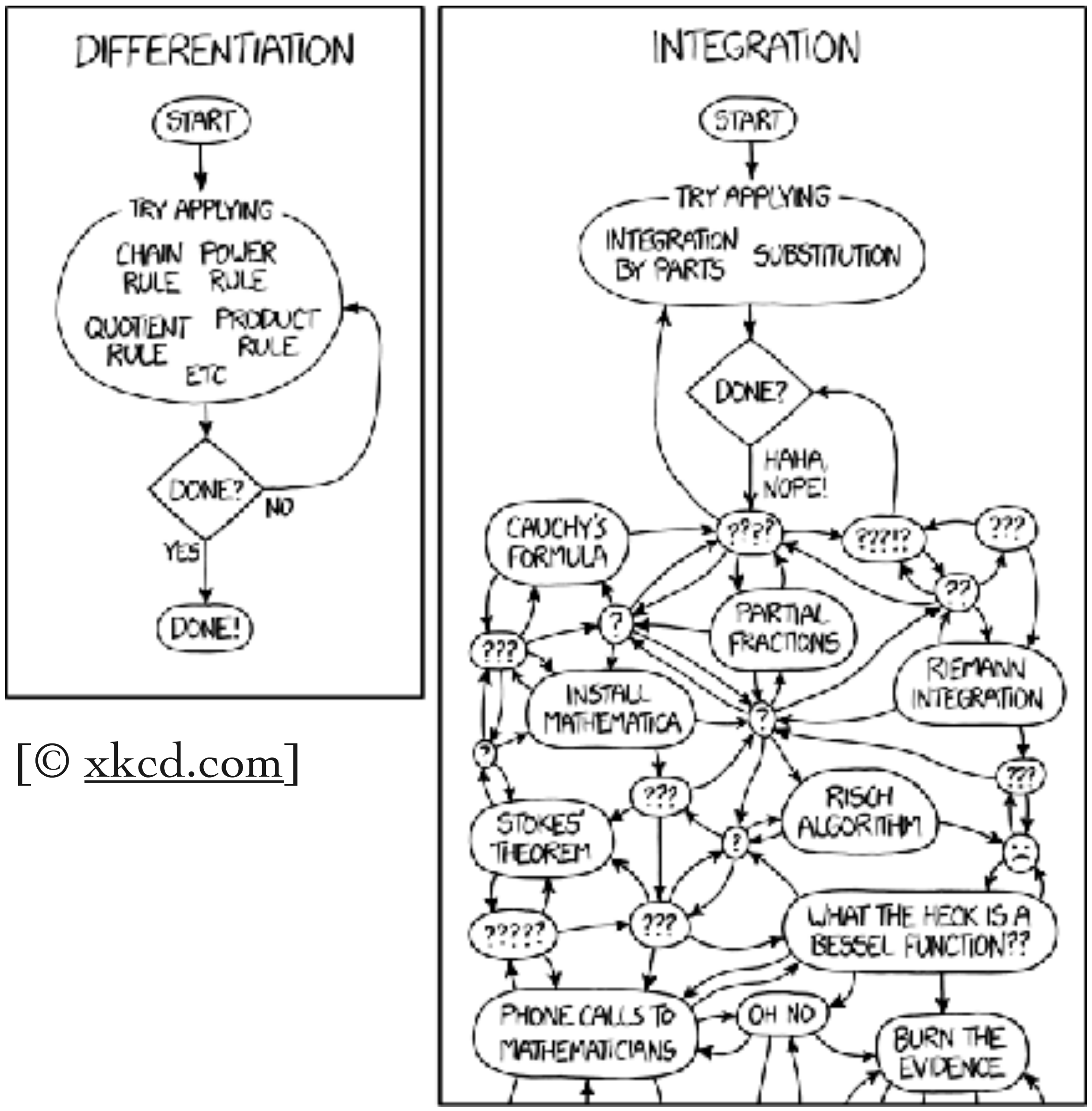
- Combing real and virtual corrections:
  - ➡ NLO: usually doable.
  - ➡ NNLO: typically $2 \to 2$ and first $2 \to 3$.
  - ➡ N3LO: $2 \to 1$.

Outline

- In the rest of the talk: focus (mostly) on virtual contributions.
- Virtual corrections require the integration over momentum of unresolved particle.

$$\int d^4k$$





[© xkcd.com]

Outline of the talk:

➡ The frontier of precision computations for the LHC.

➡ The mathematics of loop computations.

The frontier of precision computations for the LHC

Loop integrations

Numerical

- ✘ Often very slow.
- ✘ Potential large cancellations & instabilities.
- ✓ In principle applicable to any integral!

Analytical

- ✓ Fast & Reliable.
- ✓ All cancellations analytic.
- ✘ Analytic computations tough: 'every integral is different'!

State of the art

2-to-2 with massive propagators.

2-to-2 with massless propagators (no virtual tops!).

Examples:

$t\bar{t}$ HH Hj

$\gamma\gamma(\text{finite } m_t)$

jj $Hj(m_t = \infty)$ $\gamma\gamma$

Vj VV VH

State-of-the-art NNLO

- Fully differential predictions for $2 \to 2$ processes at NNLO are becoming the standard, e.g.:

$$pp \to t\bar{t}$$

[Czakon, Fiedler, Mitov; Catani, Devoto, Grazzini, Kallweit, Mazzitelli]

$$pp \to \gamma + j$$

[Chen, Gehrmann, Glover, Höfer, Huss]

$$pp \to H + j$$

[Boughezal, Caola, Melnikov, Petriello; Schulze; Boughezal, Focke, Giele, Liu, Petriello; Chen, Gehrmann, Glover, Jaquier]

$$pp \to j\,j$$

[Currie, Gehrmann-de Ridder, Gehrmann, Glover, Huss, Pires; Czakon, van Hameren, Mitov, Poncelet]

$$pp \to V + j$$

[Boughezal, Focke, Liu, Petriello; Boughezal, Campbell, Ellis, Focke, Liu, Petriello; Gehrmann-de Ridder, Gehrmann, Glover, Huss, Morgan]

$$pp \to \gamma\,\gamma$$

[Catani, Cieri, de Florian, Ferrera, Grazzini]

$$pp \to V\,V'$$

[Cascioli, Gehrmann Grazzini, et al.; Gehrmann, Grazzini, Kallweit, et al.; Grazzini, Kallweit, Wiesemann, Yook]

$$pp \to V\,H$$

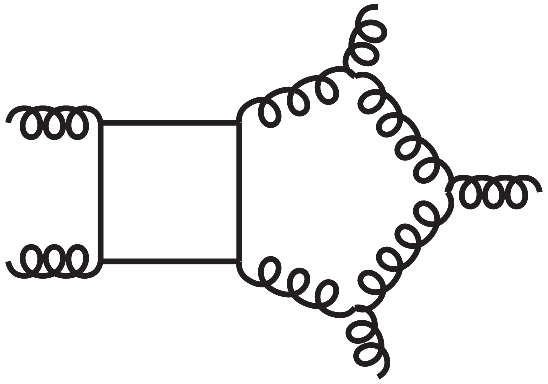
[Ferrera, Grazzini, Tramontano; Gauld, Gehrmann-de Ridder, Glover, Huss, Majer]

- The relevant two-loop virtual integrals are mostly known (analytically or numerically).

  ➡ Frontier: two-loop computations with massive propagators.

5-parton scattering

- Algebraic complexity (‘bookkeeping of algebraic expressions’):
  - ➡ Increase in the number of scales!
  - ➡ Swell in algebraic complexity: standard computer algebra tools cannot handle this!



✓ Numerical ways of dealing with algebraic complexity.
[Badger et al.; Ita et al.; Peraro; …]

✓ Heavily inspired by computational algebraic geometry.

- Analytic complexity (‘doing the integrals’):
  - ➡ Complicated special functions in many variables.

