Matching, Merging & Higher-Order Corrections

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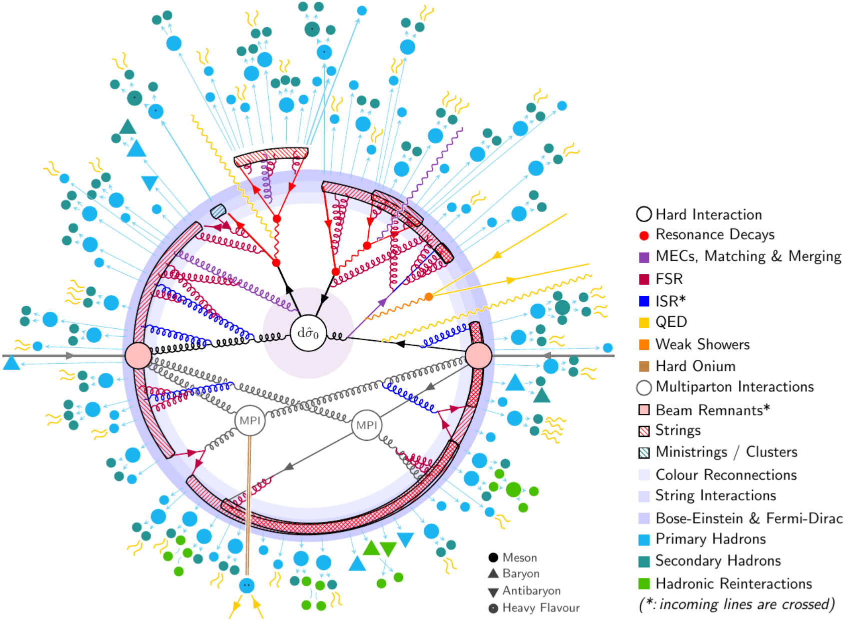

dσ̂₀
MPI
MPI
Meson
Baryon
Antibaryon
Heavy Flavour
Hard Interaction
Resonance Decays
MECs, Matching & Merging
FSR
ISR*
QED
Weak Showers
Hard Onium
Multiparton Interactions
Beam Remnants*
Strings
Ministrings / Clusters
Colour Reconnections
String Interactions
Bose-Einstein & Fermi-Dirac
Primary Hadrons
Secondary Hadrons
Hadronic Reinteractions
(*: incoming lines are crossed)

Suggested reading

- R. K. Ellis, W. J. Stirling, B. R. Webber
  QCD and Collider Physics
  Cambridge University Press, 2003
- M. E. Peskin, D. V. Schroeder
  An Introduction to Quantum Field Theory
  Westview Press, 1995
- L. Dixon, F. Petriello (Editors)
  Journeys Through the Precision Frontier
  Proceedings of TASI 2014, World Scientific, 2015
- Pythia collaboration
  A comprehensive guide to the physics and usage of PYTHIA 8.3
  SciPost Phys. Codebases 8 (2022)

Additional references provided on the slides.

Matrix Elements vs. Parton Showers

Matrix Elements

Fixed order good for hard jets

- + Contains all terms in given order of α_{s}
- + Valid also for high relative $p_{\perp}^2$
- - Only feasible for a few emissions

Parton Showers

Approx. excl. multi-parton cross section

- + Always finite
- + Can produce any number of emissions
- - Is only valid in soft/collinear regions

Combine strengths of Matrix Elements and Parton Showers

Experiments measure both high and low $p_{\perp}^2$ phenomena

- Describe hard emissions by fixed order predictions
- Add further emissions and include no-emission probabilities from PS

Matching & Merging Overview

Combine Matrix Element calculations and Parton Showers. Improve in different ways:

Matrix Element Corrections Oldest scheme, correct first emission of parton shower according to full process-dependent real emission calculation

Multi-jet Merging Improve radiation pattern of parton shower by adding higher-multiplicity matrix elements

NLO Matching Improve the perturbative precision by one higher order (NLO in α_s) cross section matched to parton showers

NLO Multi-jet Matching/Merging Combine multiple higher-multiplicity and higher-order cross sections in parton shower

Also: NLO splittings in parton showers and comments on shower accuracy

Recap:

Parton Showers

Recap: Parton Showers

Start from hard $2 \to 2$ scattering, dress with extra partons to get exclusive $2 \to n$ cross section

$$\mathrm{d}\sigma_n^{\mathrm{ex}} = F_0^+ F_0^- |M_0|^2 \mathrm{d}\phi_0 \times \left[\prod_{i=1}^{n} \frac{\alpha_\mathrm{s}(\rho_i)}{2\pi} \frac{F_i}{F_{i-1}} P_i \frac{\mathrm{d}\rho_i}{\rho_i} \mathrm{d}z \Pi_{i-1}(\rho_{i-1}, \rho_i) \right] \Pi_n(\rho_n, \rho_{\mathrm{min}})$$

- $|M_0|^2 \mathrm{d}\phi_0$: Born-level ME and phase space
- $F_i = x_i f_i(x_i, \rho_i)$: PDF's from both sides of i-parton state, $\pm$ for $\pm p_z$ beams
- $P_i \mathrm{d}z \mathrm{d}\rho_i / \rho_i$: Differential emission rate, correct for soft/collinear splittings
- ρ, z: Splitting variables, ρ jet resolution scale, z energy/momentum fraction
- $\Pi(\rho_{i-1}, \rho_i)$: No-emission probabilities
- ρ_{min}: Minimal resolution scale / shower cut-off scale

Recap: No-emission Probabilities

$$\Pi_i(\rho_i, \rho_{i+1}) = \exp\left(- \int_{\rho_{i+1}}^{\rho_i} \frac{\mathrm{d}\rho}{\rho} \frac{\alpha_\mathrm{s}(\rho)}{2\pi} \int_{z_\mathrm{min}}^{z_\mathrm{max}} \mathrm{d}z \frac{F_{i+1}}{F_i} P_i(z) \right)$$

- Probability of not having any emissions harder than ρ_{i+1} when starting shower from ρ_i
- Introduces all order corrections in α_s
- F_{i+1}/F_i only included for ISR
- Exclusive description of final state needs no-emission probabilities

Unitarity of Parton Shower: Fixed Order Expansion

Expand to $\mathcal{O}(\alpha_s^2)$

Use $\frac{1}{2\pi\rho}\frac{F_{i+1}}{F_i}P_i(z) = \bar{P}_i$ for ISR, $\frac{1}{2\pi\rho}P_i(z) = \bar{P}_i$ for FSR to simplify notation

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \left[1 - \alpha_\mathrm{s}\int_{\rho_\mathrm{min}}^{\rho_0} \mathrm{d}\rho\mathrm{d}z\bar{P}_1 + \frac{\alpha_s^2}{2}\left(\int_{\rho_\mathrm{min}}^{\rho_0} \mathrm{d}\rho\mathrm{d}z\bar{P}_1\right)^2\right]$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s}\mathrm{d}\rho_1\mathrm{d}z_1\bar{P}_1 \left[1 - \alpha_\mathrm{s}\int_{\rho_1}^{\rho_0} \mathrm{d}\rho\mathrm{d}z\bar{P}_1 - \alpha_\mathrm{s}\int_{\rho_\mathrm{min}}^{\rho_1} \mathrm{d}\rho\mathrm{d}z\bar{P}_2\right]$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s}^2\mathrm{d}\rho_1\mathrm{d}z_1\bar{P}_1\mathrm{d}\rho_2\mathrm{d}z_2\bar{P}_2\Theta(\rho_1 - \rho_2)$$

$\Rightarrow$ Unitarity in every order of α_s, total cross-section

$$\frac{\mathrm{d}\sigma_0^{\mathrm{inc}}}{\mathrm{d}\phi_0} = \frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} + \int \frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} + \int\int \frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2$$

But 1-jet cross section not correct for hard/wide-angle emissions

Matrix Element Corrections

Matrix Element Corrections / Tree-level Matching

Want improved parton shower with full matrix elements for hard emissions
$\Rightarrow$ First step: Use full real-emission matrix element for hardest emission, process-dependent!

$$\alpha_\mathrm{s} \bar{P}_i \to \alpha_s \bar{P}_i^{\mathrm{ME}} \equiv \frac{|M_i|^2 \mathrm{d}\phi_i}{|M_{i-1}|^2 \mathrm{d}\phi_{i-1} \mathrm{d}\rho \mathrm{d}z}$$

- Old, but very good! [Bengtsson, Sjöstrand (1987)]
- $+$ Natural and efficient within PS: Use modified acceptance probability
- - Difficult to generalize beyond one emission
- Vincia & Dire parton showers exponentiate n-parton matrix elements

  [Giele, Kosower, Skands (2008)] [Fischer, Prestel (2017)]

