Scheme Dependence in pQCD at the Four Loop Level

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Abstract

We consider measurable quantities calculated in massless perturbative QCD in a variety of schemes, including the symmetric momentum subtraction schemes, up to the four-loop level in order to investigate scheme dependence in the perturbative series as a theory lab. Appropriate error values should be attached to estimates of physical results which can be done by asserting that renormalization group invariance is true to order in truncation. Scheme invariance is investigated as a measure of error using measurements to calculate $\alpha_{\overline{\text{MS}}}(M_Z)$ from different schemes as a point of comparison for the results.

Renormalization

Renormalization is the procedure through which divergences in interacting Quantum Field Theories are removed by mapping variables onto renormalized parameters e.g.

$$\psi \to Z_\psi^{-\frac{1}{2}} \psi_0 = \psi^R, \tag{1}$$

and requiring the bare Green's functions are finite.

The conditions ensuring this is realised define the scheme, practically this means selecting for the finite components of the renormalization group functions e.g. Z_ψ.

Measurable quantities should not depend on the choice of scheme as this is a calculational artifact, however in perturbation theory this can only be ensured to order in truncation, i.e., the difference of the values calculated for a measurable ρ in two schemes S_1 and S_2 up to order N is

$$\rho_{(N)}^{S_1}(Q) - \rho_{(N)}^{S_2}(Q) = \mathcal{O}(a_0(Q)^{N+1}) \tag{2}$$

where $a_0(Q)$ is the leading order coupling constant.

Kinematic and Non-Kinematic Schemes

The Minimal Subtraction scheme (MS) sets the finite contribution to the counter-terms to zero and the modified Minimal Subtraction scheme ($\overline{\text{MS}}$) changes MS by defining counter-terms to remove factor of $4\pi e^{-\gamma}$ from calculations.

These schemes are computationally simple because the finite order contributions from the ϵ expansion in dimensional regularisation do not need to be calculated.

However, these schemes offer no reference to kinematics and can be expensive on the lattice making matching calculations more difficult.

Additionally, MS schemes are in many ways special, so it is difficult to draw universal conclusions about the underlying theory from these schemes alone therefore it is worthwhile considering other schemes.

One category of kinematic scheme under particular consideration here are the symmetric momentum subtraction schemes, which are defined by requiring

- Gluon two point function has no loop corrections at a characteristic momenta $p^2 = -M^2$
- Characteristic vertex function has no finite order corrections in symmetric momentum configuration $p_i^2 = -M^2$

where the triple gluon vertex is used to define MOMg, quark-gluon is used to define MOMq and ghost-gluon is used to define MOMc.

Changing Schemes

Four loop series for measurable ρ in scheme S_1:

$$\rho(Q) \approx \rho_0^{S_1} + \rho_1^{S_1} a_{S_1}(Q) + \rho_2^{S_1} a_{S_1}^2(Q) + \rho_3^{S_1} a_{S_1}^3(Q) + \rho_4^{S_1} a_{S_1}^4(Q) \tag{3}$$

Process of changing schemes:

- Apply perturbative expansion of coupling constant conversion functions calculated from $a^{S_1} = (Z_g^{S_2}/Z_g^{S_1})^2 a^{S_2}$and truncate series to current loop order

$$\rho(Q) \approx \rho_0^{S_1} + \rho_1^{S_1} a_{S_2}(Q) + (\rho_2^{S_1} + c_1^{S_1,S_2} \rho_1^{S1}) a_{S_2}^2(Q) + (...) a_{S_2}^3(Q) + (...) a_{S_2}^4(Q) \tag{4}$$

- Write running in terms of renormalization scale

$$a^S(Q, \Lambda^S) \approx \frac{1}{\beta_0^S L^s}\left[1 - \frac{\beta_1^S \ln(L^S)}{\beta_0^{S^2} L^S} + \frac{\beta_1^{S^2}(\ln^2(L^S) - \ln(L^S) - 1) + \beta_0^S \beta_2^S}{\beta_0^{S^4} L^{S^2}} + ...\right] \tag{5}$$

which is calculated by perturbatively solving from the beta function $\frac{d}{dl}a = \beta(a)$.

- Find common unit with Λ ratio

$$\frac{\Lambda_{S_1}}{\Lambda_{S_2}} = \exp\left(\frac{c_1^{S_1,S_2}}{2\beta_0}\right) \tag{6}$$

which is exact [4] and can be found by comparing expansion $a^S(Q, \Lambda^S)$ and the coupling constant conversion functions.