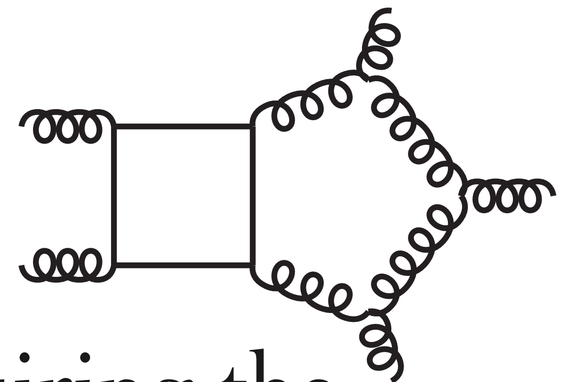
✓ Better understanding of how to perform loop integrals. [Henn]

✓ Some developments were inspired by modern mathematics and/or more formal areas of physics. [Chicherin, Henn, Sokatchev; Chicherin, Henn, Mitev; …]

The 2-to-3 frontier

- Two-loop integrals for 5-point functions (with massless propagators) are slowly becoming available.

  [Gehrmann, Henn, Lo Presti; Papadopoulos, Tommasini, Wever; Gehrmann, Henn, Wasser, Zhang, Zoia; Abreu, Ita, Moriello, Page, Tschernow]



  ➡ Extremely challenging computation, often requiring the development of novel computational techniques and/or new insight from mathematics.

- This opens the way for two-loop amplitudes for $2 \to 3$ processes at the LHC:

$$pp \to 3j$$

[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia; Abreu, Dormans, Frebres Cordero, Ita, Page, Sotnikov]

$$pp \to 3\gamma$$

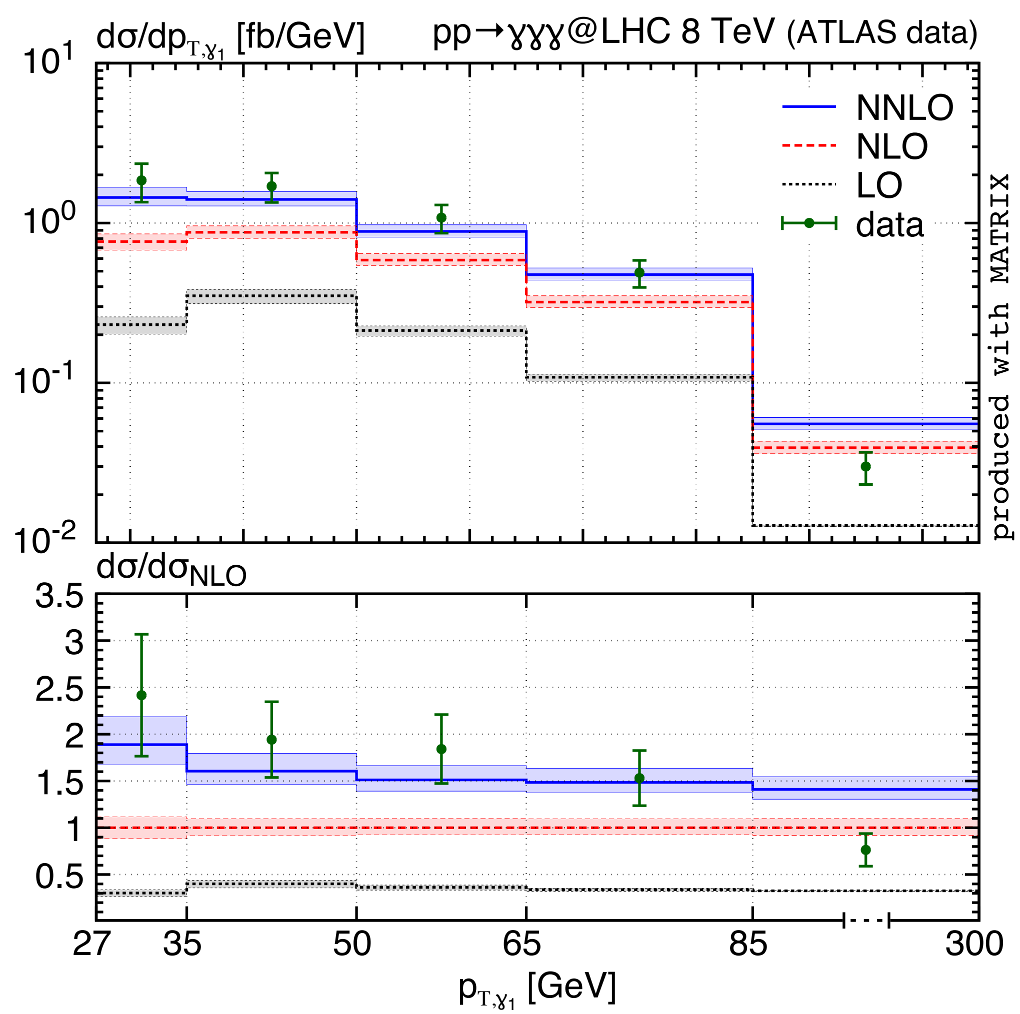
[Abreu, Page, Pascual, Sotnikov; Chawdhry, Czakon, Mitov, Poncelet]

$$pp \to 2\gamma + j$$

[Agarwal, Buccioni, von Manteuffel, Tancredi; Chawdhry, Czakon, Mitov, Poncelet]

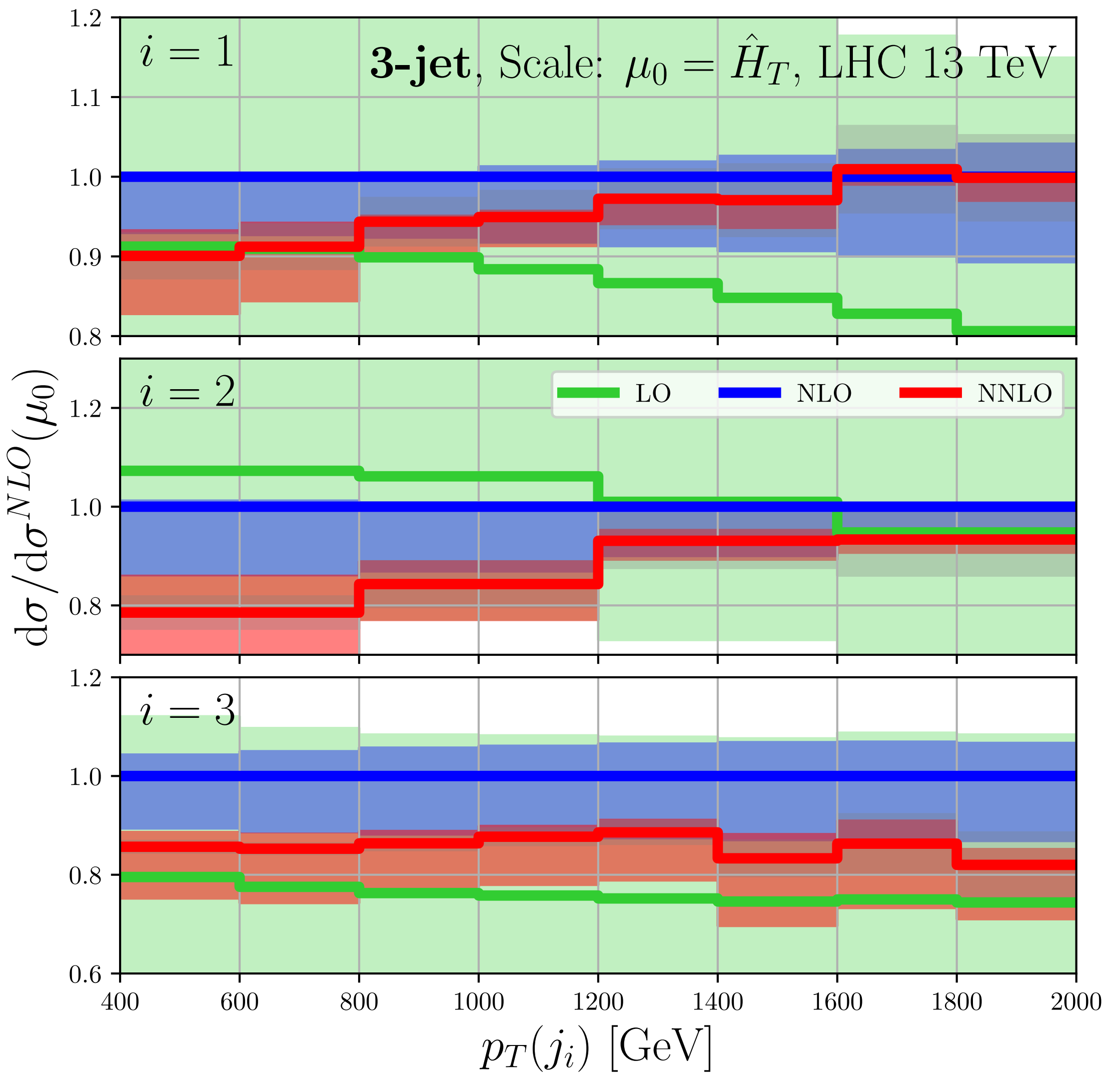
The 2-to-3 frontier

- Over the last year, the first NNLO predictions for $p\,p \to 3\gamma$, $p\,p \to 2\gamma + j$ and $p\,p \to 3j$ have been published.