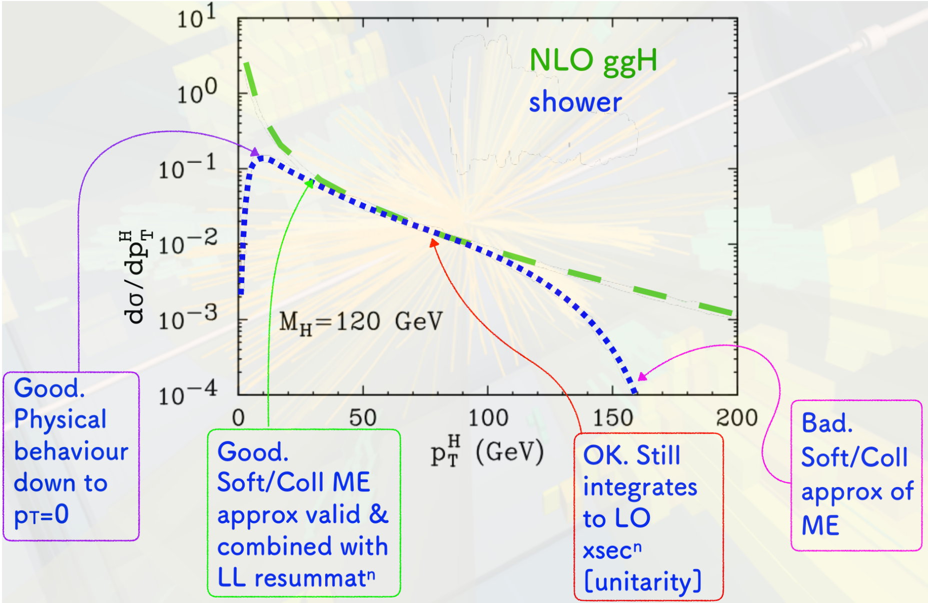
Matrix Element Corrections Preserve PS Unitarity

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \left[1 - \alpha_s \int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1^{\mathrm{ME}} + \frac{\alpha_s^2}{2} \left(\int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1^{\mathrm{ME}}\right)^2\right]$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_{\mathrm{s}} \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \left[1 - \alpha_{\mathrm{s}} \int_{\rho_1}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1^{\mathrm{ME}} - \alpha_{\mathrm{s}} \int_{\rho_{\min}}^{\rho_1} \mathrm{d}\rho \mathrm{d}z \bar{P}_2\right]$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_{\mathrm{s}}^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \mathrm{d}\rho_2 \mathrm{d}z_2 \bar{P}_2 \Theta(\rho_1 - \rho_2)$$

- Still unitary to all orders in α_{s}
- Valid in whole shower emission phase space, down to scale $\rho_{\min}$



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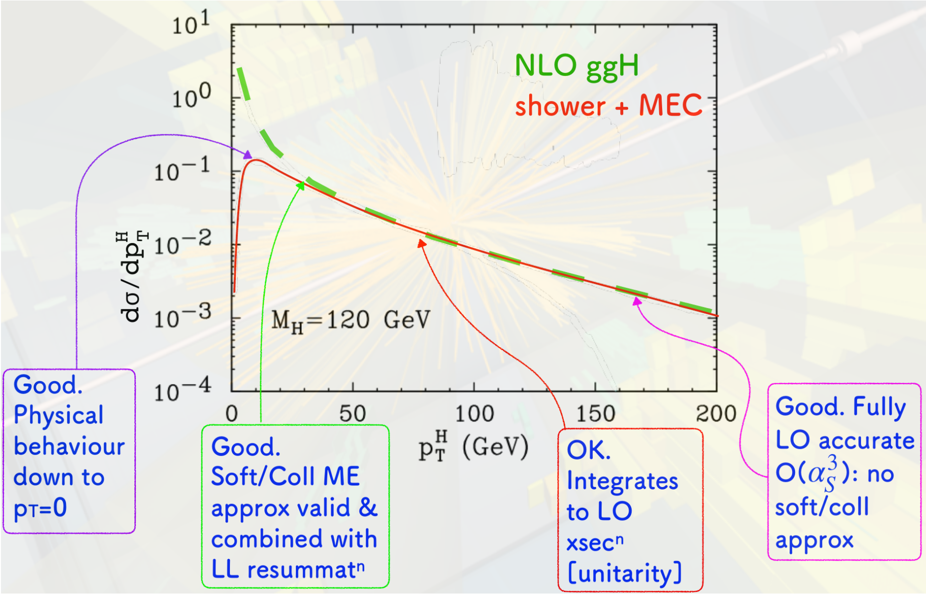


borrwed from Keith Hamilton

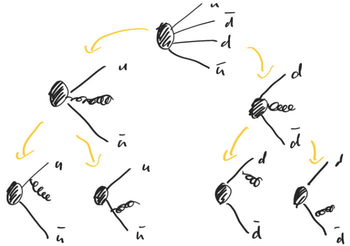
Iterative Matrix Element Corrections [Giele, Kosower, Skands (2008)][Fischer, Prestel (2017)]

- Consider matrix element state $|\mathcal{M}(\Phi_0)|^2$
- Parton-shower produces branching according to $P(\Phi_1/\Phi_0)|\mathcal{M}(\Phi_0)|^2\mathrm{d}\Phi_1$
- Apply MEC factor to correct weight of Φ_1 to full fixed-order matrix element

$$\mathcal{R}(\Phi_1) = \frac{|\mathcal{M}(\Phi_1)|^2}{\sum_{\Phi_0'} P(\Phi_1/\Phi_0')|\mathcal{M}(\Phi_0')|^2}$$

- Iterate, taking all possible PS histories into account

$$\mathcal{R}(\Phi_2) = \frac{|\mathcal{M}(\Phi_2)|^2}{\sum_{\Phi_1'} P(\Phi_2/\Phi_1')\mathcal{R}(\Phi_1') \sum_{\Phi_0'} P(\Phi_1'/\Phi_0')|\mathcal{M}(\Phi_0')|^2}$$



Leading Order Multi-Jet Merging

Multi-jet Merging: The Naive (and Wrong) Way

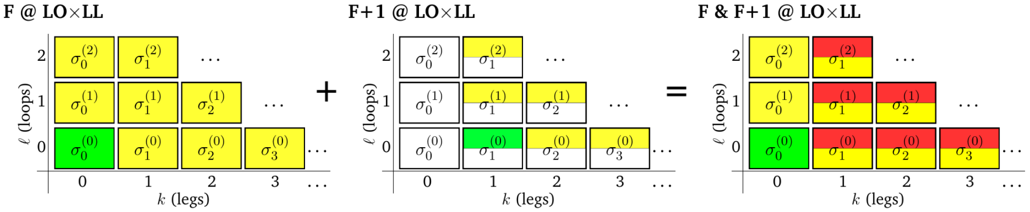
Want to improve PS emissions for more than hardest emission. Naive approach:

- Generate $[X]_{\mathrm{ME}}$ + parton shower
- Generate $[X + 1\text{jet}]_{\mathrm{ME}}$ + parton shower
- Generate $[X + 2\text{jet}]_{\mathrm{ME}}$ + parton shower
- ...

And combine everything into one sample. Does not work, double counting!

- $[X]_{\mathrm{ME}}$ + parton shower is inclusive
- $[X + 1\text{jet}]_{\mathrm{ME}}$ + parton shower is inclusive
- ...

See also Skands: Introduction to QCD



Multi-jet Merging: Exclusive Description without Double-counting

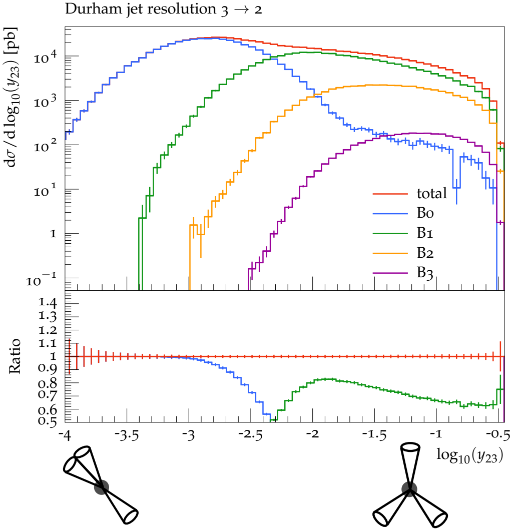
Solve double-counting issue by dividing phase space in "hard and soft region":