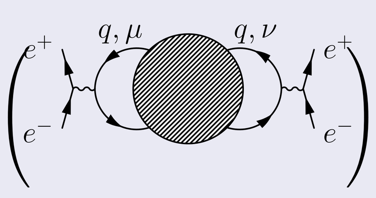
R-Ratio

We will here focus on the R-Ratio, which is a measurable quantity calculable in perturbative QCD based on the cross-section of electron-positron annihilation resulting in hadronic products, normalised by the cross-section of electron-positron annihilation resulting in a muon-anti-muon pair, given by

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{i=0} R_i a_S^i = R_0 + R_1 a_{RR}. \tag{7}$$

Typically this is calculated using the optical theorem to find the corresponding square matrix element

$$|M(e^+e^- \to \text{partons}|^2 = -2\text{Re}$$



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Theory Error In pQCD

Quantities calculated in perturbation theory are only approximations, so a measure of accuracy should be applied to these estimates. Measurable objects are only renormalization group invariant up to order in truncation and any deviations from it will only exist at $\mathcal{O}(a^{N+1})$. This can be used to generate a theory error.

For example, for measurable ρ we have $\mu\frac{d}{d\mu}\rho(Q,\mu) = 0$ to all orders, so by considering the quantity at different scales we can construct the scale error, the standard theory error quoted.

As has been explored in e.g. [7], we can also use scheme difference as a measure of error. Consider we have calculated the measurable in two schemes to order N

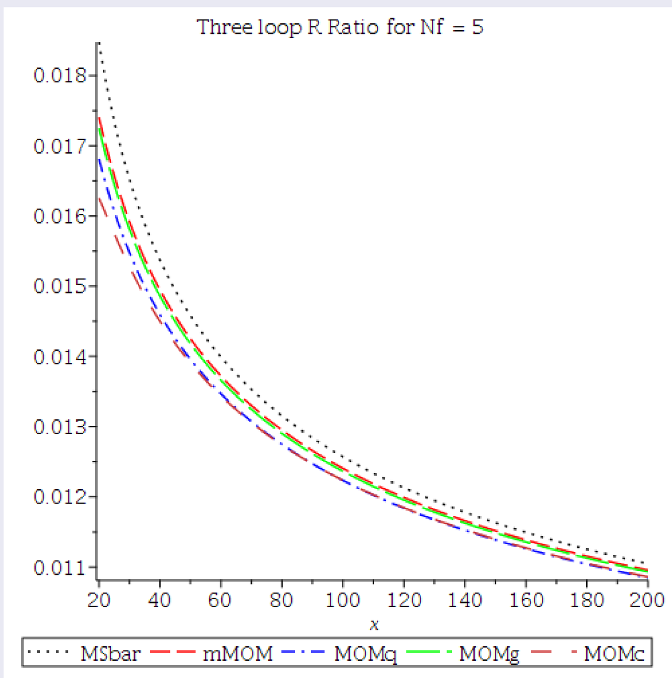
$$\rho(Q) = \rho_{(N)}^{S_1} + \Delta(\rho^{S_1}, N, Q) = \rho_{(N)}^{S_2} + \Delta(\rho^{S_2}, N, Q) \tag{9}$$

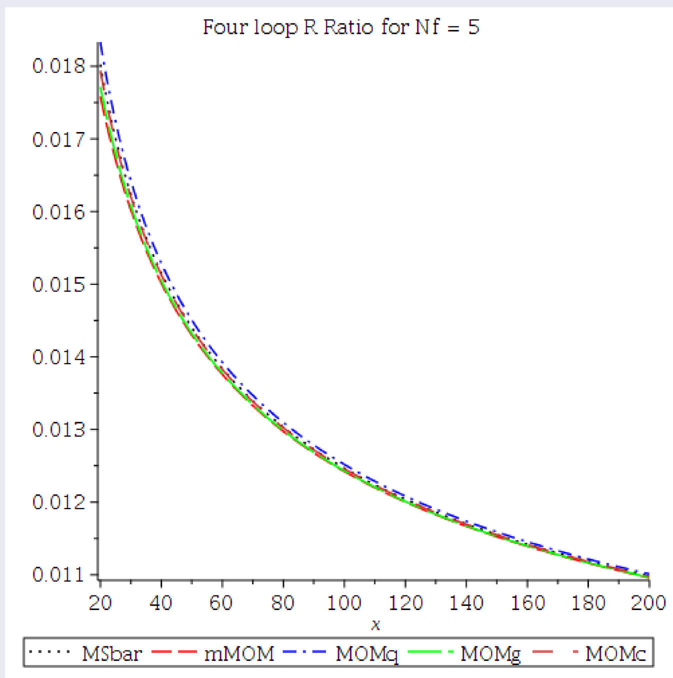
where $\Delta(\rho^S, N, Q)$ represents our ignorance in the higher order of the series.