$p\,p \to 3\gamma$

[Kallweit, Sotnikov, Wiesemann]

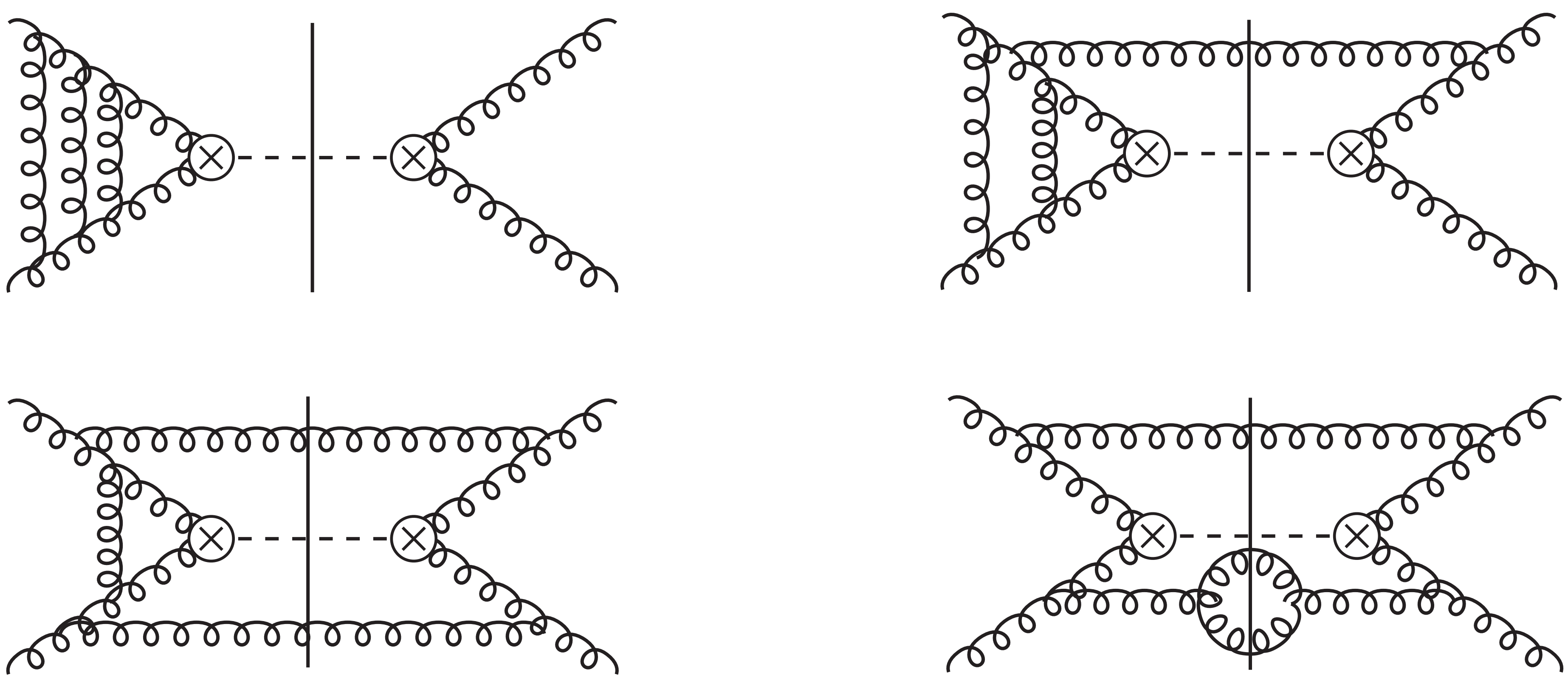


$p\,p \to 3j$

[Czakon, Mitov, Poncelet]

The dawn of N^3LO

- First N^3LO computation for the LHC: Higgs production in gluon fusion (in large m_t - limit).
- Algebraic complexity under control:
  - ➡ ~100.000 diagrams & 1.028 integrals.

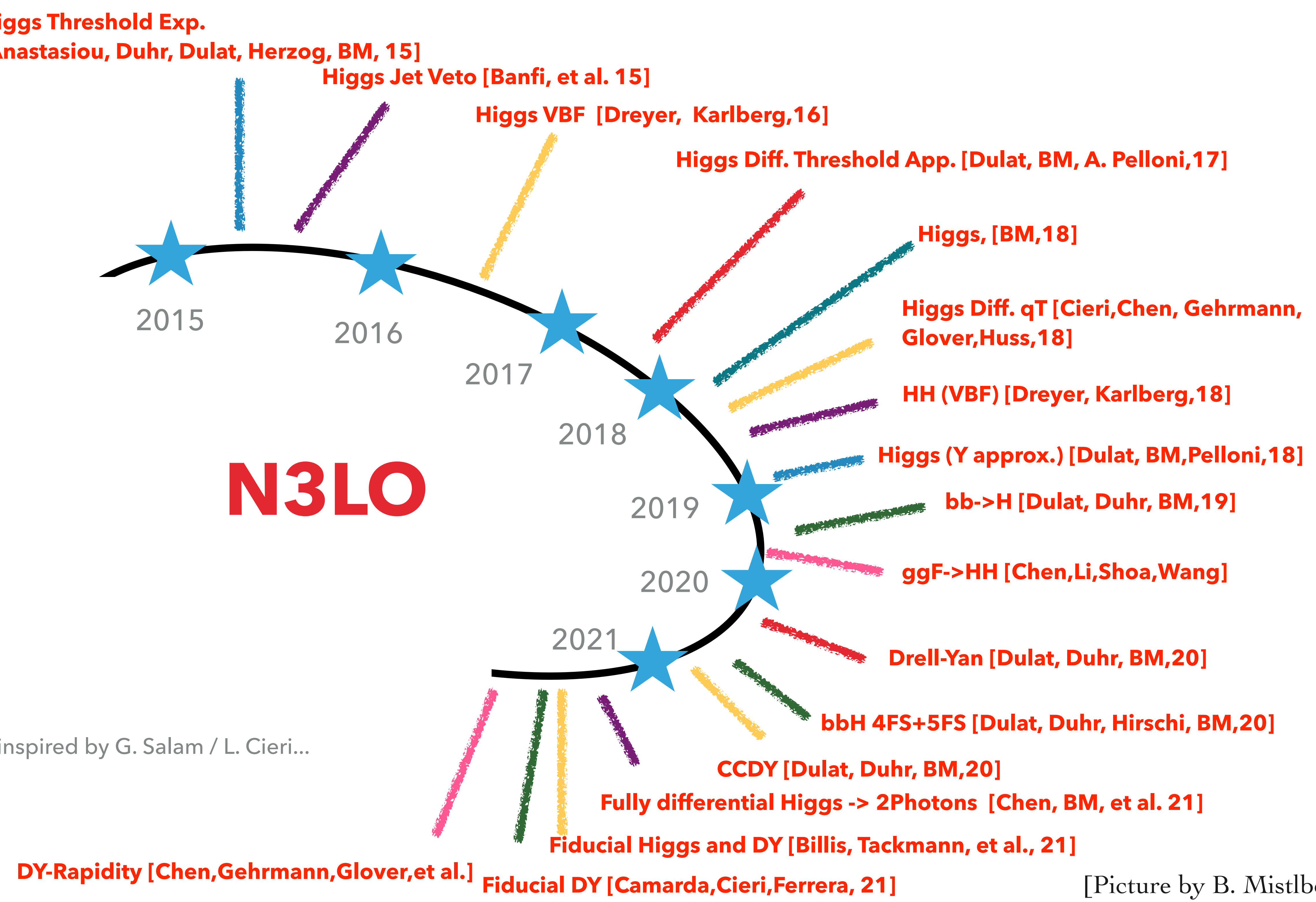


[Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger]

- Analytic complexity an issue: Elliptic functions show up…

The dawn of N^3LO

- N^3LO corrections to $2 \to 1$ processes are mature.