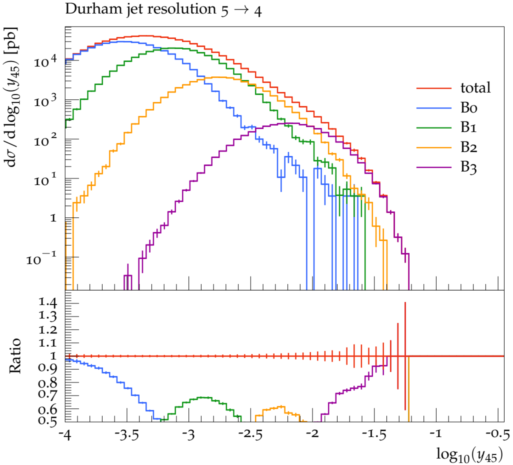
- Generating inclusive few jet samples according to exact tree-level $F_n^+ F_n^- |M_n|^2 \equiv B_n$ in "hard region"
- Using some merging scale ρ_{ms} to cut off divergences
- Making exclusive by reweighting with no-emission probabilities (and α_{s} and PDF ratios), i.e. how would PS have produced this configuration
- Using normal shower in "soft region" below ρ_{ms}

Remaining issues:

- Merging scale dependence
- Merging scale might not be defined in terms of shower evolution variable
- Might break unitarity of shower

Multi-jet Merging: $e^+e^- \to q\bar{q}$ + jets example

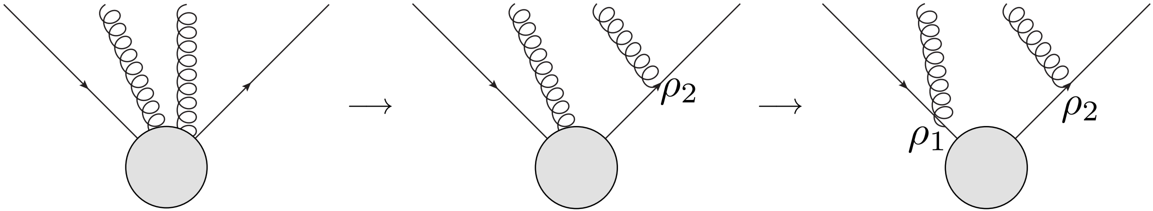




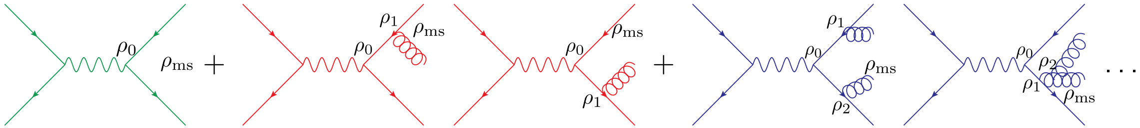
How to Reweight: Parton Shower Histories

Want to apply no-emission probabilities and scale dependent ratios $\Rightarrow$ need ρ_i. Two ways:

- Find unique history by applying sequential $2 \to 1$ jet algorithm
- Find all possible parton shower histories by $3 \to 2$ clustering, choose one according to product of splitting probabilities
  - Choose one history according to product of splitting probabilities
  - Combine partons according to parton shower kinematics



Multi-jet Merging: Illustration in FSR



Combine MEs with different multiplicities, avoid overlap by reweighting

$$\langle \mathcal{O} \rangle = \int d\phi_0 \left\{ \mathcal{O}_0 B_0 w_0 + \int d\phi_1 \mathcal{O}_1 B_1 w_1 + \int d\phi_1 \int d\phi_2 \mathcal{O}_2 B_2 w_2 \right\}$$

with the weights

$$w_0 = \Pi_0(\rho_0, \rho_{\mathrm{ms}}) \, , \; w_1 = \Pi_0(\rho_0, \rho_1) \frac{\alpha_s(\rho_1)}{\alpha_s(\mu_R)} \Pi_1(\rho_1, \rho_{\mathrm{ms}}) \, ,$$

$$w_2 = \Pi_0(\rho_0, \rho_1) \frac{\alpha_s(\rho_1)}{\alpha_s(\mu_R)} \Pi_1(\rho_1, \rho_2) \frac{\alpha_s(\rho_2)}{\alpha_s(\mu_R)}$$

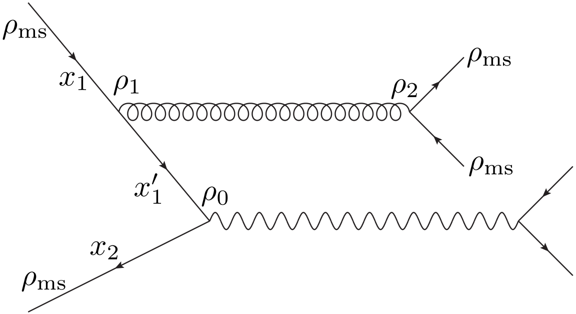
Multi-jet Merging: Illustration in ISR

Inclusive Matrix Element:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{in}}}{\mathrm{d}\phi_{0+2}} = F_1(x_1, \rho_0) F_2(x_2, \rho_0) |M_2|^2$$

Exclusive Parton Shower:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{ex}}}{\mathrm{d}\phi_0 \mathrm{d}\phi_{1,2}} = F_1'(x_1', \rho_0) F_2(x_2, \rho_0) |M_0|^2 \Pi_0(\rho_0, \rho_1) \frac{\alpha_{\mathrm{s}}(\rho_1)}{2\pi} \frac{F_1(x_1, \rho_1)}{F_1'(x_1', \rho_1)} \frac{P_1}{\rho_1} \Pi_1(\rho_1, \rho_2) \frac{\alpha_{\mathrm{s}}(\rho_2)}{2\pi} \frac{P_2}{\rho_2} \Pi_2(\rho_2, \rho_{\mathrm{ms}})$$



Find weight to make inclusive matrix element exclusive:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{ex}}}{\mathrm{d}\phi_0 \mathrm{d}\phi_{1,2}} = w \frac{\mathrm{d}\sigma_2^{\mathrm{in}}}{\mathrm{d}\phi_{0+2}}$$

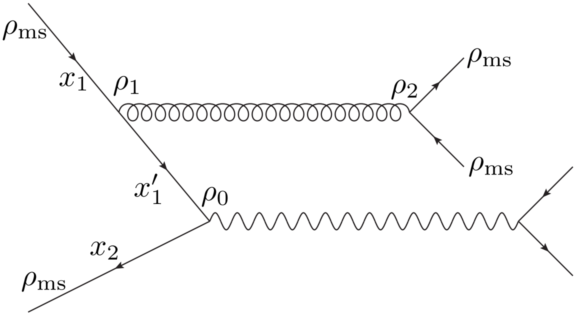
Multi-jet Merging: Merging Weight in ISR

$$w = w_{\alpha_s} w_{\mathrm{pdf}} w_{\mathrm{no-em}}$$

$$w_{\alpha_s} = \frac{\alpha_s(\rho_1)}{\alpha_s(\rho_0)} \frac{\alpha_s(\rho_2)}{\alpha_s(\rho_0)}$$

$$w_{\mathrm{pdf}} = \frac{f(x_1', \rho_0)}{f(x_1', \rho_1)} \frac{f(x_1, \rho_1)}{f(x_1, \rho_0)}$$

$$w_{\mathrm{no-em}} = \Pi_0(\rho_0, \rho_1)\Pi_1(\rho_1, \rho_2)\Pi_2(\rho_2, \rho_{\mathrm{ms}})$$



Multi-Jet Merging Algorithm

Summary of general multi-jet merging procedure:

1. Calculate inclusive cross sections for $X + n$ partons (with kinematic cut ρ_{ms} to avoid singularities)
2. Cluster according to jet algorithm or find parton shower history to find scales for no-emission probabilities and scale dependent ratios
3. Multiply with merging weight: α_{s}-ratios, no-emission probabilities (and PDF ratios)
4. If $n < N$, with N highest fixed order multiplicity, multiply no-emission probability towards merging scale ρ_{ms}
5. Allow further parton shower emissions below ρ_{ms}, for $n = N$ also above

CKKW Merging [Catani, Krauss, Kuhn, Webber (2001)]

- Cluster according to $k_\perp$ jet algorithm
- Apply analytic Sudakov factors (NLL accuracy) as no-emission probabilities
- Perform "truncated showering", since parton shower evolution variable not exactly identical to merging scale cut: Start shower from ρ_0, but forbid emissions above t_{ms}. Handle hard emissions (in ρ) below t_{ms} with care!
    - $+$ Appealing theoretical treatment
    - \- Requires dedicated PS implementation
    - \- Mismatch between analytical Sudakov and parton shower
    - Implemented in Sherpa (v 1.1) [Krauss (2002)]

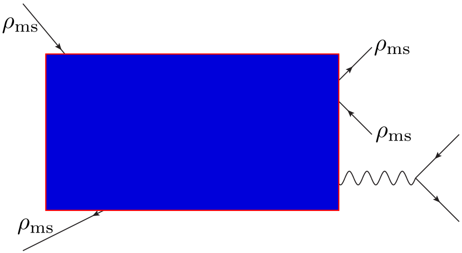
CKKW-L Merging [Lönnblad (2001)]

