Higher-spin symmetry in asymptotically flat space-time

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Based on:
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- 2211.16498 with Xavier Bekaert and Andrea Campoleoni

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Free higher-spin fields

- Higher-spin irreducible representations of the Poincaré algebra [Wigner '39]

- Simplest system: massless symmetric gauge fields

$$A_a\,,\;g_{(a_1a_2)}\,,\;\varphi_{(a_1a_2a_3)}\,,\;\ldots\quad\to\quad\varphi_{a(s\geq1)}\quad\text{(shorthand)}$$

- Free gauge theory on background of constant curvature with arbitrary constant Λ [Fronsdal '78, '79]

$$\mathscr{F}_{a(s)}\equiv\bar\nabla^2\varphi_{a(s)}-\bar\nabla_a\bar\nabla\cdot\varphi_{a(s-1)}+\bar\nabla_a\bar\nabla_a\varphi'_{a(s-2)}+\mathscr{O}(\Lambda)=0\quad\text{with}\quad\varphi''_{a(s-4)}=0$$

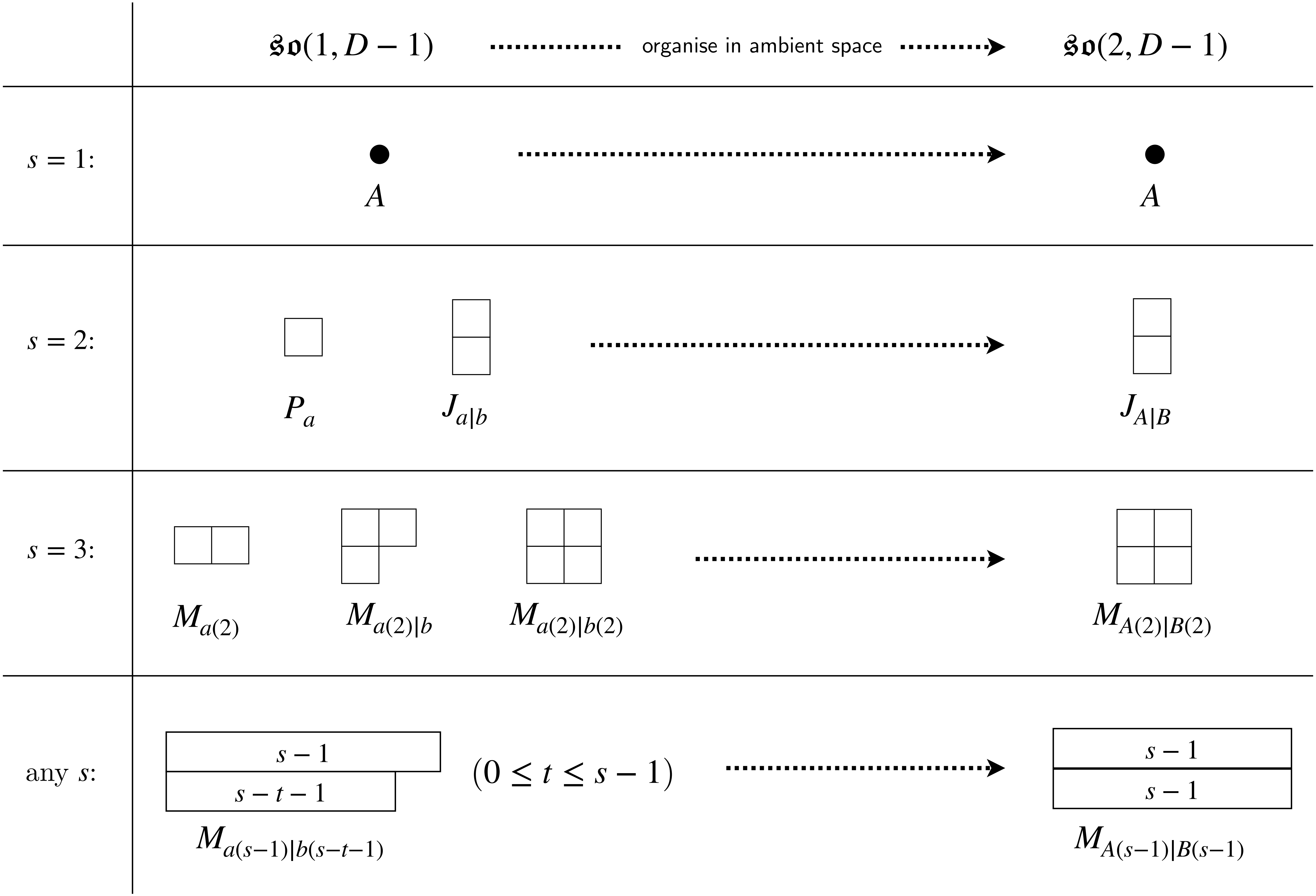
gauge transformations preserving the free e.o.m.