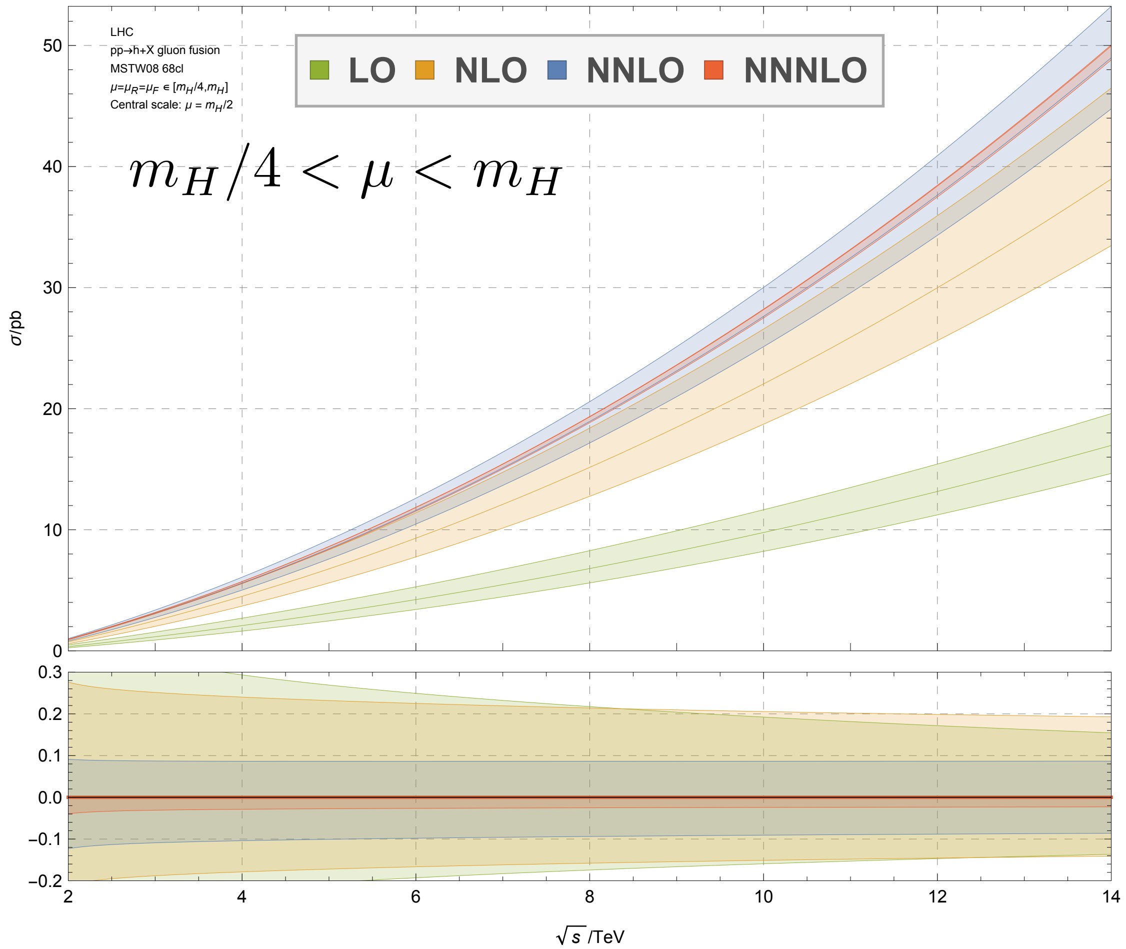


Slide inspired by G. Salam / L. Cieri...

[Picture by B. Mistlberger]

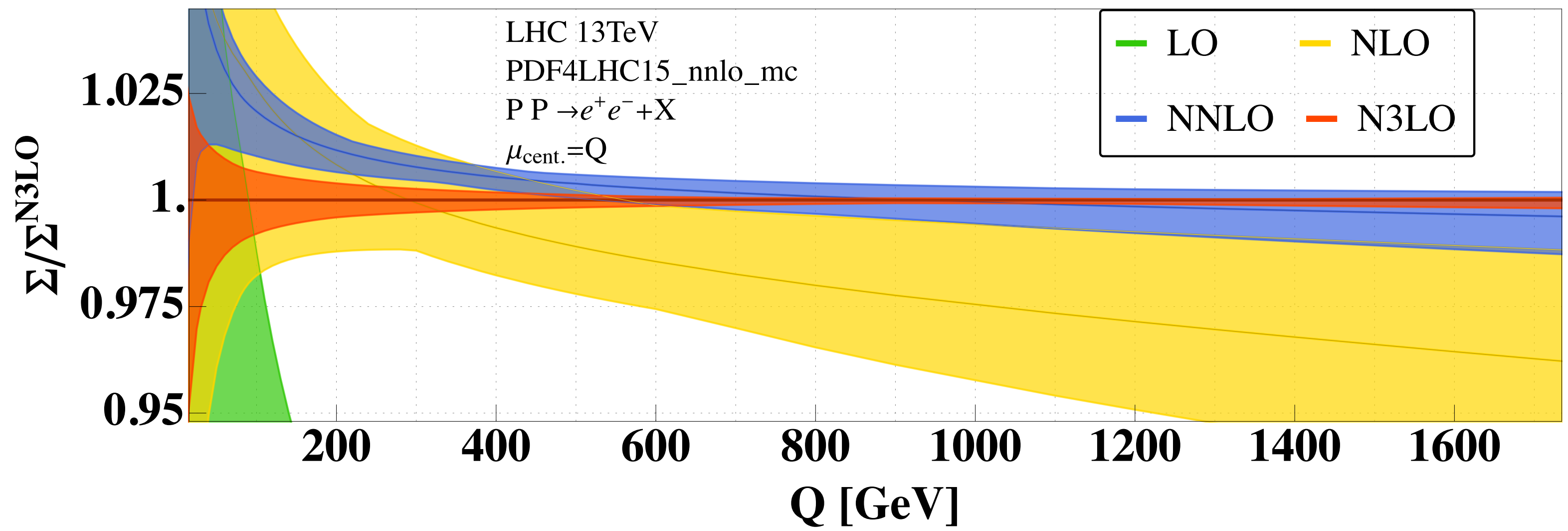
N^3LO for 2-to-1 processes

Gluon fusion: $g\,g \to H$



[Anastasiou, CD, Dulat, Herzog, Mistlberger; Mistlberger]

Drell-Yan: $q\,\bar{q} \to Z/\gamma^* \to e^+\,e^-$



[CD, Dulat, Mistlberger]

	Q [GeV]	K-factor	δ(scale) [%]	$\delta(\text{PDF}+\alpha_S)$	δ(PDF-TH)
$gg \to$ Higgs	m_H	1.04	$^{+0.21\%}_{-2.37\%}$	$\pm 3.2\%$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	0.978	$^{+3.0\%}_{-4.8\%}$	$\pm 8.4\%$	$\pm 2.5\%$
NCDY	30	0.952	$^{+1.53\%}_{-2.54\%}$	$^{+3.7\%}_{-3.8\%}$	$\pm 2.8\%$
	100	0.979	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm 2.5\%$
CCDY(W^+)	30	0.953	$^{+2.5\%}_{-1.7\%}$	$\pm 3.95\%$	$\pm 3.2\%$
	150	0.985	$^{+0.5\%}_{-0.5\%}$	$\pm 1.9\%$	$\pm 2.1\%$
CCDY(W^-)	30	0.950	$^{+2.6\%}_{-1.6\%}$	$\pm 3.7\%$	$\pm 3.2\%$
	150	0.984	$^{+0.6\%}_{-0.5\%}$	$\pm 2\%$	$\pm 2.13\%$

[Table from Snowmass '21 Whitepaper [2203.06730]]

➡ K-factors (N^3LO/NNLO) ~ 2-5 %.