If the sign of $\Delta(\rho^{S_1}, N, Q)$ and $\Delta(\rho^{S_2}, N, Q)$ are different then $\rho_{(N)}^{S_1}$ and $\rho_{(N)}^{S_2}$ bound the true value $\rho(Q)$, otherwise it will not be. By considering more schemes we may improve the probability that true series lies in envelope given by the largest and smallest value of $\rho_{(N)}^{S_i}$.

Graph

The four-loop R-Ratio calculation was completed for $\overline{\text{MS}}$ in [6] and the conversion functions [3] could then be used to change the scheme for comparison. Below are graphs comparing the R-Ratio effective coupling constant with $N_f = 5$ active quarks evaluated at $Q = x\Lambda_{\overline{\text{MS}}}$ in the $\overline{\text{MS}}$, mMOM, and MOMi schemes to the three- and four-loop level.





Clearly there is a large reduction in scheme difference between three- and four-loop. A quantitative comparison can be made by solving numerically for the formal parameter $\alpha_{\overline{\text{MS}}}(M_Z) = 4\pi a_{\overline{\text{MS}}}(M_Z)$ using experimental data of $R = R^*$ at momentum $Q = Q_E$, as is given below where the 'Average' represents the scheme envelope of the measurement.

$\alpha_{RR} = 0.13697 \pm 0.01225$ at $Q_E = 82.15838$ GeV [5]					
Scheme	N	$\alpha_{\overline{\text{MS}}}(M_Z)$	Scheme	N	$\alpha_{\overline{\text{MS}}}(M_Z)$
$\overline{\text{MS}}$	2	$0.12725^{+0.01066}_{-0.01053}$	MOMc	2	$0.12677^{+0.01054}_{-0.01039}$
	3	$0.12982^{+0.01126}_{-0.01124}$		3	$0.13201^{+0.01203}_{-0.01237}$
	4	$0.13056^{+0.01149}_{-0.01154}$		4	$0.13059^{+0.01150}_{-0.01156}$
mMOM	2	$0.12635^{+0.01044}_{-0.01027}$	MOMg	2	$0.12637^{+0.01045}_{-0.01028}$
	3	$0.13085^{+0.01162}_{-0.01174}$		3	$0.13115^{+0.01171}_{-0.01187}$
	4	$0.13081^{+0.01159}_{-0.01169}$		4	$0.13077^{+0.01157}_{-0.01166}$
MOMq	2	$0.12650^{+0.01048}_{-0.01031}$	Average	2	$0.12680 \pm 0.00045^{+0.01066}_{-0.01027}$
	3	$0.13216^{+0.01203}_{-0.01230}$		3	$0.13099 \pm 0.00117^{+0.01203}_{-0.01124}$
	4	$0.13024^{+0.01137}_{-0.01137}$		4	$0.13053 \pm 0.00028^{+0.01159}_{-0.01137}$

Note the small two-loop scheme difference is due to three loop term offering the same sign convention so the envelope does may not include the true value of the series. The large experimental error value means we cannot provide an experimentally interesting result, therefore applying this process to more up to date data may be of phenomenological interest.

Conclusions

In this project we have

- Calculated the perturbative series of the R-Ratio, Adler D function and Bjorken Sum rule up to the four-loop level in the $\overline{\text{MS}}$, mMOM and MOMi schemes and considered their running.
- Investigated scheme dependence and found expected reduction as loop level is increased.
- Considered the meaning of the envelope formed by considering series in different schemes.

To consider

- Running of measurables outside of the Landau gauge for linear-covariant gauge fixing,
- Using more up to date experimental data, applying methods to make perturbation theory more valid at lower energies considered.

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