$$\delta\varphi_{a(s)}=\bar\nabla_a\epsilon_{a(s-1)}\quad\text{with}\quad\epsilon'_{a(s-3)}=0\;\;\text{and}\;\;\bar\nabla\cdot\epsilon_{a(s-2)}=0$$

- Vacuum-preserving diffeomorphisms (traceless Killing tensors)

$$\bar\nabla_a\epsilon_{a(s-1)}=0$$

Vacuum-preserving symmetries



The road to an interacting theory

- Noether procedure: deform action and gauge transformations

$$S = S_2 + g\,S_3 + \mathcal{O}(g^2) \quad , \qquad \delta = \delta_0 + g\,\delta_1 + \mathcal{O}(g^2)$$

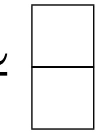
- Cubic interactions are higher-derivative [Berends, van Holten, van Nieuwenhuizen '79] [Aragone, Deser '80]
  There is no $2 - s - s$ term with two derivatives for $\Lambda = 0$ (there is one for $\Lambda \neq 0$ with F-V mechanism)

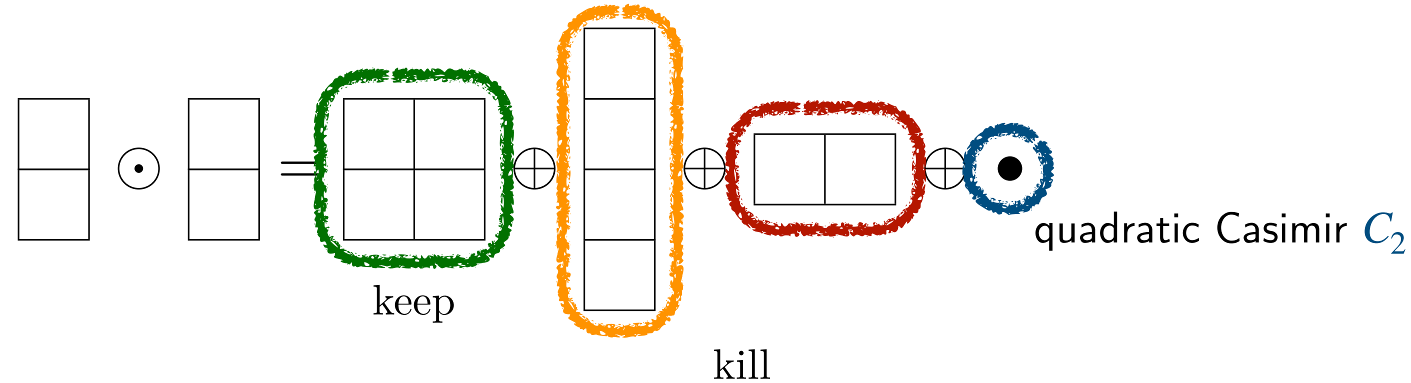
  we now have in general (starting at cubic level)

$$\left[\delta_1\,,\,\delta_0\right]\varphi_{a(s)} = \delta_{0'}\,\varphi_{a(s)}$$

- Defines an associative algebra (in general an algebroid since structure constants can depend on $\varphi_{a(s)}$)
  - for $\Lambda \neq 0$, this algebra was constructed explicitly in [Fradkin, Vasiliev '87]
  - for $\Lambda = 0$, no such algebra exists in the Fronsdal formulation $\rightarrow$ *use a different formulation?*
- Interactions beyond cubic level $\rightarrow$ perturbatively interacting gauge theory in AdS$_D$ $\rightarrow$ *locality?*
- AdS/CFT $O(N)$ vector model [Klebanov, Polyakov '02] [Sezgin, Sundell '02] [Maldacena, Zhiboedov '11] [Bekaert, Erdmenger, Ponomarev, Sleight '15]

Higher-spin algebras: coset construction

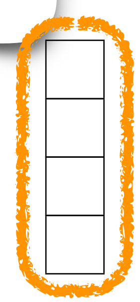
Start with the AdS$_D$ vacuum isometry algebra $\mathfrak{g} = \mathfrak{so}(2, D-1)$ with generator $J \simeq$ 



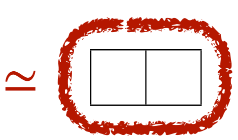
[Nikitin '91]
[Shapovalov, Shirikov '92]
[Eastwood '02]
[Segal '02]
[Vasiliev '03]
[Iazeolla, Sundell '08]
[Boulanger, Skvortsov '11]
[Joung, Mkrtchyan '14, '15]
...