➡ Scale dependence & PDF uncertainties ~ few %.

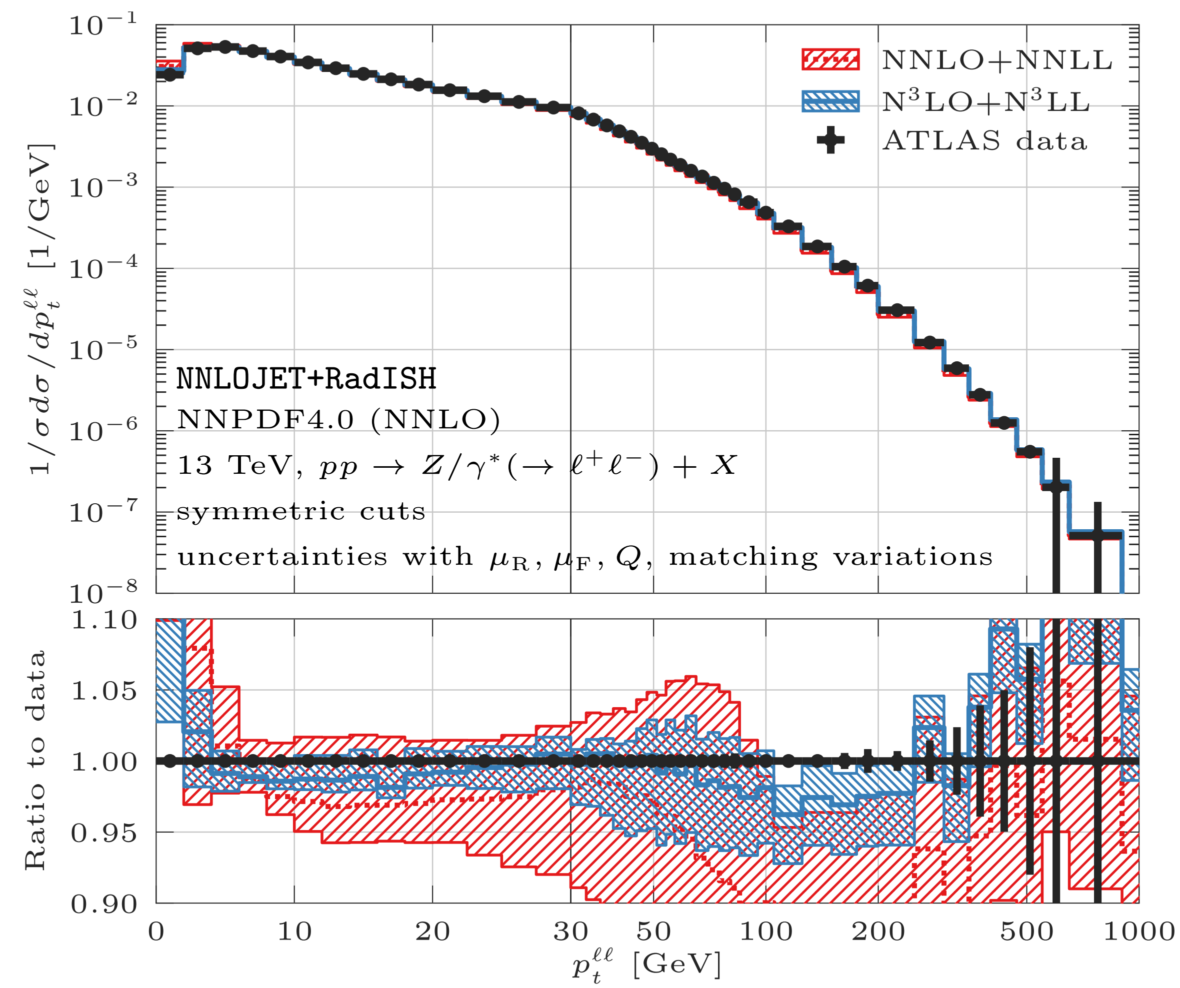
N^3LO for 2-to-1 processes

- We are also starting to see the first differential predictions for $2 \to 1$ processes at N^3LO .
  - ➡ For example fiducial cross section for Drell-Yan production in the fiducial volume at N^3LO+N^3LL:

$$66\,\text{GeV} < m_{\ell\ell} < 116\,\text{GeV}$$
$$|\eta^{\ell^\pm}| < 2.5$$
$$|\vec{p}_T^{\,\ell^\pm}| > 27\,\text{GeV}$$

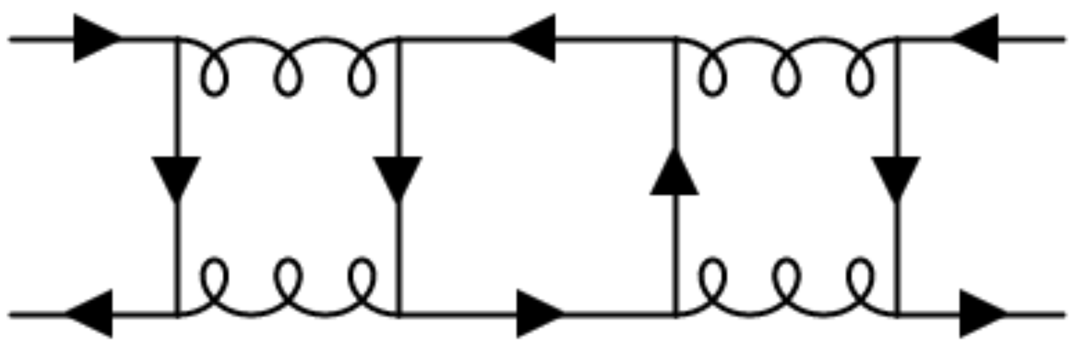
Order k	σ [pb] Symmetric cuts N^kLO	N^kLO+N^kLL
0	$721.16^{+12.2\%}_{-13.2\%}$	—
1	$742.80(1)^{+2.7\%}_{-3.9\%}$	$748.58(3)^{+3.1\%}_{-10.2\%}$
2	$741.59(8)^{+0.42\%}_{-0.71\%}$	$740.75(5)^{+1.15\%}_{-2.66\%}$
3	$722.9(1.1)^{+0.68\%}_{-1.09\%} \pm 0.9$	$726.2(1.1)^{+1.07\%}_{-0.77\%}$

[Chen, Gehrmann, Glover, Huss, Monni, Rottoli, Re, Torielli]



The 3-loop frontier

- Recently also the first 3-loop integrals contributing to $2 \to 2$ scattering have started to appear. [Henn, Mistlberger, Smirnov, Wasser]



- We also have first results for 3-loop processes at the LHC:

$$q\,\bar{q} \to \gamma\gamma \qquad q\,\bar{q} \to q\,\bar{q} \qquad g\,g \to g\,g$$

[Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi]

➡ Together with the 2-loop corrections to $p\,p \to 2\gamma + j$, these are all the loop corrections needed to compute diphoton production at $\mathrm{N^3LO}$.

The mathematics of loop computations

Scattering amplitudes

- What kind of objects are scattering amplitudes?
- Scattering amplitudes are functions of the energies and momenta of the scattered particles.

➡ What kind of functions?

- Unitarity: Probabilities must add up to 1.

➡ Consequence: Amplitudes must have discontinuities:

$$\text{Disc}\,\mathcal{A} \neq 0$$

➡ They cannot be simple polynomials or rational functions.

The logarithm & beyond

- One-loop integrals:

$$\log z = \int_1^z \frac{dt}{t} \qquad\qquad \mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t}\log(1-t)$$

➡ Extension to two-loop integrals with massless partons.

