Quantum Field Theory – Lecture 9

Last time we were evaluating the $2 \to 2$ amplitude $A_{2\to 2}$ and we had arrived to the expression

$$A_{2\to2} = \langle 0 | \hat{T}(\hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2} \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2}) | 0 \rangle$$
$$- \frac{i\lambda}{4!} \int d^4x \, \langle 0 | \hat{T}(\hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2} \hat{\phi}^4(x) \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2}) | 0 \rangle$$
$$+ \mathcal{O}(\lambda^2).$$

If scattering really does happen and $|\vec{p}_1 \vec{p}_2\rangle \neq |\vec{k}_1 \vec{k}_2\rangle$, then $\langle \vec{p}_1 \vec{p}_2 | \vec{k}_1 \vec{k}_2 \rangle = \langle 0 | \hat{T}(\hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2} \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2}) | 0 \rangle = 0$. Therefore, the order-$\lambda^0$ term in $A_{2\to2}$ is zero. For the order-λ term we want to apply Wick's theorem to write

$$A^{(\lambda)}_{2\to2} = \underbrace{\langle 0 | \hat{N}(\hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2} \hat{\phi}^4(x) \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2}) | 0 \rangle}_{0} + (D_F \text{ terms}).$$

For the D_F terms we have two kinds:

$$\overbracket{\phi(x)\,\phi(y)} \longrightarrow D_F(x-y)$$
$$\overbracket{\phi(x)\,\hat{a}_{\vec{k}}}\,, \quad \overbracket{\phi(x)\,\hat{a}^\dagger_{\vec{k}}} \longrightarrow ?$$

We have

$$\overbracket{\phi(x)\,\hat{a}^\dagger_{\vec{k}}} = \langle 0 | \hat{T}\, \hat{\phi}(x)\, \hat{a}^\dagger_{\vec{k}} | 0 \rangle$$
$$= \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (\hat{a}_{\vec{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}} e^{ip\cdot x})\, \hat{a}^\dagger_{\vec{k}} | 0 \rangle$$
$$= \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}\, \hat{a}_{\vec{p}}\, e^{-ip\cdot x}\, \hat{a}^\dagger_{\vec{k}} | 0 \rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}\, e^{-ip\cdot x} \langle 0 | [\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{k}}] | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip\cdot x} (2\pi)^3 \, 2\omega_{\vec{p}} \, \delta^{(3)}(\vec{p}-\vec{k})$$

$$= e^{-ik\cdot x},$$

since $\omega_{\vec{k}} = \omega_{\vec{p}}$ when $\vec{k} = \vec{p}$. Similarly,

$$\hat{a}_{\vec{k}} \, \phi(x) = e^{ik\cdot x}$$

Pictorially we represent

$\phi(x)|k\rangle = \phi(x)\, \hat{a}^{\dagger}_{\vec{k}} \longrightarrow$  $e^{-ik\cdot x}$

$\langle k|\phi(x) = \hat{a}_{\vec{k}}\, \phi(x) \longrightarrow$ $e^{ik\cdot x}$

These are external lines.

Back to our computation

$$A^{(\lambda)}_{2\to 2} = -\frac{i\lambda}{4!} \int d^4x \langle \vec{p}_1 \, \vec{p}_2 | \phi(x)\,\phi(x)\,\phi(x)\,\phi(x) | \vec{k}_1 \, \vec{k}_2 \rangle$$

$$= -\frac{i\lambda}{4!} \int d^4x \langle \vec{p}_1 \, \vec{p}_2 | \phi(x)\,\phi(x)\,\phi(x)\,\phi(x) | \vec{k}_1 \, \vec{k}_2 \rangle$$

+ (every way of contracting the ϕ's with p's and k's)

4! = 24 ways

$$- \frac{i\lambda}{4!} \int d^4x \langle \vec{p}_1 \, \vec{p}_2 | \phi(x)\,\phi(x)\,\phi(x)\,\phi(x) | \vec{k}_1 \, \vec{k}_2 \rangle$$

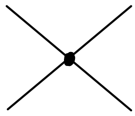
+ (every way of contracting ϕ's among them and p's with k's).

The first set of terms correspond to connected diagrams, while the second to disconnected ones. Connected terms are of the form

momentum conserving delta function

$$-\frac{i\lambda}{4!} \int d^4x \, e^{ip_1\cdot x} e^{ip_2\cdot x} e^{-ik_1\cdot x} e^{-ik_2\cdot x} + \cdots = -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2).$$

This is represented by



$-i\lambda$

and it is the Feynman rule for our interaction vertex.

We have finished the computation of $A_{2\to2}$ up to order λ in ϕ^4 theory:

$$A_{2\to2} = -i\lambda\,(2\pi)^4\delta^{(4)}(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2) + \mathcal{O}(\lambda^2)$$

where we have included the connected part only.