Coset construction of HS algebras using the Universal Enveloping Algebra

$$\mathfrak{hs}_D \equiv \frac{\mathscr{U}(\mathfrak{g})}{\langle I \rangle} \qquad \langle I \rangle = \mathscr{U}(\mathfrak{g}) \odot I \odot \mathscr{U}(\mathfrak{g}) \;, \quad I = I_{[ABCD]} \oplus I_{(AB)}$$

$$I_{[ABCD]} = J_{[AB} \odot J_{CD]} \simeq$$ 

and

$$I_{(AB)} = J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\eta_{AB}\, C_2 \simeq$$  ,

As a result of factorizing both $I_{[ABCD]}$ and $I_{(AB)}$, the Casimir $C_2 \equiv \frac{1}{2} J_{AB} \odot J^{BA}$ is fixed

$$C_2 \sim -\frac{(D+1)(D-3)}{4}$$

Only known algebra reproducing known non-Abelian AdS cubic interactions [Boulanger, Ponomarev, Skvortsov, Taronna '13]

Candidate ideal for the flat algebra

"Open" the ideal in ambient space, multiply expressions by suitable power of ℓ (linear combinations are allowed) and send $\ell \to \infty$ at the level of associative algebra [Campoleoni, SP '21]

$$J_{ab}\,,\quad P_a \qquad \text{with} \qquad \left[J_{ab}, J_{cd}\right] = 4\,\eta_{[c[a}\,J_{b]d]}\quad,\qquad \left[J_{ab}, P_c\right] = 2\,\eta_{c[a}\,P_{b]}\quad,\qquad \left[P_a, P_b\right] = 0$$

$$I^\flat_{[ABCD]} \;\Leftrightarrow\; \begin{cases} I^\flat_{[abcd]} \equiv & J_{[ab} \odot J_{cd]} \\ I^\flat_{[abc]} \equiv & J_{[ab} \odot P_{c]} \end{cases}$$

$$I^\flat_{(AB)} \;\Leftrightarrow\; \begin{cases} I^\flat_{ab} \equiv & P_a \odot P_b - \frac{1}{D}\,\eta_{ab}\,P^2 \\ I^\flat_{a} \equiv & P^b \odot J_{ba} \\ I^\flat \equiv & P^2 \end{cases}$$

$$J^2 + \frac{(D-1)(D-3)}{4}$$

The whole ideal is generated by only factorising the following two elements

$$I^\flat_{[abcd]} \equiv J_{[ab} \odot J_{cd]}$$

$$J^2 + \frac{(D-1)(D-3)}{4}$$

Flat HS algebra: coset construction and unicity

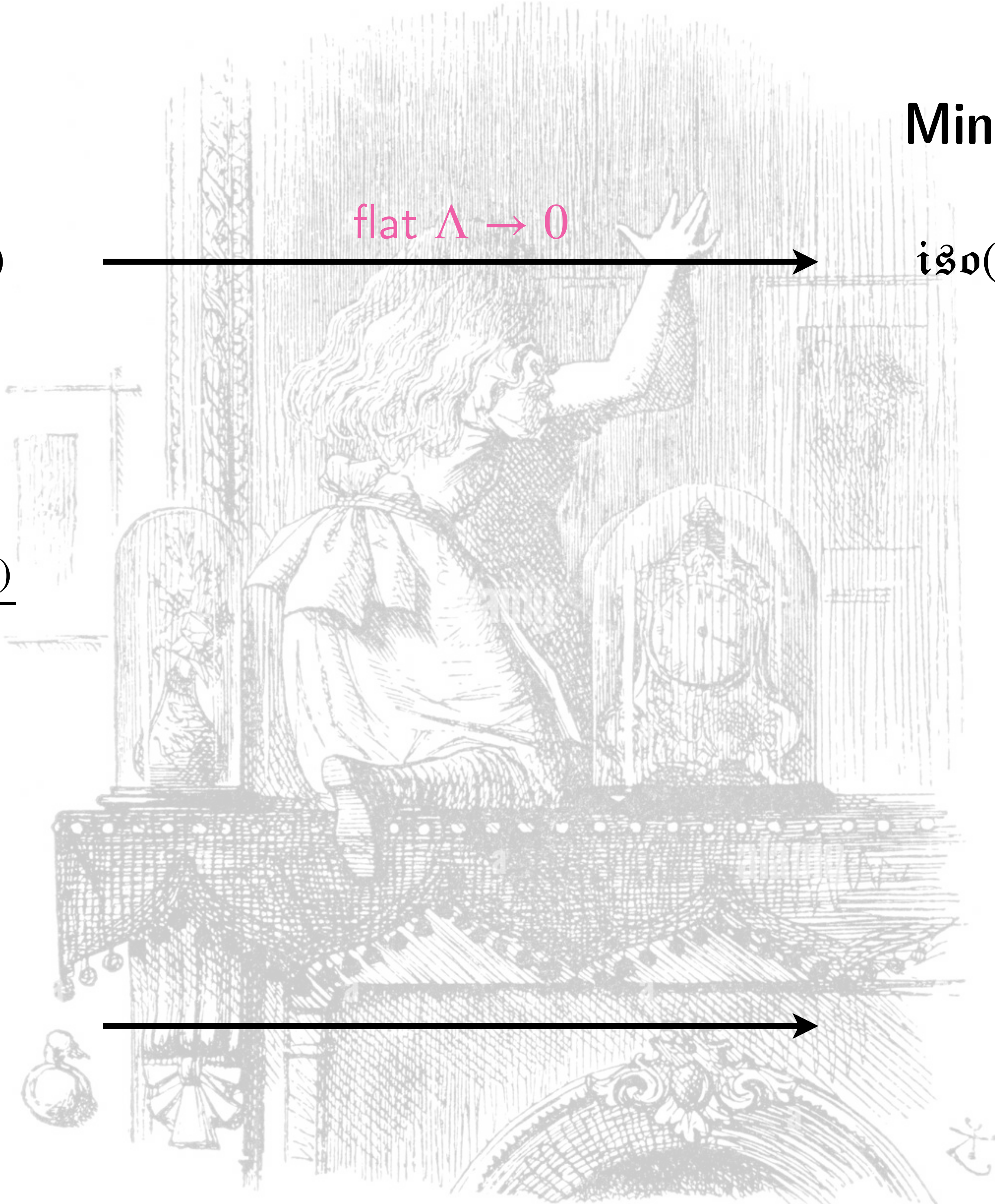
$$\mathfrak{ihs}_D \equiv \frac{\mathscr{U}\left(\mathfrak{iso}(1,D-1)\right)}{\left\langle I^\flat_{[ABCD]} \oplus I^\flat_{(AB)} \right\rangle}$$