- Beyond one loop, also multiple polylogarithms appear:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

$$G(0;z) = \log z$$

$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0,1;z) = -\mathrm{Li}_2(z)$$

[Poincaré; Kummer; Lappo-Danilevsky; Goncharov; …]

➡ Well understood mathematics! [Goncharov; Brown; Goncharov, Spradling, Vergu, Volovich; CD; Panzer; …]

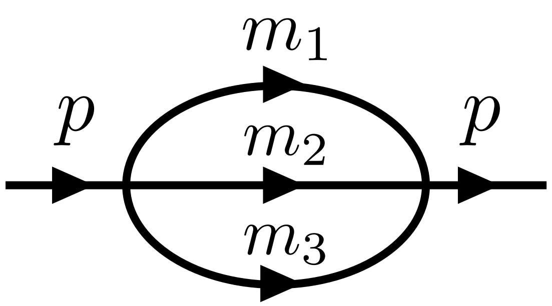
From mathematics to physics

- Recent pheno applications (non-exhaustive!):
  - ➡ N^3LO cross sections (H, DY, W). [Anastasiou et al.; CD, Dulat, Mistlberger]
  - ➡ 2-to-3 processes at 2 loops. [Papadopoulos et al.; Abreu et al.; Badger et al.]
  - ➡ 4-loop cusp anomalous dimension. [Henn, Korchemsky, Mistlberger]
  - ➡ 3-loop Soft Anomalous Dimensions. [Almelid, CD, Gardi]

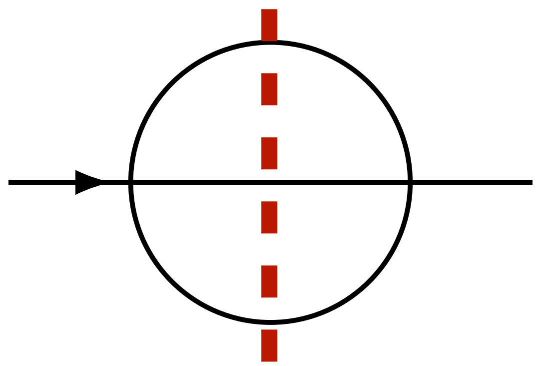
- New insight into (planar) N=4 SYM:
  - ➡ 6-point planar amplitude through 7 loops.
  - ➡ 7-point planar amplitude through 4 loops. [Dixon et al.]
  - ➡ 3-loop 4-point non-planar amplitude. [Henn, Mistlberger, Smirnov, Wasser]

The elliptic sunrise

- Polylogarithms are not the end of the story.
- Prototype example: the massive sunrise graph.



➡ Look at maximal cut (in D=2): [Laporta, Remiddi]



$$= \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{dx}{\sqrt{R_2(x,m_2^2,m_3^2)R_2(s,x,m_1^2)}}$$

$$R_2(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$$

➡ Evaluates to a (complete) elliptic integral of the 1st kind:

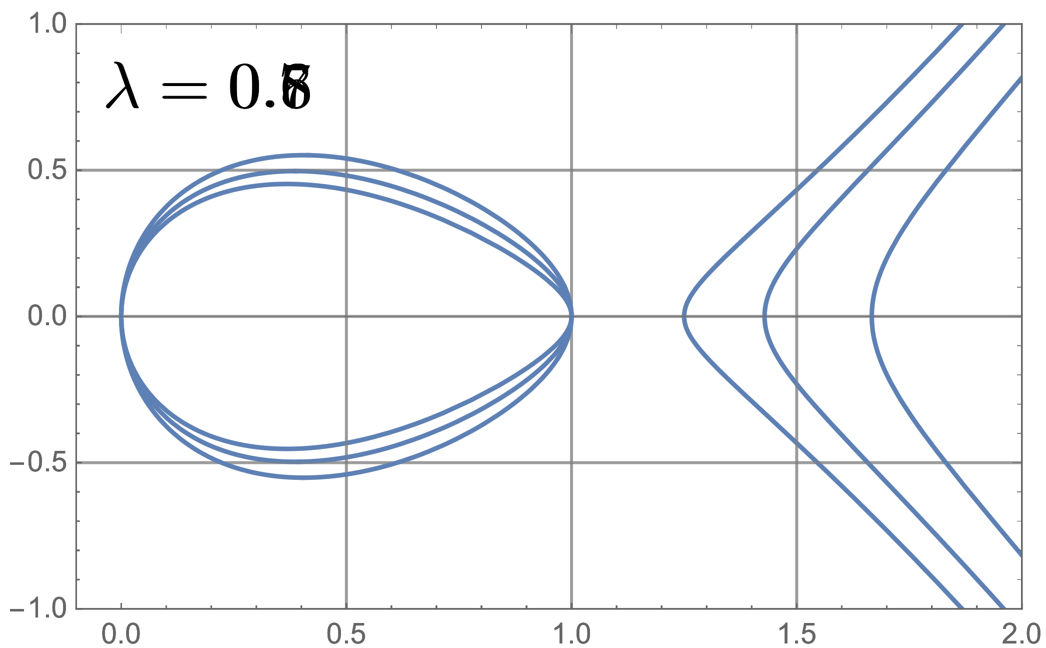
$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$$

The elliptic sunrise

- Elliptic curve ~ set of points (x, y) that satisfy an equation of the form $y^2 = P(x)$, where $P(x)$ is a polynomial of degree 3 or 4.
- Example:

$$y^2 = x(x-1)(x-1/\lambda) \qquad y^2 = R_2(x, m_2^2, m_3^2) R_2(s, x, m_1^2)$$

➡ cf. the elliptic integral: $\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$



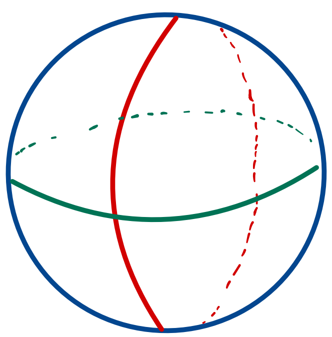
➡ Kinematics encoded in 'shape' of the elliptic curve.

➡ When looked over the complex numbers, it becomes torus.

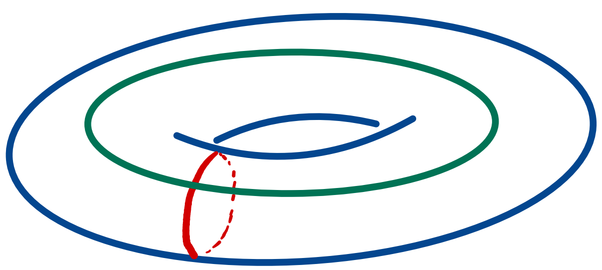
Beyond elliptic

- It does not stop with elliptic curves!
  - ➡ Riemann sphere (~complex plane): polylogarithms.
  - ➡ Elliptic curves: elliptic polylogarithms.
  - ➡ K3 surfaces, Calabi-Yau manifolds,…

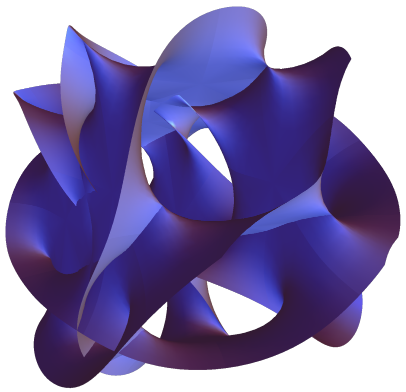
[Brown, Schnetz; Bloch, Kerr, Vanhove; Bourjaily, McLeod, von Hippel, Vergu, Volk, Wilhelm; Bönisch, CD, Fischbach, Klemm, Nega, Safari; CD, Klemm, Loebbert, Nega, Porkert; CD, Klemm, Nega, Tancredi; Pögel, Wang, Weinzierl]



Sphere



Torus (elliptic curve)



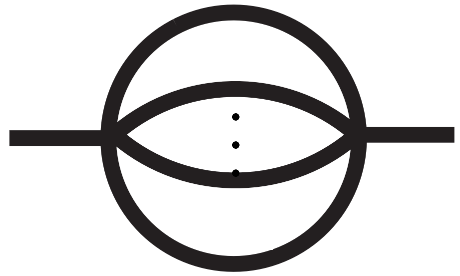
[Picture drawn with Wolfram Demonstrations Project]

Calabi-Yau manifold

- New mathematical structures, with connections to string theory.

Bananas and ice-cones

- The equal-mass banana integrals (in 2D) can be expressed in terms of iterated integrals involving the periods of Calabi-Yau.



