Quantum Field Theory - Tuesday Problems

Christopher McCabe (King's College London)

1 Classical Physics

1.1 Consider a transformed Lagrangian L', which is related to another Lagrangian L as follows:

$$L'(\dot{q}, q, t) = L(\dot{q}, q, t) + \frac{dF(q,t)}{dt} \ . \tag{1}$$

Here, F is an arbitrary function of q and t but is not a function of $\dot{q}$.
Show that the Euler-Lagrange equations are invariant under this transformation.
What does this imply about the uniqueness of the Lagrangian for a given physical system (e.g. the Lagrangian for the Simple Harmonic Oscillator)?

1.2 Show that if the Hamiltonian does not depend on time explicitly (i.e. $\partial H/\partial t = 0$), then H is a constant of motion.
In many cases when H is a constant of the motion, it is identified with a well known quantity. Which quantity?

1.3 Verify that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E(\mathbf{k})}} \left\{ e^{ik\cdot x} a(\mathbf{k}) + e^{-ik\cdot x} b(\mathbf{k}) \right\}$$

is a solution of the Klein-Gordon equation if $E(\mathbf{k})^2 = \mathbf{k}^2 + m^2$.
Show that a real scalar field $\phi^*(x) = \phi(x)$ requires the condition $b(\mathbf{k}) = a^*(\mathbf{k})$.

1.4 The Lagrangian density for classical 'ϕ^4-theory' is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ .$$

Use the Euler-Lagrange equations to find the field equation that ϕ satisfies.

1.5 Derive the components P_0, $\mathbf{P}$ of the energy-momentum four-vector P^μ for classical ϕ^4-theory.

1.6 Calculate the Hamiltonian density $\mathcal{H}$ for ϕ^4-theory. Is this Hamiltonian density Lorentz invariant?

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1 Free quantum fields: working with commutators

1.1 Consider the Heisenberg equation of motion for the momentum operator $\hat{p}$ of the harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right),$$

and show that it is equivalent to Newton's law for the position operator $\hat{x}$.

1.2 Starting from the expression

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E(\mathbf{k})}}\left(e^{ik\cdot x}\hat{a}^\dagger(\mathbf{k}) + e^{-ik\cdot x}\hat{a}(\mathbf{k})\right) ,$$

find the corresponding expression for the canonical momentum operator $\hat{\pi}(x) = \partial_0\hat{\phi}(x)$.

1.3 Show that the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)\right] = i\delta^3(\mathbf{x}-\mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{x}',t)\right] = \left[\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)\right] = 0 ,$$

imply that

$$[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{p})] = 0 .$$

(A similar method would also show that $\left[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{p})\right] = 0$.)

1.4 In this problem, we want to show that the scalar field operator $\hat{\phi}(\mathbf{x},t)$ satisfies the Klein-Gordon equation:

$$\partial_\mu\partial^\mu\hat{\phi}(\mathbf{x},t) + m^2\hat{\phi}(\mathbf{x},t) = 0 .$$

We know already that

$$\hat{\pi}(\mathbf{x},t) = \partial_t\hat{\phi}(\mathbf{x},t) .$$

We now need to find an equation for $\partial_t\hat{\pi}(\mathbf{x},t)$. This can be done with the Heisenberg equation of motion, which for a general field operator $\hat{O}$ is

$$\frac{\partial}{\partial t}\hat{O} = i[\hat{H}, \hat{O}] ,$$

where

$$\hat{H} = \frac{1}{2}\int d^3x \left\{\hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right\} .$$

Assuming the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)\right] = i\delta^3(\mathbf{x}-\mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{x}',t)\right] = \left[\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)\right] = 0 ,$$

evaluate $[\hat{H}, \hat{\pi}(\mathbf{x},t)]$ to show that

$$\partial_t\hat{\pi}(\mathbf{x},t) = \nabla^2\hat{\phi}(\mathbf{x},t) - m^2\hat{\phi}(\mathbf{x},t) .$$

[Hint: You will need to integrate terms involving $(\nabla\hat{\phi})^2$ by parts and assume that the fields vanish at the boundary of space (spatial infinity).]