Remarks:

- all Casimir operators of Poincaré are zero ($P^2 \sim 0$, Pauli-Lubanski $I^\flat_{abc} \sim 0$)
  $\Rightarrow$ *massless* representation in the bulk (not a unitary one because $P_a P_b \sim 0$)
- rather general method of constructing algebras, also works for other kinematical spaces [SP in progress]

- Non-Abelian isometry algebras for massless HS fields propagating on Minkowski background
  - in $D=3$, previously-known one-parameter family $\mathfrak{ihs}_3[\lambda]$, with truncations for $\lambda \in \mathbb{Z}$
  - in $D=5$, only one single deformation $\mathfrak{ihs}_5[0^+]$
- Unicity of the construction in dimension $D \geq 4$ under the assumptions of
  - Lorentz-covariance
  - admitting $\mathfrak{iso}(1,D-1)$ as a sub-algebra
  - spectrum of isometry generators corresponding to massless fields
  - built as a coset of the UEA of the vacuum isometry algebra
- This points to a 'dual' or alternative formulation to the one of Fronsdal (see, e.g. [Skvortsov, Tran, Tsulaia '18])

Construction (bulk)



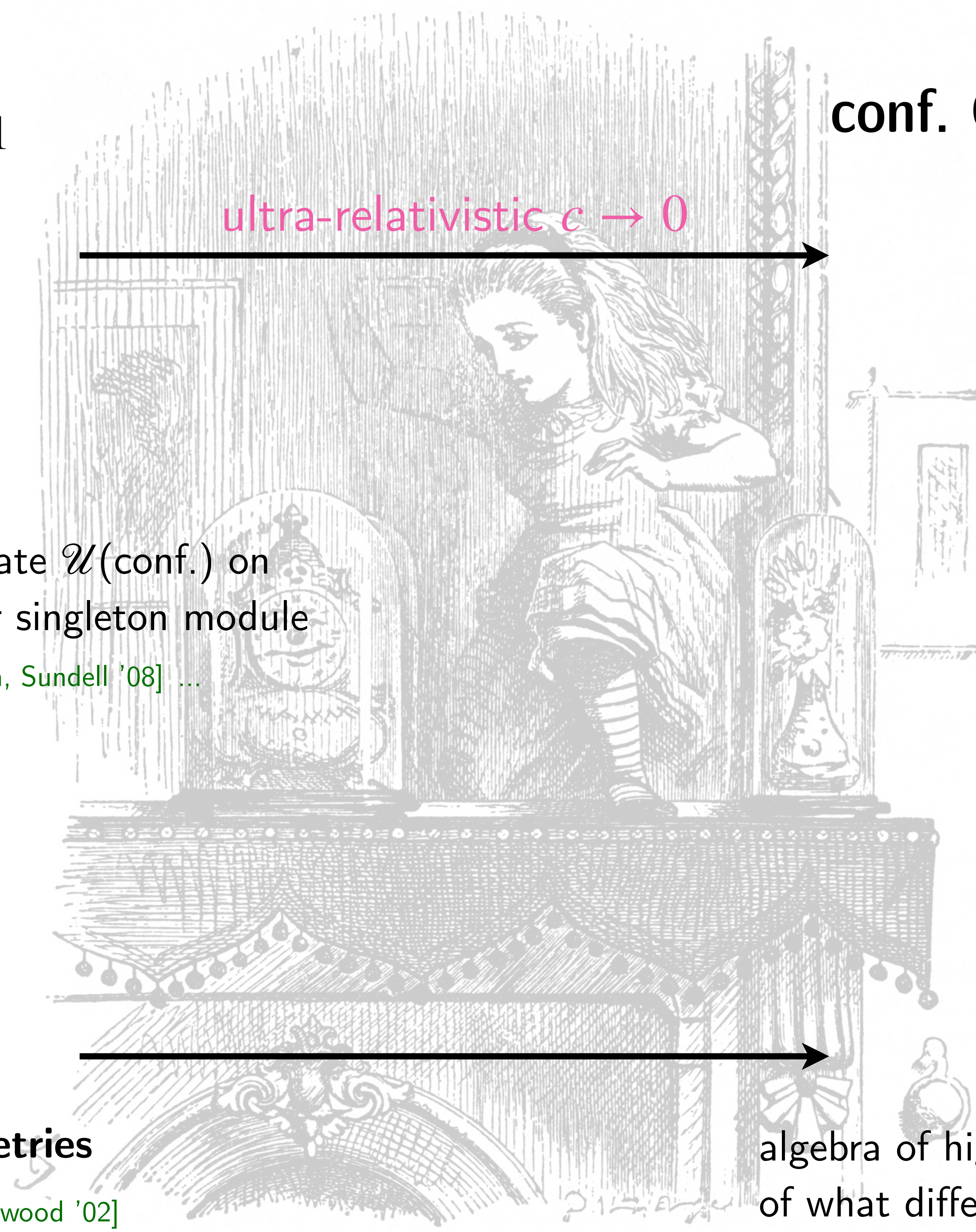
AdS$_D$ **Minkowski**$_D$

$$\begin{array}{ccc}
\mathfrak{so}(2,D-1) & \xrightarrow{\text{flat } \Lambda \to 0} & \mathfrak{iso}(1,D-1) \\
\Big\downarrow \frac{\mathcal{U}(\,\cdot\,)}{\langle I \rangle} & & \Big\downarrow \frac{\mathcal{U}(\,\cdot\,)}{\langle I^\flat \rangle} \\
\mathfrak{hs}_D & \longrightarrow & \mathfrak{ihs}_D