- Cluster back to parton shower history according to splitting probabilities in PS
- Generate of no-emission probabilities using parton shower
- Perform showering step-by-step for each step in history, starting from respective clustering scale
- Veto event if emission at larger scale than next clustering scale or ρ_{ms} in last step
- Keep PS emissions below ρ_{ms} (and between ρ_n and ρ_{ms} at highest multiplicity)
  - $+$ Agreement between Sudakov and shower by construction $\Rightarrow$ Reduced merging scale dependence
  - $+$ Use simple veto in shower if ρ_{ms} in terms of PS evolution variable
  - - Requires dedicated PS implementation
  - Implemented in Sherpa ($\geq$1.2) [Höche, Krauss, Schumann, Siegert (2009)], Pythia8 [Lönnblad, Prestel (2012)] and Herwig7 [Bellm, Gieseke, Plätzer (2018)]

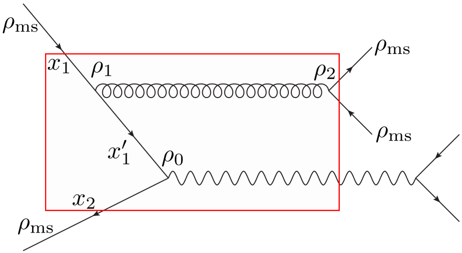
MLM [Mangano (2002)] [Mangano, Moretti, Piccinini, Treccani (2007)]

- Simplest way to estimate Sudakov suppression: Run shower on ME state without prior reclustering, starting from ρ_0
- Perform jet clustering, and reject if PS emits any jets harder than original partons or partons that are not clustered to hard partons
- No reconstructed history $\Rightarrow$ Sudakov factor corresponds to final partons only, not taking into account intermediate states
- Approximation turns out to be good enough
  - $+$ Simplest available scheme
  - $+$ Matching with any shower algorithm without specific implementation
  - - Sudakov suppression not exact $\Rightarrow$ mismatch with shower

Sudakov Factor: MLM vs. CKKW-L



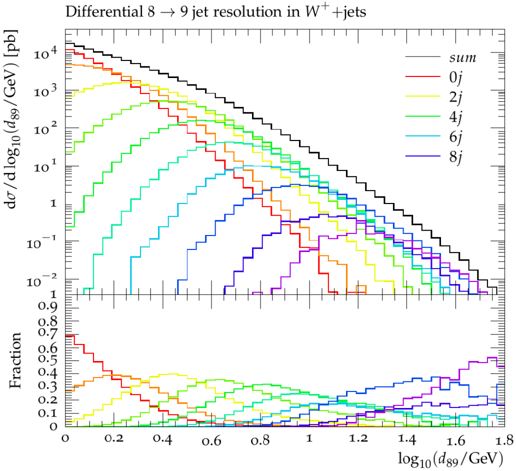
- First shower from ρ_0 to $\rho_{\rm ms}$
- Then do jet clustering to veto if hard emissions occured
- Resulting no-emission probability: $\Pi_q^2(\rho_0, \rho_{\rm ms})\Pi_q^2(\rho_0, \rho_{\rm ms})$



- First construct parton shower history
- Then do trial shower on reconstructed history, veto event if emission above merging scale
- Resulting no-emission probability: $\Pi_q^2(\rho_0, \rho_2)\Pi_g(\rho_1, \rho_2)\Pi_q^4(\rho_2, \rho_{\rm ms})$

What about high multiplicities?

- Merging requires reconstruction of parton shower histories, grow factorially
- For high multiplicities beyond 5 or so: need to be creative
- One way: winner-takes-all: go for highest probability in first clusering steps [Höche, Prestel, Schulz (2019)]
- Another way: sector showers, i.e., unique histories [Brooks, Preuss (2021)]



Unitarity in Multi-jet Merging

Unitarity in Multi-jet Merging

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \left[1 - \alpha_s \int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 + \frac{\alpha_s^2}{2} \left(\int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 \right)^2 \right]$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s} \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \left[1 - \alpha_\mathrm{s} \int_{\rho_1}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 - \alpha_\mathrm{s} \int_{\rho_{\min}}^{\rho_1} \mathrm{d}\rho \mathrm{d}z \bar{P}_2 \right]$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s}^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \mathrm{d}\rho_2 \mathrm{d}z_2 \bar{P}_2^{\mathrm{ME}} \Theta(\rho_1 - \rho_2)$$

- Unitarity of parton shower broken in multi-jet merging schemes mentioned above
- Inclusive cross-section only preserved if splitting probabilities in no-emission probability identical to full fixed order splitting probabilities

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

Start from CKKW-L scheme, want to restore PS unitarity. Use:

$$\Pi_n(\rho_n, \rho_{\mathrm{ms}}) = 1 - \int_{\rho_{\mathrm{ms}}}^{\rho_n} \mathrm{d}\rho \mathrm{d}z \alpha_{\mathrm{s}} \bar{P}_{n+1}^{\mathrm{ME}}(\rho, z) \Pi_n(\rho_0, \rho)$$

i.e. probability of no emission is 1 - probability of at least one emission

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \Pi_0(\rho_0, \rho_{\mathrm{ms}})$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_1^+ F_1^- |M_1|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \Pi_1(\rho_1, \rho_{\mathrm{ms}})$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_2^+ F_2^- |M_2|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \mathrm{d}\rho_2 \mathrm{d}z_2 \Pi_1(\rho_1, \rho_2)$$

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

Start from CKKW-L scheme, want to restore PS unitarity. Use:

$$\Pi_n(\rho_n, \rho_{\rm ms}) = 1 - \int_{\rho_{\rm ms}}^{\rho_n} {\rm d}\rho {\rm d}z \alpha_{\rm s} \bar{P}^{\rm ME}_{n+1}(\rho, z) \Pi_n(\rho_0, \rho)$$

i.e. probability of no emission is 1 - probability of at least one emission

$$\frac{{\rm d}\sigma_0^{\rm ex}}{{\rm d}\phi_0} = F_0^+ F_0^- |M_0|^2 \cancel{\Pi_0(\rho_0, \rho_{\rm ms})} - \int F_1^+ F_1^- |M_1|^2 {\rm d}\rho_1 {\rm d}z_1 \Pi_0(\rho_0, \rho_1)$$

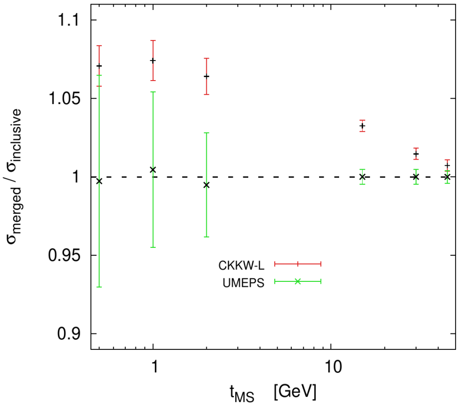
$$\frac{{\rm d}\sigma_1^{\rm ex}}{{\rm d}\phi_0} = F_1^+ F_1^- |M_1|^2 {\rm d}\rho_1 {\rm d}z_1 \Pi_0(\rho_0, \rho_1) \cancel{\Pi_1(\rho_1, \rho_{\rm ms})}$$

$$- {\rm d}\rho_1 {\rm d}z_1 \Pi_0(\rho_0, \rho_1) \int F_2^+ F_2^- |M_2|^2 {\rm d}\rho_2 {\rm d}z_2 \Pi_1(\rho_1, \rho_2)$$

$$\frac{{\rm d}\sigma_2}{{\rm d}\phi_0} = F_2^+ F_2^- |M_2|^2 {\rm d}\rho_1 {\rm d}z_1 \Pi_0(\rho_0, \rho_1) {\rm d}\rho_2 {\rm d}z_2 \Pi_1(\rho_1, \rho_2)$$

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

- Still add CKKW-L reweighted samples
- Instead of last Sudakov, subtract +1 parton integrated sample
  $\Rightarrow$ Individual multiplicities still exclusive
- Can still add normal PS below merging scale
- + Procedure does not change inclusive cross section
- - UMEPS introduces negative weights $\Rightarrow$ less efficient



Summary Lecture I

Summary Lecture I

Goal: Combine matrix elements and parton showers. The Problem:

- Parton showers generate singular terms of higher-order matrix elements
- Same terms present in X + jet(s) matrix elements
- Combination must not double count

ME Corrections

- Oldest scheme, correct PS emissions to match full real emission ME
- Can be iterated beyond one emission
- Developments: higher multiplicity, NLO in VINCIA

