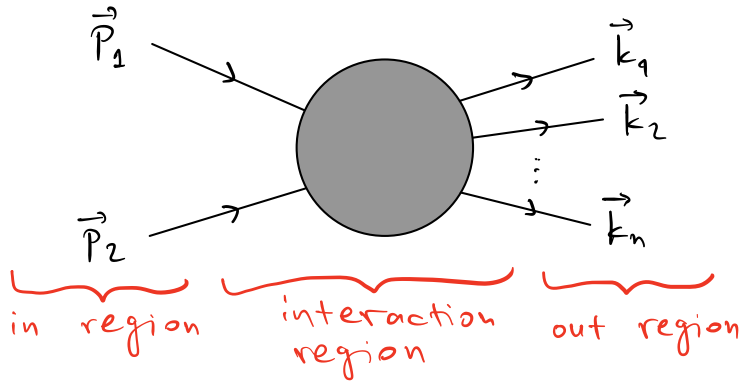
Quantum Field Theory – Lecture 8

Scattering

The scattering of particles, e.g. that of two particles with momenta $\vec{p}_1, \vec{p}_2$ into n particles with momenta $\vec{k}_1, \dots, \vec{k}_n$, can be represented by



In the in and out regions the particles are free, while the interactions happen in the interaction region and typically last a very short time.

In the in region, we may define

$$\hat{\phi}_{in}(x) = \text{"}\lim_{t\to-\infty}\text{"}\ \hat{\phi}(x)$$

and similarly

$$\hat{\phi}_{out}(x) = \text{"}\lim_{t\to\infty}\text{"}\ \hat{\phi}(x).$$

These are free fields:

$$(\partial^2+m^2)\,\hat{\phi}_{in} = (\partial^2+m^2)\,\hat{\phi}_{out} = 0.$$

They can be written as

$$\hat{\phi}_{in}(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k},in}\,e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k},in}\,e^{ik\cdot x}\right),$$

$$\hat{\phi}_{out}(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k},out}\,e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k},out}\,e^{ik\cdot x}\right).$$

The operators $\hat{a}^{+}_{\vec{k},in}$ create in states and the operators $\hat{a}^{+}_{\vec{k},out}$ create out states. Since

$$\langle in|\,\hat{\phi}_{in}\,|in\rangle = \langle out|\,\hat{\phi}_{out}\,|out\rangle,$$

same thing in different bases

it must be that there exists some operator $\hat{S}$ such that

$$\hat{\phi}_{in} = \hat{S}\,\hat{\phi}_{out}\,\hat{S}^{\dagger}, \quad |in\rangle = \hat{S}|out\rangle, \quad \langle in| = \langle out|\,S^{\dagger}, \quad \hat{S}^{\dagger}\hat{S} = 1.$$

This operator $\hat{S}$ is called the S-matrix.

Returning to our scattering picture above,

$$\begin{aligned}\langle \vec{k}_1 \cdots \vec{k}_n | \vec{p}_1 \vec{p}_2 \rangle &= \langle \vec{k}_1 \cdots \vec{k}_n (t \to +\infty) | \hat{U}(\infty,0)\,\hat{U}(0,-\infty) | \vec{p}_1 \vec{p}_2 (t \to -\infty)\rangle \\ &= {}_{free}\langle \vec{k}_1 \cdots \vec{k}_n | \underbrace{\hat{U}(\infty,0)\,\hat{U}(0,-\infty)}_{\hat{S}} | \vec{p}_1 \vec{p}_2 \rangle_{free}\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{S} &= \hat{U}(\infty,0)\,\hat{U}(0,-\infty) \\ &= \hat{U}(\infty,-\infty) \\ &= \hat{T}\,e^{-i\int_{-\infty}^{\infty}\hat{H}_{int,I}(t)\,dt}.\end{aligned}$$

Note that if $\hat{H}_{int} = 0$, then $\hat{S} = 1$ as would be expected.

Dyson's expansion

Our expression for $\hat{S}$ as a time-ordered exponential will be understood in perturbation theory:

$$\hat{S} = \hat{T}\left(1 - i\int_{-\infty}^{\infty} dt\,\hat{H}_{int,I}(t) + \cdots\right).$$

We will use

$$\hat{H}_{int,I}(t) = \int d^3x\,\hat{\mathcal{H}}_{int,I}(t)$$

Hamiltonian density

to write

$$\hat{S} = T\left(1 - i\int d^4x\,\hat{\mathcal{H}}_{int,I}(t) + \cdots\right)$$

In ϕ^4 theory,

$$\hat{\mathcal{H}}_{int} = \frac{1}{4!}\lambda\phi^4,$$

so

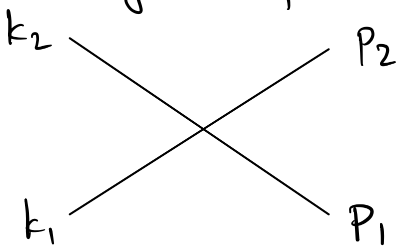
$$\hat{\mathcal{H}}_{int,I}(t) = e^{i\hat{H}_0 t}\,\hat{\mathcal{H}}_{int}(t=0)\,e^{-i\hat{H}_0 t}$$

$$= \frac{1}{4!}\lambda\, e^{i\hat{H}_0 t}\,\hat{\phi}^4\, e^{-i\hat{H}_0 t}$$

$$= \frac{1}{4!}\lambda\, e^{i\hat{H}_0 t}\,\hat{\phi}\,\underbrace{e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}}_{1}\,\phi \cdots \hat{\phi}\, e^{-i\hat{H}_0 t}$$

$$= \frac{1}{4!}\lambda\,\hat{\phi}_I^4, \qquad \hat{\phi}_I(t) = e^{i\hat{H}_0 t}\,\hat{\phi}(t=0)\,e^{-i\hat{H}_0 t}.$$