\end{array}$$

[Campoleoni, SP '21]

Construction (boundary)

Asymptotic symmetries
[Duval, Gibbons, Horvathy, (Zhang) '14]



conf. flat$_{D-1}$ | **conf. Carrollian$_{D-1}$**

$\mathfrak{conf}_{D-1}$ —ultra-relativistic $c \to 0$→ $\mathfrak{ccarr}_{D-1}$

evaluate $\mathscr{U}$(conf.) on scalar singleton module
[Iazeolla, Sundell '08] ...

evaluate $\mathscr{U}$(Carr. conf.) on what module?

$\mathfrak{chs}_{D-1}$ → $\mathfrak{cchs}_{D-1}$ [Campoleoni, SP '21]

algebra of **higher symmetries** of Laplacian in $\mathbb{R}^{D-1}$ [Eastwood '02]

algebra of higher symmetries of what differential operator?

Asymptotic symmetries?
[Bekaert, Campoleoni, SP '22]

The scalar singleton representation

Scalar singleton is an infinite-dimensional, ultra-short representation of $\mathfrak{so}(2, D-1)$

It is described by a free massless scalar in the bulk, whose Fefferman-Graham expansion is exactly z^{Δ}

$\rightarrow$ *the singleton is effectively a boundary field*

Bulk higher-spin fields couple to bilinears on the boundary $|Rac\rangle \otimes |Rac\rangle = \sum_{s \geq 0} D(s+1, s)$ [Flato, Fronsdal '78]

It is generated by the free scalar field $\phi(x^{\mu})$ on the conformal boundary [Angelopoulos, Flato, Fronsdal, Sternheimer '81]

$$S = \frac{1}{2} \int d^{D-1}x \; \phi^* \Box_x \phi \quad \text{with} \quad \Delta = \frac{D-3}{2}$$