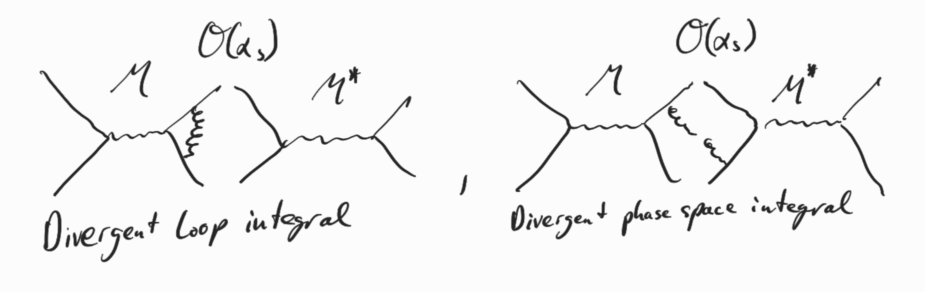
Multi-jet Merging

- Combine multiple LO ME samples by reweighting
- Separate phase space regions to deal with divergence
- Different schemes available

NLO Matching

Towards NLO

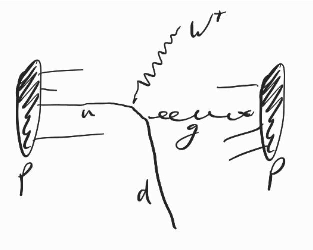
- So far, considered only higher multiplicities σ_0, σ_1, σ_2, ..., i.e., legs, no loops
- In consistent expansion in coupling parameter, need to consider loops as well!



- IR singularities cancel between different multiplicities in *inclusive* cross sections
- Measurements that ensure singularity cancellation are *IR safe*

What is NLO?

- Leading order is first order in α_s that gives non-zero result *for given observable*
- NLO is next order. If large correction $\rightarrow$ need NNLO
- Can be tricky: consider $W + j$, and measure azimuthal angle between W and (leading) jet.
- Need second jet for non back-to-back $\Rightarrow$ implicitly two-jet, so only described at LO for NLO $W + j$
- What if $\Delta\phi < \frac{2\pi}{3}$?



Finite Numerical NLO Cross Section

NLO prediction for observable $\mathcal{O}$ given by

$$\langle \mathcal{O} \rangle = \int \mathrm{d}\phi_n (B_n + V_n) \mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{n+1} B_{n+1} \mathcal{O}_{n+1}(\phi_{n+1})$$

but both V_n and B_{n+1} separately divergent, only sum is finite.
Use universal subtraction terms to get finite results: [Frixione, Kunszt, Siegner (1996)] [Catani, Seymour (1997)]

$$\langle \mathcal{O} \rangle = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1) \mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{n+1} (B_{n+1} \mathcal{O}_{n+1}(\phi_{n+1}) - B_n \otimes D_1 \mathcal{O}_n(\phi_{n+1}))$$

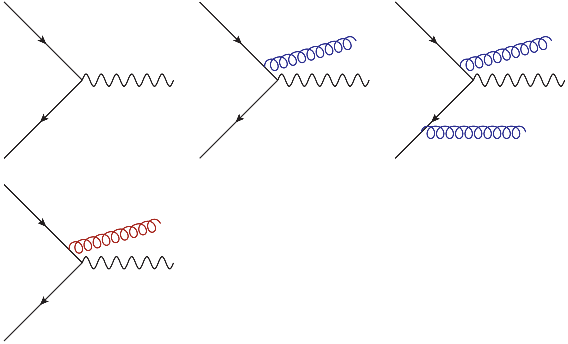
Event interpretation not yet possible, $\mathcal{O}_n$ and $\mathcal{O}_{n+1}$ contributions must be finite separately

Matching of NLO Matrix Elements & Parton Showers

We want precision predictions: Combine NLO fixed order calculations with Parton showers.

- Again problem of double counting of emissions by real emission matrix element and emissions generated by parton shower
- Also double counting of virtual terms through virtual corrections and Sudakov factors

Parton Shower ⟶



Real emission

Shower Subtraction

Want to attach shower (include factor α_s in $\bar{P}$)

$$
\begin{aligned}
\mathcal{O}_n(\phi_n) \to \mathcal{F}_n(\mathcal{O}, \phi_n) &= \Pi(\rho_n, \rho_\mathrm{min})\mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{+1} \Pi(\rho_n, \rho_{n+1})\bar{P}_{n+1}\mathcal{F}_{n+1}(\mathcal{O}, \phi_{n+1}) \\
&\overset{\mathcal{O}(\alpha_\mathrm{s})}{\to} 1 - \int \mathrm{d}\phi_{+1} \bar{P}_{n+1}\mathcal{O}_n(\phi_{n+1}) + \int \mathrm{d}\phi_{+1} \bar{P}_{n+1}\mathcal{O}_{n+1}(\phi_{n+1})
\end{aligned}
$$

But $B_n\mathcal{F}_n$ contains $\mathcal{O}(\alpha_\mathrm{s})$ terms $\Rightarrow$ subtract shower terms to first order in α_s such that accuracy of NLO not spoiled by shower

MC@NLO [Frixione, Webber (2002)]

With shower subtraction, arrive at MC@NLO prescription

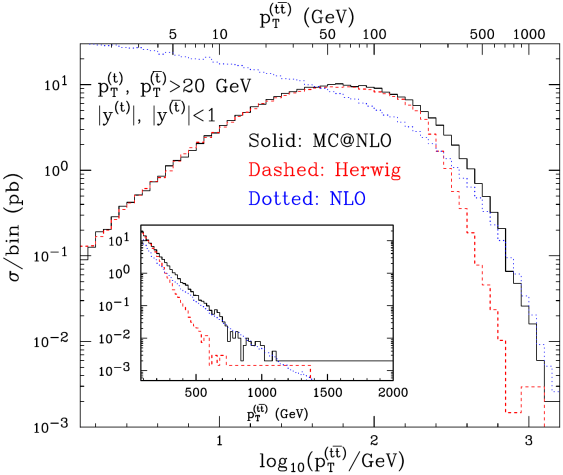
$$\langle\mathcal{O}\rangle_{\mathrm{MC@NLO}} = \int \mathrm{d}\phi_n(B_n + V_n + B_n \otimes I_1)\mathcal{F}_n(\mathcal{O},\phi_n)$$ Born + subtracted virtual

$$+ \int \mathrm{d}\phi_{n+1}(B_n \bar{P}_{n+1} - B_n \otimes D_1)\mathcal{F}_n(\mathcal{O},\phi_{n+1}))$$ Shower virtual - subtraction

$$+ \int \mathrm{d}\phi_{n+1}(B_{n+1} - B_n \bar{P}_{n+1})\mathcal{F}_{n+1}(\mathcal{O},\phi_{n+1})$$ Real - shower real

- Event generation possible since $\mathcal{O}_n$ and $\mathcal{O}_{n+1}$ separately finite
- Sudakov supression agrees with shower prediction
- Distribution correct only if parton shower is attached to cancel MC counterterms
- Can lead to many events with negative weights
- Needs to be implemented for each shower separately
- Automated in Sherpa [Höche, Krauss, Schönherr, Siegert (2012)] and aMC@NLO [Frederix, Frixione, Hirschi, Maltoni, Pittau, Torrielli (2012)]

MC@NLO



Plot form [Nason, Webber (2012)]

- MC@NLO gives smooth transition between real emission pattern at high scales and parton shower at low scales
- Inclusive cross section correct at NLO

POWHEG [Nason (2004)] [Frixione, Nason, Oleari (2007)]

Positive Weight Hardest Emission Generator

$$\langle\mathcal{O}\rangle_{\mathrm{POWHEG}} = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1)\mathcal{F}_n^{\mathrm{HI}}(\mathcal{O}, \phi_n) \qquad \text{Born + subtracted virtual}$$

$$+ \int \mathrm{d}\phi_{n+1}(B_{n+1} - B_n \otimes D_1)\mathcal{F}_n^{\mathrm{HI}}(\mathcal{O}, \phi_{n+1})) \qquad \text{Shower virtual - subtraction}$$

Based on MC@NLO, modify shower to get “shower real” = “real” for hardest emission (similar to matrix element corrections)

- Less negative weights ⇒ Improved efficiency
- Hardest emission modified ⇒ Differences compared to MC@NLO, but both NLO correct
- Implementation process by process, but independent of attached shower

MC@NLO-Δ

- Let's look at MC@NLO again:

$$\langle \mathcal{O} \rangle_{\mathrm{MC@NLO}} = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1) \mathcal{F}_n(\mathcal{O}, \phi_n)$$
Born + subtracted virtual

$$+ \int \mathrm{d}\phi_{n+1} (B_n \bar{P}_{n+1} \Delta - B_n \otimes D_1$$
Shower virtual - subtraction

$$B_{n+1}(1 - \Delta)) \mathcal{F}_n(\mathcal{O}, \phi_{n+1}))$$

$$+ \int \mathrm{d}\phi_{n+1} (B_{n+1} - B_n \bar{P}_{n+1}) \Delta \mathcal{F}_{n+1}(\mathcal{O}, \phi_{n+1})$$
Real - shower real

- With $\Delta \to 0$ in soft/collinear limit, $\Delta \to 1$ in hard regions. Use shower no-emission probability (between hard scale and scale of emission)
- Also: optimize shower starting scales and sampling
- $\Rightarrow$ Reduces fraction of negative weights [Frederix, Frixione, Prestel, Torrelli (2020)]

There's more

- Matching: discussed matrix element corrections a.k.a. LO multiplicative matching, MC@NLO and POWHEG
- Multiplicative matching also possible for NLO: KrkNLO [Jadach, Płazcek, Sapeta, Siódmok, Skrzypek (2015)]
  - Can be applied as weights $\rightarrow$ fast, but also efficient?
  - No negative weights
  - Hard to extend to generic processes
- MAcNLOPS: Multiplicative for $P_{\mathrm{exact}} < P_{\mathrm{shower}}$, MC@NLO otherwise [Nason, Salam (2022)]
  - No negative weights
  - Unrestricted applicability?

