Non-Linear Solutions and the Double Copy

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Friday 16th December 2022

Double Copy: The Colour-Kinematics duality

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- Within the last 2 decades, a duality has been (known as the *Double Copy*) written down for amplitudes between Quantum Gravity and Yang-Mills theory.
- Within the last decade, this duality has been extended to classical relativistic theories (*YM and General Relativity*).
- At a conceptual level, the double copy challenges our current understanding of the foundations for both QFT and GR.
- The double copy has also found great success as a tool for research; having been utilised in Cosmological (**Gravitational Waves**) and Optical settings.

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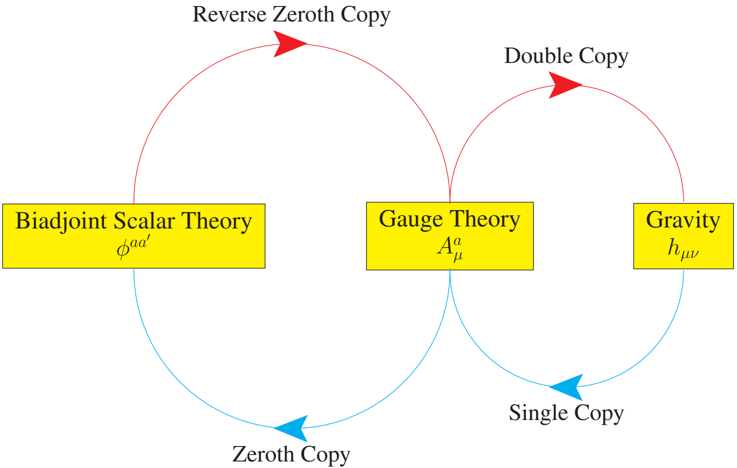
- So far little attention has been paid to studying the duality between solutions existing within Euclidean Spacetimes.
- In particular, non-linear solutions have remained frustratingly elusive.

The Kerr-Schild Double Copy



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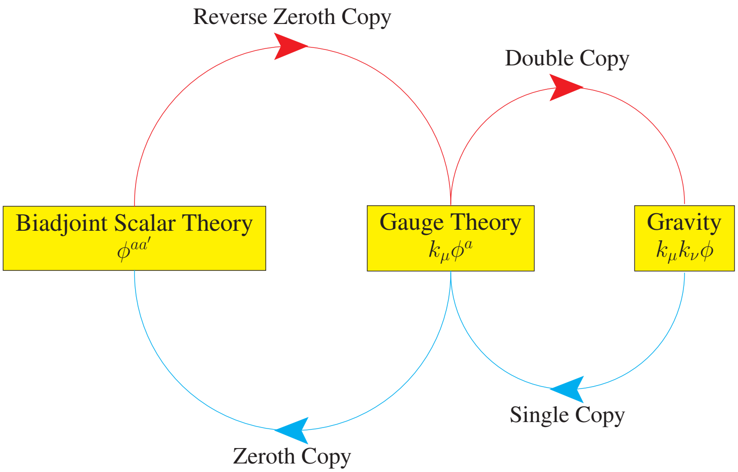

Biadjoint Scalar Theory (BAST)

In $\mathbb{R}^{(1,d-1)}$, the Lagrangian density for BAST with Gauge structure $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is given by:

$$\mathscr{L} = \frac{1}{2}\partial^{\mu}\Phi^{aa'}\partial_{\mu}\Phi^{aa'} + \frac{g}{3}\epsilon^{abc}\epsilon^{a'b'c'}\Phi^{aa'}\Phi^{bb'}\Phi^{cc'} \qquad (1)$$

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Performing the Wick rotation ($\tau \to \mathrm{i}t$) on (1), we derive the E.O.M for BAST in $\mathbb{R}^{d}$ to be:

$$\triangle\Phi^{aa'} + g\epsilon^{abc}\epsilon^{a'b'c'}\Phi^{bb'}\Phi^{cc'} = 0 \tag{2}$$