The ideal $\langle I_{[ABCD]} \oplus I_{(AB)} \rangle$ corresponds to the ambient space *annihilator* (aka the trivial symmetries) of the scalar singleton module in the dual CFT [Iazeolla, Sundell '08]

- $I_{[ABCD]} \sim 0$ $\leftrightarrow$ scalar
- $I_{(AB)} \sim 0$ $\leftrightarrow$ massless
- $J^2 \sim -\frac{(D-3)(D+1)}{4}$ $\leftrightarrow$ quadratic Casimir $C_2 = \Delta(\Delta - D + 1) + s(s + D - 3)$

Ambient construction of the singleton

- Conformal boundary of AdS_D is defined through a null projection

$$X^2 = 0 \quad , \quad X^A \sim \lambda X^A \quad (\forall \lambda > 0)$$

- The singleton is a field in ambient space $\Phi(X)$ defined as

$$\Box_X \Phi(X) = 0 \quad , \quad \left(X^A \frac{\partial}{\partial X^A} + \Delta \right) \Phi(X) = 0 \quad , \quad \Phi(X) \simeq \Phi(X) + X^2 \Psi(X)$$

(the differential operators acting on Φ satisfy the $\mathfrak{sp}(2,\mathbb{R})$ algebra iff Δ takes the previous value)

- Generators of isometries leaving the light-cone $X^2 = 0$ invariant with homogeneity degree zero

$$J_{AB} = 2\, X_{[A}\, \partial_{B]}$$

verifying

$$J_{[AB}\, J_{CD]}\, \Phi(X) = 0 \quad , \quad J^C{}_{(A}\, J_{B)C}\, \Phi(X) \Big|_{\text{trace-free}} \simeq 0 \quad , \quad J^2\, \Phi(X) = \Delta(\Delta + 1 - D)\, \Phi(X)$$

Algebra of higher symmetries

- The algebra $\mathfrak{hs}_D$ is the algebra of higher symmetries of the conformal scalar in $\mathbb{R}^{D-1}$ [Eastwood '02]

  i.e. symmetry algebra of both AdS [Fradkin, Vasiliev '87] [Vasiliev '03] and conformal higher-spin gravity [Segal '02]

- Higher symmetries of $\Box_x \phi = 0$ are generated by differential operators $\delta_{\hat{D}}\phi = i\hat{D}\,\phi$

  that weakly commute with the kinetic operator $\Box_x \circ \hat{D} = \hat{D}^\dagger \circ \Box_x$

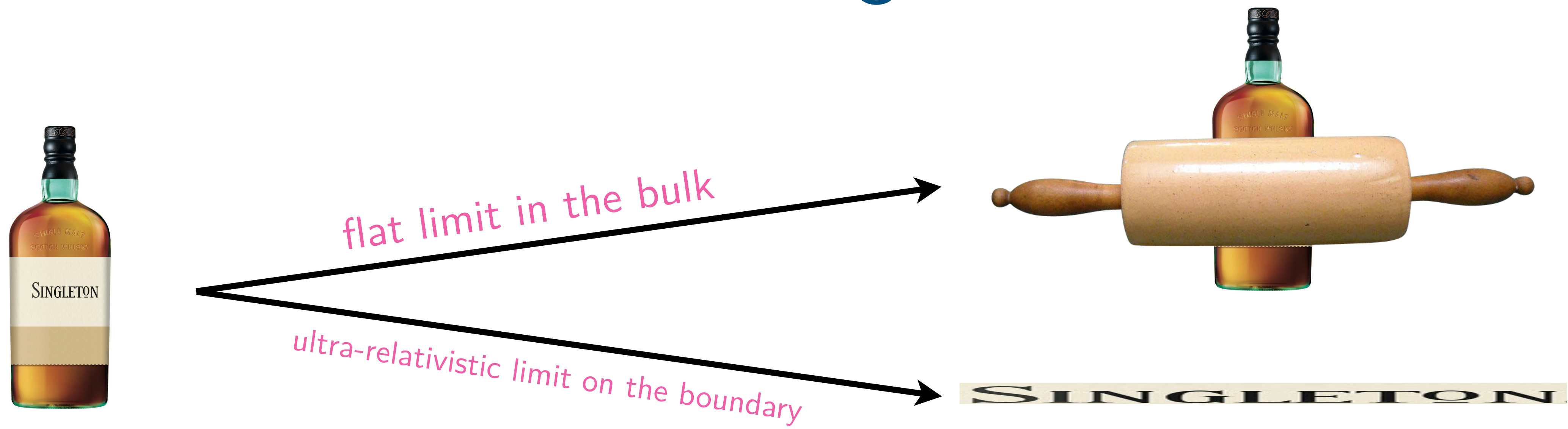
  modulo terms that vanish 'on-shell' $\hat{D} \approx \hat{D} + \hat{B} \circ \Box_x$