NLO Multi-jet Merging

Combine NLO Matching and Multi-leg Merging

Goal: Combine several NLO matrix elements for same process: NLO for X, $X+1$, $X+2$, ...
Mostly based on parton shower unitarity
Different methods available:

- UNLOPS, based on UMEPS [Lönnblad, Prestel (2013)][Plätzer (2013)]
- MiNLO, based on POWHEG [Hamilton, Nason, Zanderighi (2012)] [Frederix, Hamilton (2016)]
- FxFx, based on MC@NLO [Frederix, Frixione (2012)]
- MEPS@NLO, based on CKKWL[Höche, Krauss, Schönherr, Siegert (2013)]
- (Vincia, based on NLO MEC) [Hartgring, Laenen, Skands (2013)]
- ...

Multi-jet Merging at NLO

- UNLOPS [Lönnblad, Prestel (2013)]: Combine NLO matrix elements in unitary merging
- Subtract $\mathcal{O}(\alpha_s)$ from weights to preserve perturbative accuracy

$$\langle\mathcal{O}\rangle = \int d\phi_0 \left\{\mathcal{O}_0 \left[\bar{B}_0 - \int_S \bar{B}_{1\to 0} - \int_S B_{1\to 0}\left(w_1 - w_1|_{\mathcal{O}(\alpha_s)}\right)\right] + \int d\phi_1 \mathcal{O}_1 \left[\bar{B}_1 + B_1\left(w_1 - w_1|_{\mathcal{O}(\alpha_s)}\right)\right]\right\}$$

with $\bar{B}$ subtracted NLO cross sections, w CKKW-L weight as before

Freedom in Choice of Merging Scheme

Merging scheme should

- preserve fixed order quantum interference model
- preserve parton shower state evolution model

Define three valid variants of UNLOPS, look at 1 jet contribution

UNLOPS-1

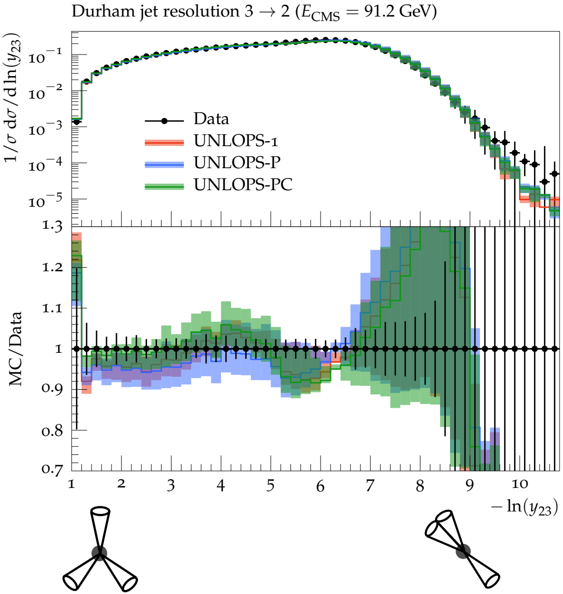
$$B_1 w_1 + \left[\bar{B}_1 - B_1 w_1|_{\mathcal{O}(\alpha_s)}\right]$$

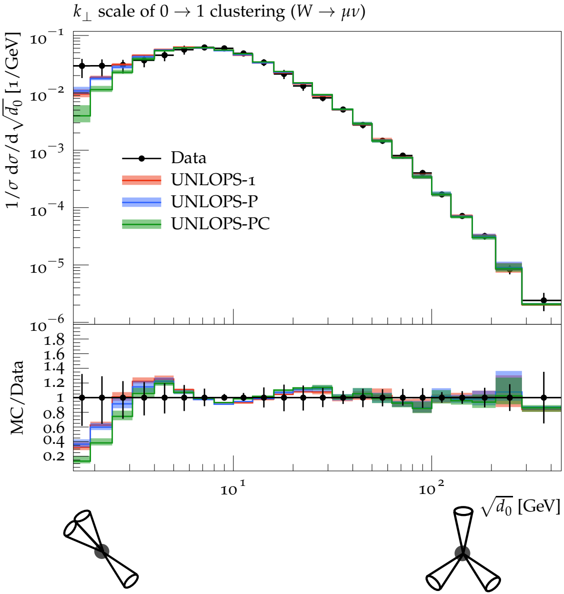
UNLOPS-P

$$B_1 w_1 + \left[\bar{B}_1 - B_1 w_1|_{\mathcal{O}(\alpha_s)}\right] \Pi_0(\rho_0, \rho_1, b)$$

UNLOPS-PC

$$B_1 w_1 + \left[\bar{B}_1 - B_1 w_1|_{\mathcal{O}(\alpha_s)}\right] \Pi_0(\rho_0, \rho_1, b) \frac{\alpha_s(b\rho_1)}{\alpha_s(b\mu_R)}$$


Durham jet resolution 3 → 2 (E_{CMS} = 91.2 GeV)
$1/\sigma\, d\sigma/d\ln(y_{23})$
Data
UNLOPS-1
UNLOPS-P
UNLOPS-PC
MC/Data
$-\ln(y_{23})$


$k_\perp$ scale of 0 → 1 clustering ($W \to \mu\nu$)
$1/\sigma\, d\sigma/d\sqrt{d_0}$ [1/GeV]
Data
UNLOPS-1
UNLOPS-P
UNLOPS-PC
MC/Data
$\sqrt{d_0}$ [GeV]

Beyond NLO

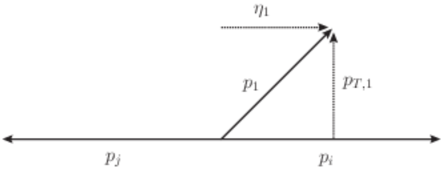
Beyond NLO

- NNLO results become available: combine with parton shower for fully exclusive predictions
- Unitarization well suited: replace lowest multiplicity to get inclusive NNLO cross section
  - UN2LOPS implemented in Sherpa [Höche, Li, Prestel (2015)]
  - Yet higher orders thinkable, see UN3LOPS/Tomte for toy implementation [Prestel (2021)]
- Fully differential NNLO+PS
  - Extension of POWHEG philosophy to NNLO
  - Born-local NNLO K-factor
  - Hardest-emission spectrum of PS given by NLO result (real-virtual and double-real corrections)
  - Proof-of-concept worked out for $e^+e^- \to 2j$ [Campbell, Höche, Li, Preuss, Skands (2023)]
- And more, e.g. MINNLOPS [Monni, Nason, Re, Wiesemann, Zanderighi (2020)], GENEVA [Alioli, Bauer, Berggren, Tackmann, Walsh (2015)]

Higher Orders in Parton Shower

The Lund plane

- Compute everything in center-of-mass frame of quarks



- Write momenta in Sudakov decomposition $\quad p_1 = p_1^+ + p_1^- + p_{T,1}$
  - On-shell condition: $p_1^2 = 2(p_1^+ p_1^- - p_{T,1}^2)$
  - "$-$"-projection: $p_1^- = 2p_i p_1 / \sqrt{2p_i p_j}$
  - "$+$"-projection: $p_1^+ = 2p_j p_1 / \sqrt{2p_i p_j}$
- Simple expressions for transverse momentum and rapidity
  - $p_{T,1}^2 = \dfrac{2(p_i p_1)(p_j p_1)}{p_i p_j}$
  - $\eta_1 = \dfrac{1}{2} \ln \dfrac{p_i p_1}{p_j p_1}$
- Semi-classical matrix element squared $\propto 1/p_T^2$