Quantum Field Theory - Thursday Problems

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1.1 What is the normal ordered product $:\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{q})\hat{a}(\mathbf{r})\hat{a}^\dagger(\mathbf{s}):$?

1.2 After normal ordering, the conserved three-momentum $P^i = \int d^3x T^{0i}$ takes the form

$$:\hat{P}^i := \int \frac{d^3p}{(2\pi)^3} p^i \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \ .$$

Prove the commutator relation

$$\left[:\hat{P}^i:, \hat{a}^\dagger(\mathbf{k})\right] = k^i \hat{a}^\dagger(\mathbf{k}) \ .$$

1.3 Write down the general result for $\left[:\hat{P}^\mu:, \hat{a}^\dagger(\mathbf{k})\right]$ in terms of k^μ and $\hat{a}^\dagger(\mathbf{k})$. Hence show that

$$:\hat{P}^\mu: \ \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\, \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle. \tag{1}$$

Interpret the physics of this result.

1.4 The number operator is

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})$$

Prove by induction that

$$\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \ \underbrace{|\mathbf{k},\ldots,\mathbf{k}\rangle}_{n \text{ momenta}} = n \ \underbrace{|\mathbf{k},\ldots,\mathbf{k}\rangle}_{n \text{ momenta}} \ .$$

[**Hint:** induction proceeds in two steps. *i)* show that the statement is true for some starting value of n; *ii)* show that if the statement holds for some general n, then it also holds for $n+1$.]

1.5 Show that $\hat{N}$ is a constant of motion when

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} E_p \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \ .$$

1.6 We normalise our momentum eigenstates such that $\langle\mathbf{p}|\mathbf{k}\rangle = 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})$. Show that the combination $E_p\,\delta^3(\mathbf{p}-\mathbf{k})$ is Lorentz invariant.

Quantum Field Theory - Friday Problems

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1.1 Consider a theory with the Hamiltonian $H = H_0 + H_{\text{int}}$. Using the definition $U(t) = e^{iH_0 t}\, e^{-iHt}$, derive the evolution equation for $U(t)$:

$$i\frac{d}{dt}\,U(t) = H_{\text{int}}(t)\,U(t),$$

where

$$H_{\text{int}}(t) = e^{iH_0 t}\,H_{\text{int}}\,e^{-iH_0 t}.$$

1.2 Use Wick's theorem to find an expression for $T[\phi(x_1)\phi(x_2)\phi(x_3)]$ in terms of $N[\phi(x_1)\phi(x_2)\phi(x_3)]$ and the Feynman propagators $D_F(x_i - x_j)$.

1.3 Given that ϕ_{in} is a free field, obeying the Heisenberg equation of motion

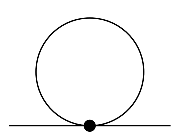
$$\dot{\phi}_{\text{in}} = i\,[H_0(\phi_{\text{in}}, \pi_{\text{in}}),\, \phi_{\text{in}}]\,,$$

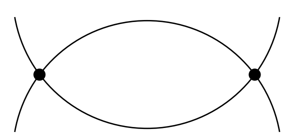
show that ϕ_{out} is also a free field, which obeys

$$\dot{\phi}_{\text{out}} = i\,[H_0(\phi_{\text{out}}, \pi_{\text{out}}),\, \phi_{\text{out}}]\,.$$

[**Hint:** use $\phi_{\text{out}} = S^\dagger \phi_{\text{in}} S$ and $\pi_{\text{out}} = S^\dagger \pi_{\text{in}} S$. Keep in mind that the S-matrix has no explicit time dependence.]

1.4 (Harder) Find the expressions corresponding to the following *momentum space* Feynman diagrams for the scattering amplitude (i.e. the truncated Green's function)





Integrate out all the δ-functions but do not perform the remaining integrals. Argue from the behaviour of the integrands as the loop momenta diverge that both of these Feynman diagrams give infinite results. Which one is more divergent?