- Realisation of $\mathfrak{hs}_D$ as the algebra of differential operators on the conformal boundary

$$\hat{D} \sim V^{\mu(s-1)}(x)\,\partial_\mu \,\dots\, \partial_\mu + \text{lower} \quad \text{with} \quad \mu = 0\,,\ \dots\,,\, d \quad (D = d+2)$$

  for any integer $s \geq 1$, there are a finite number of symbols $V^{\mu(s-1)}$, in one to one correspondence with conformal Killing tensors of rank $(s-1)$

Tentative definition of a flat singleton



The naïve flat contraction of a singleton is a collection of zero-momentum fields [Flato, Fronsdal '78]
$P_a P_b \sim 0$ is weaker, but if we insist on unitary irreducible representations we recover $p_\mu = 0$

From the boundary point of view, we are defining an ultra-relativistic cousin of the singleton

- $I^\flat_{[abcd]} \sim 0$, $(I^\flat_{[abc]} \sim 0)$ $\leftrightarrow$ scalar
- $(J_{ab} \odot P^b \sim 0$, $P_a P_b \sim 0)$ $\leftrightarrow$ satisfies $\partial_u{}^2 \phi \approx 0$
- $J^2 \sim -\dfrac{d^2-1}{4}$ $\leftrightarrow$ generalised free field in d-dimensional Euclidean CFT

The *simpleton* representation

- Consider *Carrollian* ambient space $X^A = (u, y^a) \in \mathbb{R} \times \mathbb{R}^{d+1,1}$ with null metric along u and define the locus of the light-cone and projection equivalence relation [Bekaert, Campoleoni, SP '22]

$$y^2 = 0 \quad , \qquad (u, y^a) \sim \lambda\,(u, y^a) \qquad (\forall \lambda > 0)$$

- The *simpleton* is a field in ambient space $\Phi(u, y)$

$$\partial_u{}^2\, \Phi(u,y) = 0 \quad , \qquad \left(u\,\partial_u + y^a \frac{\partial}{\partial y^a} + \Delta\right) \Phi(u,y) = 0 \quad , \qquad \Phi(u,y) \simeq \Phi(u,y) + y^2\, \Psi(u,y)$$

(here, the differential operators satisfy $\mathfrak{iso}(1{,}1)$ for any value of Δ)

- Generators of isometries leaving the light-cone $y^2 = 0$ invariant and of homogeneity degree zero

$$J_{ab} = 2\, y_{[a}\, \partial_{b]} \quad , \qquad P_a = y_a\, \partial_u$$

verify

$$J_{[ab}\, J_{cd]}\, \Phi(u,y) = 0 \quad , \qquad J_{[ab}\, P_{c]}\, \Phi(u,y) = 0 \quad ,$$

$$\left(J^2 + \frac{d^2 - 1}{4}\right) \Phi(u,y) \simeq 0 \quad , \qquad \left(J_{ab}\, P^b + P^b\, J_{ab}\right) \Phi(u,y) \simeq 0 \quad , \qquad P_a\, P_b\, \Phi(u,y) = 0$$

The Carrollian conformal scalar as a flat singleton

- Null projectivisation of the light-cone $y^2 = 0$ with $y^a \in \mathbb{R}^{d+1,1} \to$ celestial sphere $x^k \in S^d$

- Action for the action for the (electric) Carrollian conformal scalar field [Bagchi, Mehra, Nandi '19]

$$S_{c=0}[\phi] = \frac{1}{2} \int du\, d^d x \sqrt{\gamma}\, \phi^* \, \partial_u{}^2 \, \phi$$

with metric γ_{ij} on the celestial sphere (scale invariant iff $\phi(u, x^k)$ has correct scaling dimension)

Remark: scaling dimension sits between Sachs $\Delta_S = \dfrac{d}{2}$ and Wick-rotated 'Rac' $\Delta_{WR} = \dfrac{d-2}{2}$ [Bekaert, Oblak '22]

- Bulk generators acting on $\phi(u, x^k)$ realised as vectors at null infinity

$$P_a = f_a(x^k)\, \partial_u \qquad \text{with} \qquad \nabla_{(i} \nabla_{j)} f_a(x^k) = \frac{1}{d} \gamma_{ij} \nabla^2 f_a(x^k)$$

$$J_{ab} = \xi^i_{ab}(x^k)\, \partial_i + \nabla_i\, \xi^i_{ab}(x^k) \big(\Delta + u\, \partial_u\big) \qquad \text{with} \qquad \nabla_{(i}\, \xi^{ab}_{j)}(x^k) = \frac{1}{d} \gamma_{ij} \nabla \cdot \xi^{ab}(x^k)$$