The Lund plane

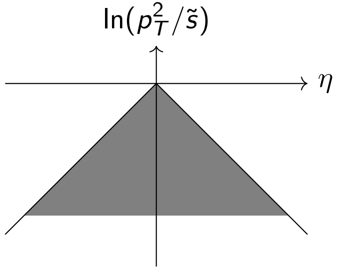
- Rewrite rapidity using transverse momentum

$$\eta_1 = \frac{1}{2}\ln\frac{p_i p_1}{p_j p_1} = \frac{1}{2}\ln\frac{s_{i1}^2}{p_{T,1}^2 s_{ij}} = \frac{1}{2}\ln\frac{p_{T,1}^2 s_{ij}}{s_{j1}^2}$$

- In momentum conserving parton branching $(\tilde{p}_i, \tilde{p}_j) \to (p_i, p_j, p_1)$

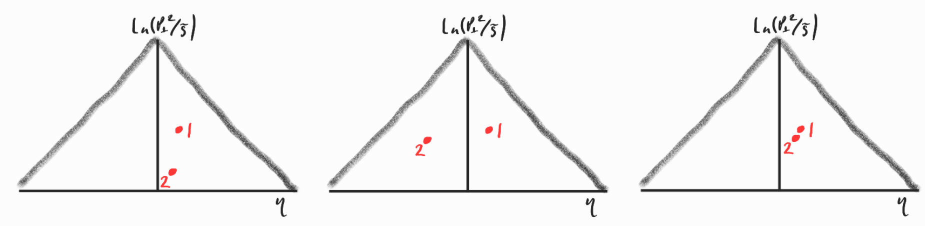
$$-\frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2} \le \eta_1 \le \frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2}$$

- Differential phase-space element $\propto \mathrm{d}p_T^2\,\mathrm{d}\eta$
- The Lund plane
  - $\eta, \ln(p_T^2/\tilde{s})$ plane
  - Phase space bounded by diagonals
  - Single-emission semi-classical radiation probability a constant



NLO Splitting Kernels: Motivation

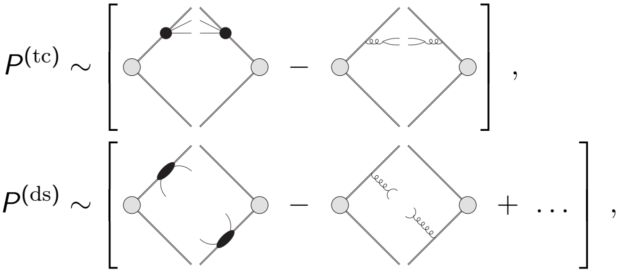
- QCD amplitudes factorise in soft and collinear limits
- Leading order shower has factorized $2 \to 3$ splitting kinematics implemented
- Higher orders in LO parton shower generated by iterating LO kernels
- Shower must reproduce the factorised amplitude for sufficiently independent emissions
- $\Rightarrow$ Any particle emitted after first one may NOT influence the kinematics of it (too much)



- First two should be correctly described by LO shower, third requires NLO splitting kernels

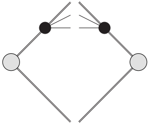
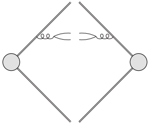
Double Soft and Triple Collinear Emissions

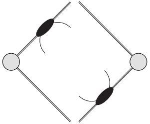
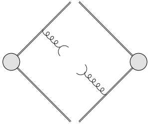
- Inclusion of double soft and triple collinear effects into NLO parton shower treated separately in [Höche, Prestel (2017)] and [Dulat, Höche, Prestel (2018) [hep-ph]]
- Two structurally different approximations. Implemented in shower as additional kernel, avoiding double counting with LO shower by subtracting iterated LO shower



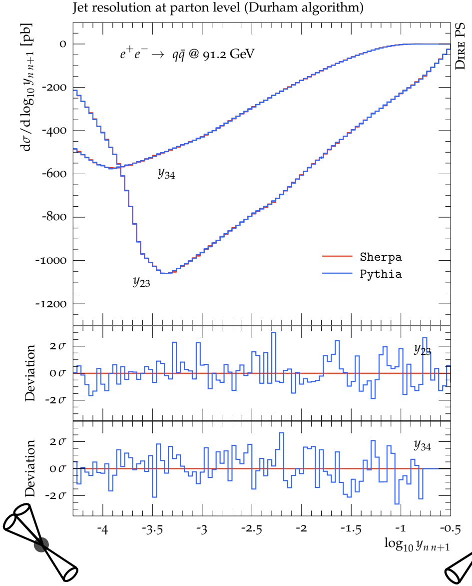
Combining Double Soft and Triple Collinear Emissions

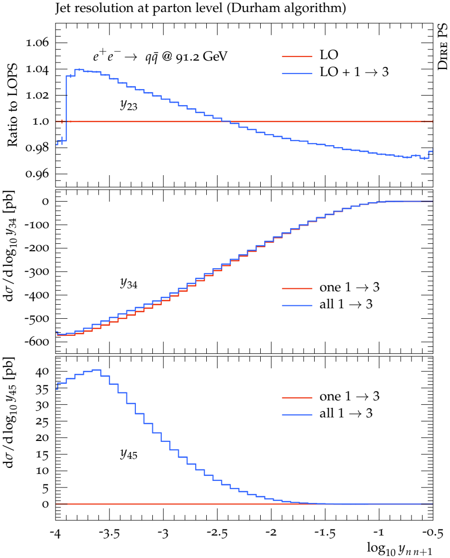
- Need both double soft and triple collinear emissions in full NLO shower, needed for NNLL/NNDL accuracy
- Remove overlap: include double soft, and subtract corresponding contribution from each triple collinear kernel [LG, Höche, Prestel (2022)]

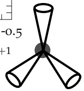
$P^{(\mathrm{tc-ds})} \sim \Big[$  $-$ 

$-$  $+$  $+ \ldots \Big].$

Validation and impact of soft-subtracted triple-collinear splittings







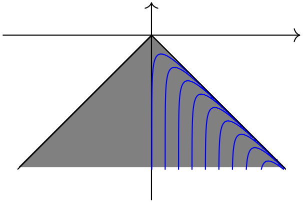
Shower Accuracy

What Determines Shower Accuracy?

- Assume we have correct splitting functions
- Freedom to choose ordering variable
- Freedom to choose recoil scheme
- ... and more
- $\Rightarrow$ Need to make careful choices!
- Problems with default dipole shower recoil, can spoil accuracy even for LO shower
  [Dasgupta,Dreyer,Hamilton,Monni,Salam] arXiv:1805.09327

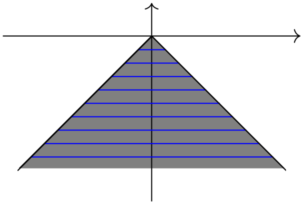
Ordering Variables in the Lund plane

Angular Ordering:
Lund plane filled from center to edges



- Dipole ends evolve separately: Parton shower
- Not ordered in $p_\perp^2$
- Color factors correct if observable insensitive to azimuthal correlations

Dipole Showers:
Lund plane filled from top to bottom

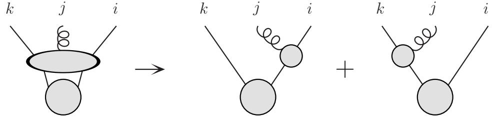


- Unified dipole and parton evolution
- Not ordered in η
- Color factors in improved leading color approximation

Structure of semi-classical matrix element

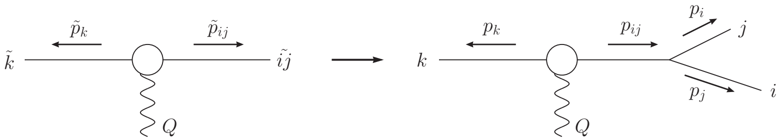
- Dipole shower approach: partial fraction matrix element & match to collinear sectors
  [Ellis,Ross,Terrano] NPB178(1981)421, [Catani,Seymour] hep-ph/9605323

$$\frac{p_i p_k}{(p_i p_j)(p_j p_k)} \to \frac{1}{p_i p_j}\frac{p_i p_k}{(p_i + p_k)p_j} + \frac{1}{p_k p_j}\frac{p_i p_k}{(p_i + p_k)p_j}$$



  - Convenient, Lorentz invariant formulation
  - Easy to integrate and use in NLO IR subtraction
  - Captures matrix element both in angular ordered and unordered region

Final state momentum mapping



- Generate off-shell momentum by rescaling

$$p_{ij}^{\mu} = \tilde{p}_{ij}^{\mu} + \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\,\tilde{p}_k^{\mu}\,, \qquad p_k^{\mu} = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^{\mu}$$