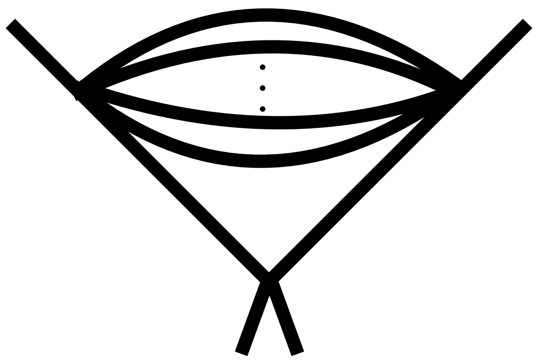
[Bönisch, CD, Fischbach, Klemm, Nega; Pögel, Wang, Weinzierl; CD, Klemm, Nega, Tancredi]

  ➡ Natural iterated integrals on the moduli space of the CY.

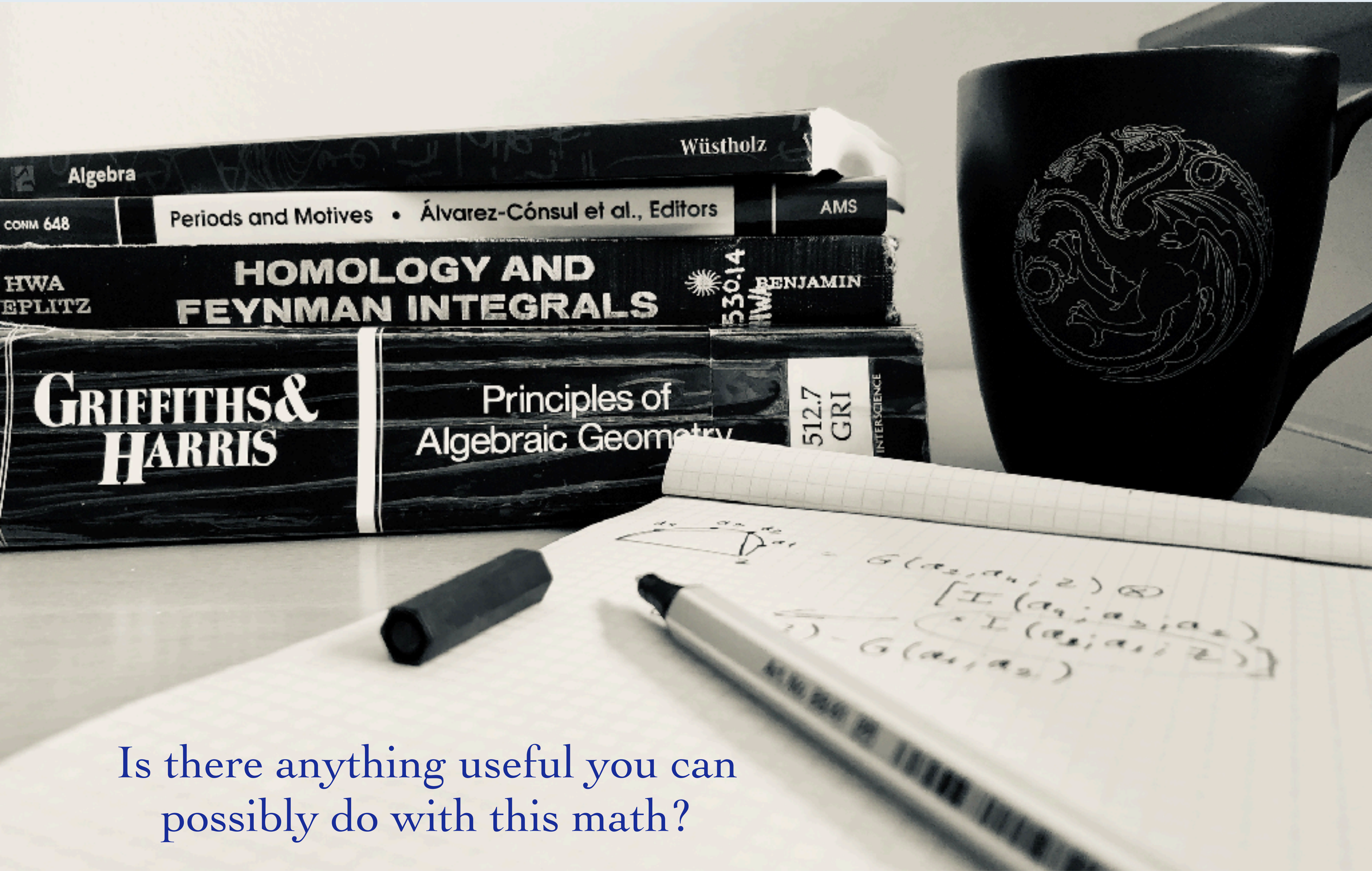
  ➡ Natural generalisation of the iterated integrals of modular forms encountered at 2 loops (and 3 loops).

- The same functions arise from the 'ice-cone' integral:

[CD, Klemm, Nega, Tancredi]



From mathematics to physics



Is there anything useful you can possibly do with this math?

Towards phenomenology

- Are these things relevant for phenomenology?

  ➡ YES!

- Elliptic curves show up in many processes of phenomenological relevance:

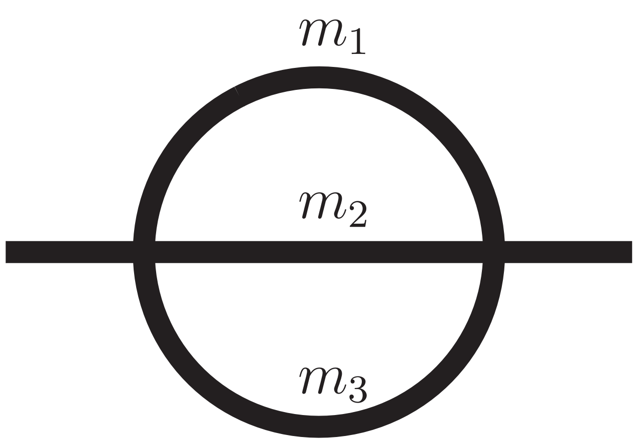
$pp \to Hj$ $\qquad$ $e^+e^- \to e^+e^-$ $\qquad$ $pp \to t\bar{t}j$

$pp \to t\bar{t}$ $\qquad$ $pp \to \eta_c$ $\qquad$ ρ-parameter

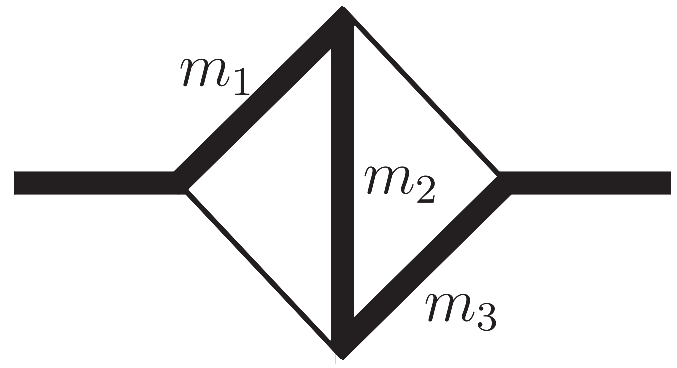
+many more!

- This is one of the major obstacles to obtaining (analytic) results for precision predictions!