Symmetries of the Carrollian conformal scalar

$$S_{c=0}[\phi] = \frac{1}{2} \int du\, d^d x \sqrt{\gamma}\, \phi^* \,\partial_u{}^2 \,\phi$$

- The higher symmetries of $S_{c=0}$ are given by differential operators $\delta\phi = i\hat{D}\,\phi$

  that weakly commute with the kinetic operator $\partial_u{}^2 \circ \hat{D} = \hat{D}^\dagger \circ \partial_u{}^2$

  modulo operators that vanish on-shell $\hat{D} \approx \hat{D} + \hat{B} \circ \partial_u{}^2$

- They span a real Lie algebra isomorphic to

$$\mathscr{H}(S^d) \otimes \mathfrak{gl}(2,\mathbb{R})$$

  i.e. arbitrary Hermitian differential operators multiplied by conformal isometries of the real line

$$\left\{ id\,,\; i\partial_u\,,\; i(u\,\partial_u - 1/2)\,,\; i(u^2\,\partial_u - u) \right\}$$

Symmetries of the Carrollian conformal scalar

- Differential operators of order 0 (spin 1 gauge transfo.) give large $\mathfrak{u}(1)$ transformation: $\delta\phi = i\alpha(x^k)\,\phi$

- Differential operators of order 1 (spin 2 gauge transfo.) acting on ϕ:

$$\delta_T\phi = T(x^k)\,\partial_u\,\phi \qquad \text{(super-translations)}$$

$$\delta_Y\phi = \left[Y^i(x^k)\partial_i + \frac{1}{d}\,\nabla_i\,Y^i(x^k)\big(u\,\partial_u + \Delta\big)\right]\phi \qquad \text{(super-rotations)}$$

$$\delta_W\phi = W(x^k)(2\,u\,\partial_u - 1)\,\phi \qquad \text{('super'-Weyl)}$$

$$\delta_Z\phi = Z(x^k)(u^2\,\partial_u - u)\,\phi \qquad \text{('super'-special-conformal)}$$

<u>Remarks:</u>

- with δ_W: isomorphic to the Weyl-BMS algebra [Freidel et al. '21]

- with δ_Z: similar to the conformal BMS algebra of [Haco, Hawking, Perry, Bourjaily '17], but $\left[\delta_T, \delta_Z\right] = \delta_{W=TZ}$

Higher symmetries of the Carrollian conformal scalar

- Differential operators of order $s-1$ (spin s gauge transfo.) generate "higher-spin BMS" transformations

$$\hat{D}_T = i^{s-2}\, T^{i(s-2)}(x^k)\, \nabla_i \cdots \nabla_i \otimes \partial_u + \text{lower} \qquad \text{('hyper'-translations)}$$

$$\hat{D}_Y = i^{s-1}\, Y^{i(s-1)}(x^k)\, \nabla_i \cdots \nabla_i \otimes id + \text{lower} \qquad \text{('hyper'-rotations)}$$

$$\hat{D}_W = i^{s-2}\, W^{i(s-2)}(x^k)\, \nabla_i \cdots \nabla_i \otimes u\, \partial_u + \text{lower} \qquad \text{('hyper'-Weyl)}$$

$$\hat{D}_Z = i^{s-2}\, Z^{i(s-2)}(x^k)\, \nabla_i \cdots \nabla_i \otimes u^2\, \partial_u + \text{lower} \qquad \text{('hyper'-special-conformal}$$

(differential operators of order $s-1$ can be obtained as a symmetrised product of $s-1$ vectors)

- All $\hat{D}_T$ and $\hat{D}_Y$ together form a sub-algebra $\mathfrak{hsbms}_{d+2}$. Decomposing their symbols $Y_s^{i(s-1)}$ and $T_s^{i(s-2)}$ into traceless components, we obtain the expected spectrum

$$Y_{i(s-1)}\Big|_{\text{trace-free}}\,,\quad T_{i(s-2)}\Big|_{\text{trace-free}}\,,\quad Y'_{i(s-3)}\Big|_{\text{trace-free}}\,,\quad T'_{i(s-4)}\Big|_{\text{trace-free}}\,,\quad \ldots$$

- Is not yet good a candidate for a flat Flato-Fronsdal theorem, since currents are max. quadratic in u
  $\rightarrow$ *need for sources?* [Donnay, Fiorucci, Herfray, Ruzziconi '22]

Future work

- Can we write down a field theory with this symmetry algebra? [wip with A. Campoleoni and N. Boulanger]
- What are the asymptotic symmetries of this theory?
- What do the extra HS asymptotic generators represent? [Haco et al. '17] [Fuentealba et al. '20] [Freidel et al. '21]
- Bulk description of the *simpleton*
- *What is all this telling us about Carrollian holography?*

Thank you for your attention!