- Then branch into two on-shell momenta

$$p_i^{\mu} = \tilde{z}\,\tilde{p}_{ij}^{\mu} + (1-\tilde{z})\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^{\mu} + k_{\perp}^{\mu}, \qquad p_j^{\mu} = (1-\tilde{z})\,\tilde{p}_{ij}^{\mu} + \tilde{z}\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^{\mu} - k_{\perp}^{\mu}$$

- On-shell conditions require that

$$\vec{k}_T^2 = p_{ij}^2\,\tilde{z}(1-\tilde{z}) \qquad \leftrightarrow \qquad \tilde{z}_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\vec{k}_T^2/p_{ij}^2}\right)$$

$\rightarrow$ for any finite $\vec{k}_T$ we have $0 < \tilde{z} < 1$

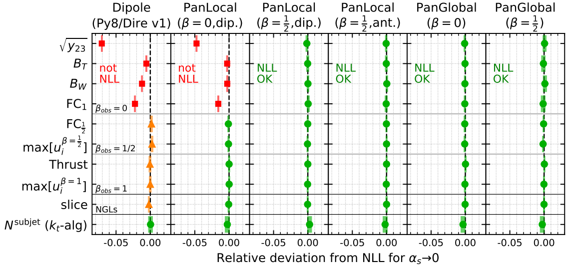
Momentum mapping in dipole-like showers

[Dasgupta,Dreyer,Hamilton,Monni,Salam,Soyez] arXiv:2002.11114

- Problem can be solved by partitioning of antenna radiation pattern and choosing a suitable evolution variable ($\beta \sim 1/2$)

$$k_T = \rho v e^{\beta|\bar{\eta}|} \qquad \rho = \left(\frac{s_i s_j}{Q^2 s_{ij}}\right)^{\beta/2}$$

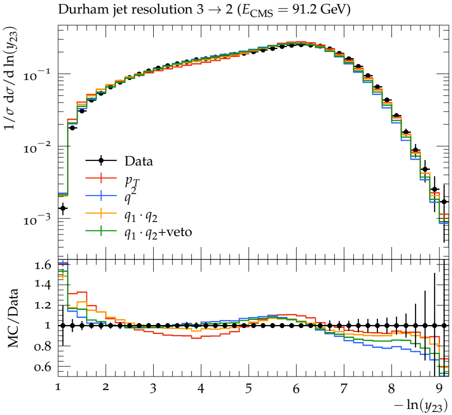
- Three different recoil schemes lead to NLL result if β chosen appropriately: Local dipole, local antenna, and global antenna
- NLL correct for global and non-global observables in $e^+e^- \to$hadrons



Momentum mapping in angular ordered showers

[Bewick,Ferrario-Ravasio,Richardson,Seymour] arXiv:1904.11866

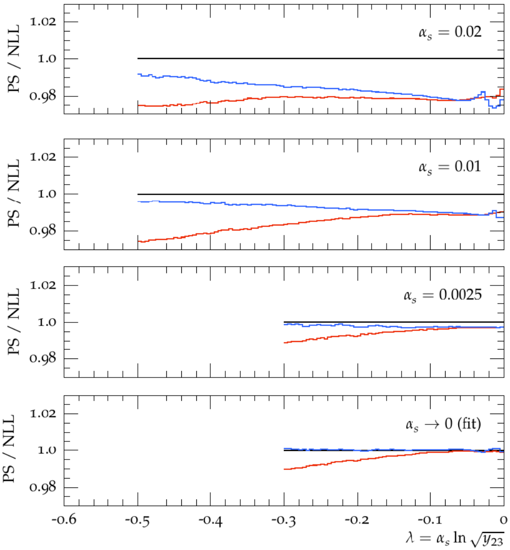
- Recoil schemes affect logarithmic accuracy but impact also phase-space coverage
- In context of angular ordered Herwig 7 (NLL accurate for global observables)
  - q_T preserving scheme:
    Maintains logarithmic accuracy
    Overpopulates hard region
  - q^2 preserving scheme:
    Breaks logarithmic accuracy
    Good description of hard region
  - Dot product preserving scheme (new):
    Maintains logarithmic accuracy
    Good description of hard radiation



Alaric

[Herren, Höche, Krauss, Reichelt, Schönherr (2022)]

- Partial fractioning of eikonal → positive definite splitting function with full phase space coverage
- Drawback: splitting kernels depend on azimuthal angle
- Global kinematics scheme enables analytic proof of NLL accuracy & numerical validation
- Right: Comparing Dire and Alaric



Summary Lecture II

Summary Lecture II

Goal: Add higher-order corrections into the picture

- NLO matrix element calculations require matching to parton showers
- NLO multi-jet merging allows for higher multiplicities at NLO
- Higher-order corrections also in shower

NLO Matching

- MC subtraction allows for NLO ME + PS
- MC@NLO and POWHEG
- Work towards NNLO

NLO Multi-jet Merging

- Combine multiple NLO ME samples
- Careful extension of LO techniques
- Different schemes available

NLO in parton shower

- Work on higher-order splitting kernels in parton showers
- It's not just about orders: recoil can spoil accuracy

Backup

Collinear Factorization

and Initial State Radiation

Collinear Factorization of QCD Cross Sections

Hadronic cross section for scattering $ab \to n$ given by

$$\sigma = \sum_{a,b} \int_0^1 \frac{\mathrm{d}x_a}{x_a} \frac{\mathrm{d}x_b}{x_b} \int x_a f_a^{h_1}(x_a, \mu_\mathrm{F}) x_b f_b^{h_2}(x_b, \mu_\mathrm{F}) \mathrm{d}\hat{\sigma}_{ab\to n}(\mu_\mathrm{F}, \mu_\mathrm{R})$$

- $\hat{\sigma}$ Partonic cross section
- $f_a^h(x_a, \mu_\mathrm{F})$ parton distribution functions (PDFs)
- x_a light cone momentum fraction $\to$ $x_a f_a$ momentum flux of parton a at x_a
- μ_F factorization scale

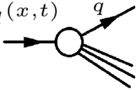
Need to take PDFs into account in initial state radiation (ISR), since they change flux

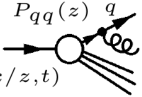
See [Collins, Soper, Sterman (1989)] for factorization theorems in QCD

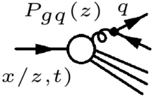
DGLAP Equations

[Dokshitzer (1977)] [Gribov, Lipatov (1972)] [Altarelli, Parisi (1977)]

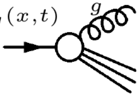
$$\frac{\mathrm{d}}{\mathrm{d}\log(t/\mu^2)} f_q(x,t) = \int_x^1 \frac{\mathrm{d}z}{z}\frac{\alpha_s}{2\pi} P_{qq}(z)\, f_q(x/z,t) + \int_x^1 \frac{\mathrm{d}z}{z}\frac{\alpha_s}{2\pi} P_{gq}(z)\, f_g(x/z,t)$$

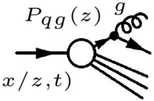


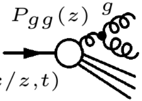




$$\frac{\mathrm{d}}{\mathrm{d}\log(t/\mu^2)} f_g(x,t) = \sum_{i=1}^{2n_f} \int_x^1 \frac{\mathrm{d}z}{z}\frac{\alpha_s}{2\pi} P_{qg}(z)\, f_q(x/z,t) + \int_x^1 \frac{\mathrm{d}z}{z}\frac{\alpha_s}{2\pi} P_{gg}(z)\, f_g(x/z,t)$$







- Coupled differential equations describing the parton flux of a hadron at different resolution scales

Initial State Radiation and PDFs

- Modify emission and no-emission probabilities to include PDFs: $x_{\mathrm{new}} = x/z$:

$$\mathrm{d}\mathcal{P}_{\mathrm{emission}}(\rho) = \frac{\mathrm{d}f_j}{f_j} = \frac{\mathrm{d}\rho}{\rho}\frac{\alpha_s}{2\pi}\int_{z_{\min}}^{z_{\max}} \mathrm{d}z P_{ij}(z) \frac{\frac{x}{z} f_i(\frac{x}{z}, \rho)}{x f_j(x, \rho)}$$

$$\mathcal{P}_{\mathrm{no-em}}(\rho_1, \rho_2) = \exp\left(-\int_{\rho_2}^{\rho_1} \frac{\mathrm{d}\rho}{\rho}\frac{\alpha_s}{2\pi}\int_{z_{\min}}^{z_{\max}} \mathrm{d}z P_{ij}(z) \frac{\frac{x}{z} f_i(\frac{x}{z}, \rho)}{x f_j(x, \rho)}\right) := \Pi(\rho_1, \rho_2)$$

- Initial state shower (more or less) reproduces DGLAP
- DGLAP evolution from small to large scale
- ISR usually uses “backwards evolution”: from large to small scales
  $\Rightarrow$ makes sure we can start from partonic process of interest at high scale [Sjöstrand (1985)]