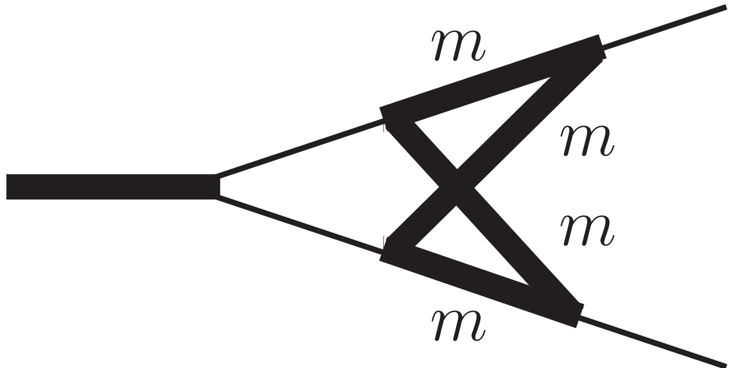
Elliptic Feynman integrals



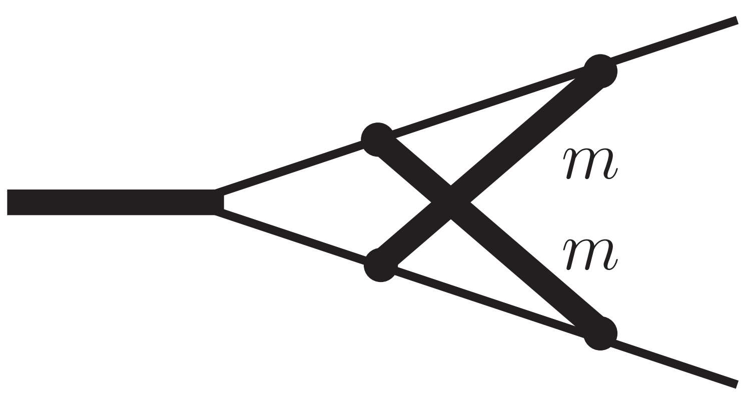
$t\,\bar{t}$



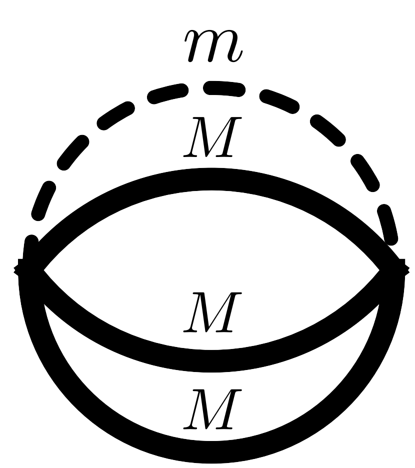
e self-energy



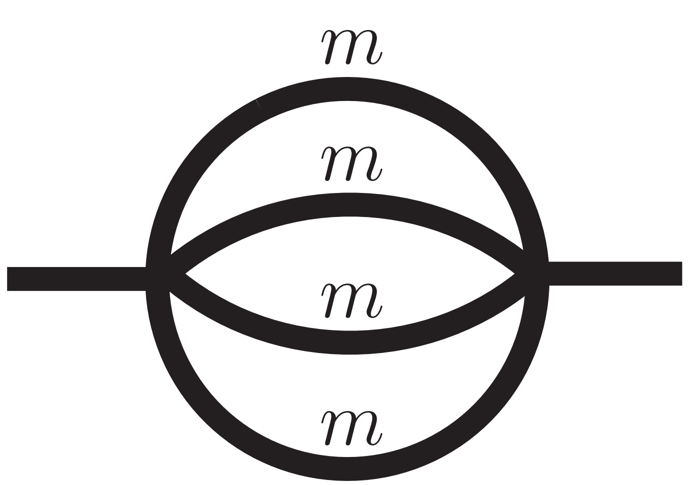
$t\,\bar{t},\ \gamma\,\gamma,$ etc.



EW form factor



ρ-parameter



$g\,g \to H$

[Bloch, Vanhove; Adams, Bogner, Chaubey, Schweitzer, Weinzierl; Brödel, CD, Dulat, Marzucca, Tancredi, Penante; Hiddings, Moriello; ...)]

The next frontier in (analytic) precision computations!

Bhabha scattering @ 2 loops

- One of the key processes at an e^+e^- collider is Bhabha scattering.
  - ➡ Elastic scattering $e^+e^- \to e^+e^-$
  - ➡ We would like to know NNLO QED corrections, retaining the complete dependence on the electron mass.
  - ➡ Missing ingredient: 2-loop corrections.
  - ➡ Open problem for 20 years!

1 Jul 2004

A complete set of scalar master integrals for massive 2-loop Bhabha scattering: where we are *

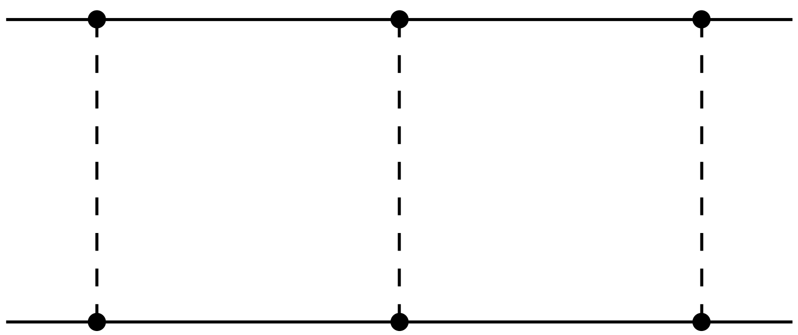
M. Czakon[ab], J. Gluza[ab] and T. Riemann[a]

[a]Deutsches Elektronen-Synchrotron, DESY Zeuthen, Platanenallee 6, 15738 Zeuthen, Germany

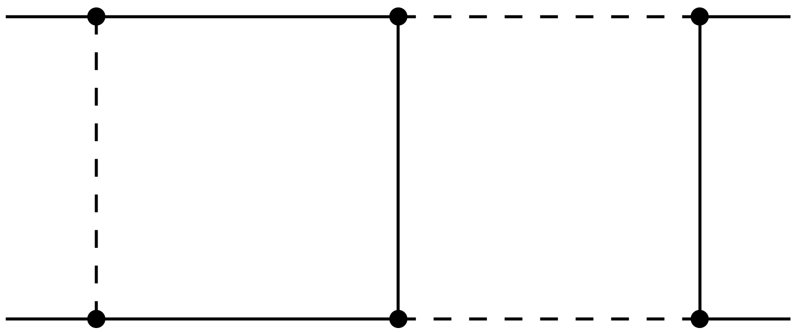
[b]Institute of Physics, University of Silesia, ul. Uniwersytecka 4, 40007 Katowice, Poland

Bhabha scattering @ 2 loops

- 3 integral families:
  - ➡ 2 planar families, expressible in terms of polylogarithms:

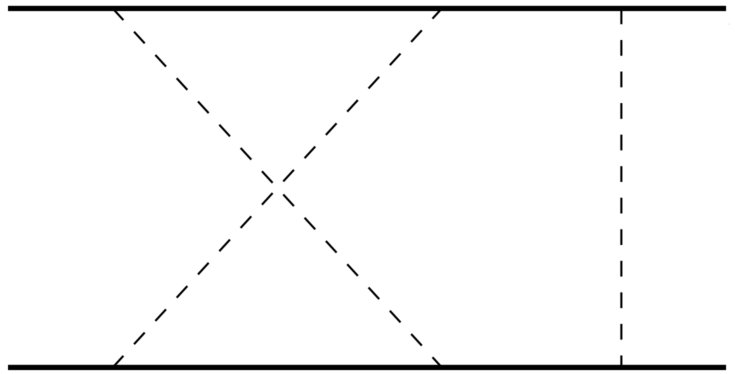


[Henn, Smirnov, Smirnov (2013); Heller, von Manteuffel, Schabinger (2019)]



[CD, Smirnov, Tancredi (2021)]

  - ➡ 1 non-planar family, involves an elliptic curve:



[Delto, CD, Tancredi, Zhu (to appear, 2023)]

Bhabha scattering @ 2 loops

- We have analytic results for the 2-loop QED corrections to Bhabha scattering, including the full dependence on the electron mass! [Delto, CD, Tancredi, Zhu (to appear, 2023)]

  ➡ Both for the polarised and unpolarised cross sections

- Analytic results involve iterated integrals over kernels associated with elliptic curves.

- We are currently comparing to existing approximate 2-loop results.

  ➡ Stay tuned!

Conclusion

- In recent years we have seen many breakthroughs in our ability to perform analytic multi-loop computations.
  - ➡ 2-to-3 processes at two loops are within reach.
  - ➡ N^3LO corrections to 2-to-1 inclusive process are under becoming available, and 3-loop 2-to-2 amplitudes are being explored!
- Loop amplitudes involve very interesting mathematics.
  - ➡ Elliptic curves, Calabi-Yau varieties, …
- Understanding this mathematics will allow us to obtain new results of phenomenological interest.
  - ➡ Example: Complete 2-loop QED corrections to Bhabha scattering.