Feynman rules for building an amplitude

- Position space

  External legs $\quad e^{\pm ip\cdot x}$

  Vertices $\quad -i\lambda$

  Propagators $\quad D_F(x-y)$

  Integrate positions of vertices over spacetime

  Account for symmetry factor

  will talk about this below

- Momentum space

  External legs $\quad 1$

  Vertices $\quad -i\lambda$

  Propagator $\quad \dfrac{i}{p^2-m^2+i\varepsilon}$

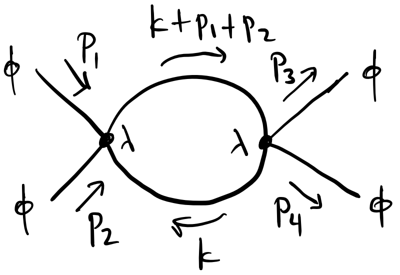
  Enforce energy-momentum conservation $\quad (2\pi)^4\delta^{(4)}(\Sigma p_i)$

  Integrate over loop momenta

  Account for symmetry factor

A one-loop example

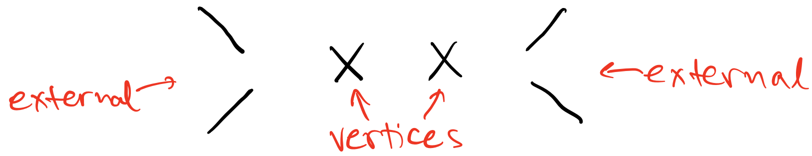
We will evaluate



From Feynan rules:

$$iM = \underbrace{\tfrac{1}{2}}_{\text{symmetry factor}} (-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(k+p_1+p_2)^2-m^2+i\varepsilon} \underbrace{\left[(2\pi)^4 \delta^{(4)}(\Sigma p_i)\right]}_{\text{momentum conservation}}$$

Let us do the calculation at zero external momenta. From the Dyson expansion we now keep the quadratic term.



Connect lines!

$$\text{symmetry factor} = \frac{1}{2}\,\frac{1}{4!}\,\frac{1}{4!}(2\times 4\times 3)(4\times 3)\times 2 = \frac{1}{2}$$

$$iM = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\varepsilon)^2}.$$

Recall that $d^4k = dk^0\, d^3k$. Define $k^0 = i k_E^0$, $\vec{k} = \vec{k}_E$. Then, $d^4k \to i\,dk_E^0\, d^3k_E = i\, d^4k_E$ and $k^2 \to i^2 (k_E^0)^2 - \vec{k}^2 = -k_E^2$.

For iM we have

$$iM = \frac{i\lambda^2}{2} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2+m^2)^2} = \frac{i\lambda^2}{2} \frac{1}{(2\pi)^4} \underbrace{\int d\Omega_4}_{2\pi^2} \int_0^\infty d|k_E| \frac{|k_E|^3}{(|k_E|^2+m^2)}$$

spherical coordinates

radius

The remaining $|k_E|$-integral is divergent. If we regularise it, i.e. if we don't integrate all the way up to ∞ but up to some $\Lambda \gg m$, then,

$$iM = \frac{i\lambda^2}{16\pi^2} \log\frac{\Lambda}{m}$$

famous log-divergence

famous $16\pi^2$

Renormalisation

Divergences like the ones we just encountered had a big impact on QFT. The pioneers of QFT had a hard time dealing with them. Suppose we restored the momenta in the calculation above. With

$$s = (p_1+p_2)^2, \qquad t = (p_1-p_3)^2, \qquad u = (p_1-p_4)^2$$

the answer, including the order-λ term, is

$$iM = -i\lambda + i\frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right) + O(\lambda^3),$$

where we have assumed that m is small relative to the external momenta.

An experimentalist who would measure such an amplitude and try to compare with our answer would be confused about what λ and Λ are. The experimentalist would measure some effective coupling $\lambda_{physical}$ and would attempt to say that

$$\underbrace{-i\lambda_{physical}}_{iM} = -i\lambda + i\frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0}\right).$$

experiment

But what are we to choose for Λ in the right-hand side? Well, the theorist says that it doesn't matter. In fact λ is a function of Λ so that the right-hand side doesn't really depend on Λ. Then,

$$\lambda = \lambda_{physical} + \frac{\lambda^2}{32\pi^2}L_0 + \dots \qquad L_0 = \log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0}$$

$$= \lambda_{physical} + \frac{\lambda_{physical}^2}{32\pi^2}L_0 + \dots$$

Now the amplitude at s, t, u can be written as

$$i\mathcal{M} = \underbrace{-i\lambda_{\text{physical}} - i\frac{\lambda_{\text{physical}}^2}{32\pi^2}L_0}_{\text{from } -i\lambda} + i\frac{\lambda_{\text{physical}}^2}{32\pi^2}L + \cdots$$

$$= -i\lambda_{\text{physical}} - i\frac{\lambda_{\text{physical}}^2}{32\pi^2}(L_0 - L) + \cdots$$

$$= -i\lambda_{\text{physical}} + i\frac{\lambda_{\text{physical}}^2}{32\pi^2}\left(\log\frac{s_0}{s} + \log\frac{t_0}{t} + \log\frac{u_0}{u}\right) + \cdots$$

This is renormalisation: the amplitude is actually something we can express in physical parameters, as opposed to the unphysical ones in our Lagrangian. That happens to all orders in the coupling. In the expression above we agree with the experimentalist at $s = s_0$, $t = t_0$, $u = u_0$ as designed (log's are zero then), and we predict what the experimentalist will measure at some other s, t, u.