Non-Linear Solutions in D-Dimension

Considering only spherical symmetric solutions, set:

$$\Phi^{aa'} = \frac{\delta^{aa'}}{gT_A} f(r) \tag{3}$$

Inputting (3) into (2), we derive:

$$\frac{1}{r^{d-1}} \frac{d}{dr} \left(r^{d-1} \frac{df(r)}{dr} \right) + f^2(r) = 0, \tag{4}$$

Non-Linear Solutions in D-Dimension

Choosing a power-like ansatz:

$$f(r) = Ar^{\alpha} \tag{5}$$

We find a general solution for $\Phi^{aa'}$ in d dimensions:

$$\Phi^{aa'} = \frac{2\delta^{aa'}}{gT_A}\frac{d-4}{r^2}. \tag{6}$$

Where for $d = 4$, we find that there exists no power-like instanton solutions.

The Mystery of $d \neq 4$

To see why this is the case, choose a more general ansatz (in d dimensions):

$$f(r) = \frac{K(r) - 1}{r^2}, \tag{7}$$

Where taking $K(r) \to 1$ yields the trivial solution.

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Where taking $K(r) \to 1$ yields the trivial solution. Subsititing (7) into (4) and performing the substitution $r = e^{-\xi}$ $(-\infty < \xi < \infty)$, we find:

$$\frac{\partial^2 K}{\partial \xi^2} - (d-6)\frac{\partial K}{\partial \xi} + (K-1)(K-2d+7) = 0. \tag{8}$$

We must employ numerical methods in order to solve this (*critical points and integral curves*).

Numerical Solutions: Part 1

Setting $\psi = \frac{\partial K}{\partial \xi}$, we receive a pair of coupled differential equations:

$$\psi = \frac{\partial K}{\partial \xi} \tag{9}$$

$$\frac{\partial \psi}{\partial \xi} = (d-6)\psi - (K-1)(K-2d+7) \tag{10}$$

Now writing both (9) and (10) into a vector field :

$$\left(\frac{\partial K}{\partial \xi}, \frac{\partial \psi}{\partial \xi}\right) = \Big(\psi, (d-6)\psi - (K-1)(K-2d+7)\Big) \tag{11}$$

Numerical Solutions: Part 2

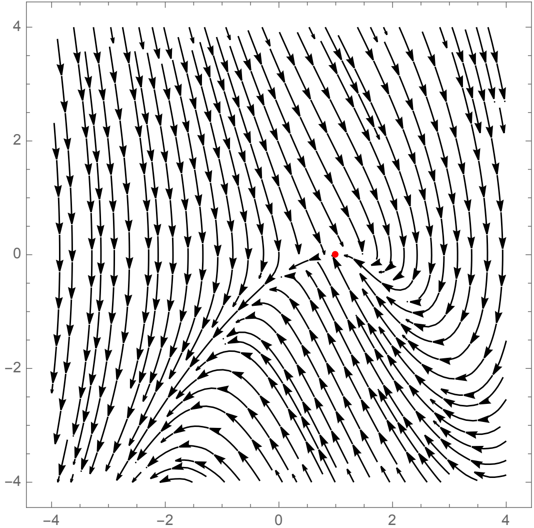
Solving (11) yields a vector field (K, ψ), where in d dimensions, we have two solutions:

$$K = 1, \psi = 0 \tag{12}$$

$$K = 2d - 7, \psi = 0 \tag{13}$$

Notice, for $d = 4$, the only solution for $\Phi^{aa'}$ corresponds to the trivial solution. Thus, there are **no non-linear power-like solutions** for $d = 4$.

$D = 4$ Solution



Conclusion and Further Work

- In $\mathbb{R}^4$, there exist no radial power-like solution to the full (non-linear) EOM for BAST.
- However, non-linear solutions do exist for $d \neq 4$.
- There exists solutions for the *linearised* EOM for BAST in $\mathbb{R}^4$. Through the Double Copy, we know that these solutions are related to the **Eguchi Hanson gravitation instanton**.
- Investigation into such solutions would require the use of the *Operator Formalism* of the Kerr-Schild Double Copy. (*10.1007/JHEP08(2022)160*)

The Kerr-Schild Double Copy: Operator Formalism