Introduction to QED & QCD
Tutorial Questions

1. Suppose we have a plane-wave solution to the Klein-Gordon equation of the form
$$\phi(\boldsymbol{x}, t) = A\, e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}.$$
Use the Klein-Gordon equation to find the *dispersion relation*, i.e. find ω in terms of $\boldsymbol{k}$. How do you interpret the two solutions?
Show that these solutions are eigenstates of the energy operator, $i\partial_t$, and the 3-momentum operator, $-i\nabla$.

2. Show that the Dirac γ-matrices defined in the lectures:
$$\gamma^0 = \beta, \qquad \gamma^k = \beta\, \alpha^k,$$
obey the hermiticity relation
$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$

3. When evaluating cross sections, you will frequently need to manipulate Dirac matrices. Using the anti-commutation relations for the γ-matrices, show that in 4 dimensions:
$$\begin{aligned}
\text{(i)}\ & \gamma^\mu \gamma_\mu = 4, \\
\text{(ii)}\ & \gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu, \\
\text{(iii)}\ & \gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\nu\lambda}, \\
\text{(iv)}\ & \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\lambda \gamma^\nu.
\end{aligned}$$
How do these change in arbitrary dimensions where $g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d$?

4. Verify the orthonormality and completeness relations for the solutions of the Dirac equation:
$$\overline{u}_r(\boldsymbol{p}) u_s(\boldsymbol{p}) = -\overline{v}_r(\boldsymbol{p}) v_s(\boldsymbol{p}) = 2m\, \delta^{rs}, \quad \overline{u}_r(\boldsymbol{p}) v_s(\boldsymbol{p}) = \overline{v}_r(\boldsymbol{p}) u_s(\boldsymbol{p}) = 0,$$

and

$$\sum_{r=1}^{2} u_r(\boldsymbol{p})\overline{u}_r(\boldsymbol{p}) = (\not{p} + m), \qquad \sum_{r=1}^{2} v_r(\boldsymbol{p})\overline{v}_r(\boldsymbol{p}) = (\not{p} - m).$$

5. Show that the Dirac hamiltonian, $H = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta\, m$, commutes with the total angular momentum operator

$$[\boldsymbol{L} + \boldsymbol{S}, H] = 0\,,$$

where $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ is the orbital angular momentum and $\boldsymbol{S}$ is the spin operator

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

6. Using the plane-wave solutions of the Dirac equation given in the lectures, show that for $\boldsymbol{p} = (0, 0, p_z)$

$$S_z u_1 = \frac{1}{2}u_1, \qquad S_z u_2 = -\frac{1}{2}u_2, \qquad S_z v_1 = \frac{1}{2}v_1 \quad \text{and} \quad S_z v_2 = -\frac{1}{2}v_2,$$

where S_z is the z-component of the spin operator.

7. Draw all the tree-level diagrams for Bhabha-scattering, $e^+(p)\, e^-(k) \to e^+(p')\, e^-(k')$ and give the expression for the scattering amplitude, $i\mathcal{M}$, in Feynman gauge. What happens in an arbitrary gauge?

8. (a) Show that the process $e^+(k')\, e^-(k) \to \mu^+(p')\, \mu^-(p)$, in the limit $m_e \to 0$, has a matrix-element-squared given by

$$\overline{|\mathcal{M}|}^2 = \frac{1}{4}\frac{e^4}{(k+k')^4}\mathrm{Tr}\,[\not{k}'\gamma^\mu \not{k}\gamma^\nu]\,\mathrm{Tr}\,[(\not{p} + M)\gamma_\mu(\not{p}' - M)\gamma_\nu]\ ,$$

when summed and averaged over final and initial spins, where M is the mass of the muon.

   (b) Show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^- \to \mu^+\mu^-} = \frac{\overline{|\mathcal{M}|}^2}{64\pi^2 s}\sqrt{1 - \frac{4M^2}{s}}\,,$$

   where $s = (k+k')^2$.

(c) The traces evaluate to (check if you have time!)

$$\overline{|\mathcal{M}|}^2 = \frac{8e^4}{s^2}\left[(pk)^2 + (pk')^2 + M^2(kk')\right] .$$

Move to the centre-of-mass frame and let the scattering angle be θ. Show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to\mu^+\mu^-} = \frac{e^4}{64\pi^2 s}\sqrt{1-\frac{4M^2}{s}}\left[1+\left(1-\frac{4M^2}{s}\right)\cos^2\theta + \frac{4M^2}{s}\right] .$$

(d) Find an expression for the total cross section in the high-energy limit where the mass of the muon can be neglected.

9. Write the amplitude for Compton scattering $e(p)\,\gamma(k) \to e(p')\,\gamma(k')$ in the form $i\mathcal{M} = M_{\mu\nu}\varepsilon^{*\mu}(k')\varepsilon^{\nu}(k)$. Verify that this is gauge-invariant.

10. In the lectures, we found the matrix element squared for unpolarised Compton scattering was

$$\overline{|\mathcal{M}|}^2 = 2e^4\left(\frac{pk}{pk'} + \frac{pk'}{pk} + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right) .$$

Working in the centre-of-mass system, in the limit where the electron mass m can be neglected, show that the matrix element squared is dominated by backward scattering, $\theta \simeq \pi$, where θ is the scattering angle of the photon.

11. Use the matrix-element squared for Compton scattering to obtain the matrix-element squared for the annihilation process $e^+\,e^- \to \gamma\,\gamma$. Again work in the centre-of-mass frame and show that, in the high-energy limit $E \gg m$,

$$\overline{|\mathcal{M}|}^2 \simeq 4e^4\frac{1+\cos^2\theta}{\sin^2\theta} .$$

12. Consider the diagrams in figure 1. Show that the colour factors are given by

$$(a)\; t^c t^a t^b \delta^{bc} = -\frac{1}{2N_c}t^a, \quad \text{and} \quad (b)\; i f^{abc} t^b t^c = -\frac{1}{2}C_A t^a$$

respectively.

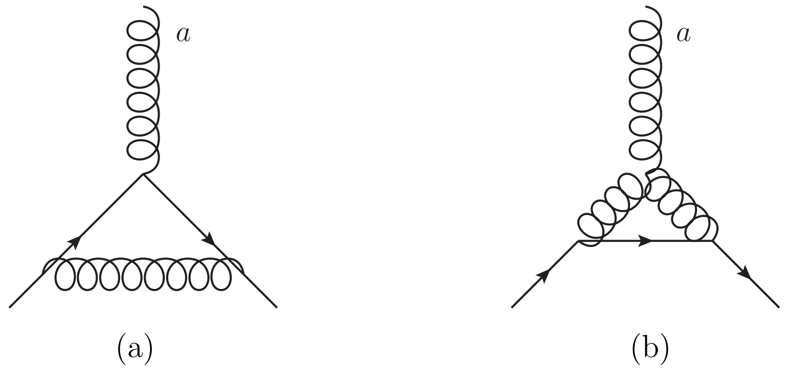

Figure 1: One-loop corrections to a $q\bar{q}g$-vertex.

13. Calculate the summed and averaged matrix-element squared, $\overline{|\mathcal{M}|}^2$, for the quark-scattering process $ud \to ud$.

14. Solve the one-loop β-functions for QCD and QED:

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = -\frac{11C_A - 2n_f}{12\pi}\alpha_s^2, \qquad \text{and} \qquad \mu^2 \frac{d\alpha}{d\mu^2} = \frac{1}{3\pi}\alpha^2\,,$$

using as initial condition the value of the couplings at the Z mass. Sketch the solutions as a function of μ^2.