Let's compute the scattering amplitude

$A_{2\to 2} =$ 

1. Use vacuum states:

$$A_{2\to 2} = {}_{free}\langle \vec{p}_1\, p_2 |\, \hat{S}\, | \vec{k}_1 \vec{k}_2 \rangle_{free}$$

$$= \langle 0 |\, \hat{a}_{\vec{p}_1}\hat{a}_{\vec{p}_2}\,\hat{S}\,\hat{a}^{+}_{\vec{k}_1}\hat{a}^{+}_{\vec{k}_2} | 0 \rangle$$

We assume that in and out states have the same vacuum.

2. Work to leading order in λ:

drop label I but recall interaction picture

$$\hat{S} = \hat{T}\left(1 - i\frac{\lambda}{4!}\int d^4x\, \hat{\phi}^4(x) + \cdots\right)$$

3. Plug this into $A_{2\to 2}$:

$$A_{2\to 2} = \hat{T}\Big(\langle 0 |\, \hat{a}_{\vec{p}_1}\hat{a}_{\vec{p}_2}\hat{a}^{+}_{\vec{k}_1}\hat{a}^{+}_{\vec{k}_2} | 0 \rangle$$

$$- \frac{i\lambda}{4!}\int d^4x\, \langle 0 |\, \hat{a}_{\vec{p}_1}\hat{a}_{\vec{p}_2}\,\hat{\phi}^4(x)\,\hat{a}^{+}_{\vec{k}_1}\hat{a}^{+}_{\vec{k}_2} | 0 \rangle$$

$$+ \cdots \Big)$$

4. Use Wick's theorem to simplify expression.

Wick's theorem

Wick's theorem says that time-ordered products of operators are given by

$$\hat{T}(\phi_1 \cdots \phi_N) = \hat{N}(\phi_1 \cdots \phi_n)$$
$$+ \hat{N}(\text{all possible contractions of } \phi_1, \ldots, \phi_N),$$

where $\hat{N}$ denotes the normal-ordered product (which we denoted by $: :$ previously). A contraction replaces two ϕ's with a Feynman propagator:

$$\overline{\phi(x_1)\,\phi(x_2)} \longrightarrow D_F(x_1 - x_2).$$

For example,

$$\hat{T}(\phi_1(x_1)\,\phi_2(x_2)) = \hat{N}(\phi_1(x_1)\,\phi_2(x_2)) + \overline{\phi_1(x_1)\,\phi_2(x_2)}$$
$$= \hat{N}(\phi_1(x_1)\,\phi_2(x_2)) + D_F(x_1 - x_2),$$

$$\hat{T}(\phi_1\phi_2\phi_3\phi_4) = \hat{N}(\phi_1\phi_2\phi_3\phi_4) + \overline{\phi_1\phi_2}\,\hat{N}(\phi_3\phi_4)$$
$$+ \overline{\phi_1\phi_3}\,\hat{N}(\phi_2\phi_4) + \cdots$$
$$+ \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \overline{\phi_1\overline{\phi_2\phi_3}\phi_4}$$
$$= \hat{N}(\phi_1\phi_2\phi_3\phi_4) + D_F(x_1 - x_2)\,\hat{N}(\phi_3\phi_4)$$
$$+ D_F(x_1 - x_3)\,\hat{N}(\phi_2\phi_4) + \cdots$$
$$+ D_F(x_1 - x_2)\,D_F(x_3 - x_4) + D_F(x_1 - x_3)\,D_F(x_2 - x_4)$$
$$+ D_F(x_1 - x_4)\,D_F(x_2 - x_3).$$

Then,

$$\langle 0|\hat{T}(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \langle 0|D_F(x_1 - x_2)\,D_F(x_3 - x_4)|0\rangle$$
$$+ \langle 0|D_F(x_1 - x_3)\,D_F(x_2 - x_4)|0\rangle$$
$$+ \langle 0|D_F(x_1 - x_4)\,D_F(x_2 - x_3)|0\rangle,$$

because

$$\langle 0|\hat{N}(\text{anything})|0\rangle = 0$$

since $\hat{a}_{\vec{k}}|0\rangle = 0$.

Now $D_F(x-y)$ is not an operator so it can simply come out of $\langle 0|\ldots|0\rangle$. Using $\langle 0|0\rangle = 1$, we find

$$\langle 0|\hat{T}(\phi_1\phi_2)|0\rangle = D_F(x_1-x_2)\langle 0|0\rangle = D_F(x_1-x_2),$$

$$\begin{aligned}\langle 0|\hat{T}(\phi_1\phi_2\phi_3\phi_4)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &+ D_F(x_1-x_3)D_F(x_2-x_4)\\ &+ D_F(x_1-x_4)D_F(x_2-x_3).\end{aligned}$$

Thus we have completed the computation of vacuum expectation values of time-ordered products of scalar operators. These results will be used to finish the computation of $A_{2\to2}$. Before going back to that, let us introduce a pictorial representation of our results above.

5. Interpret vacuum expectation values as Feynman diagrams.

Propagators = lines

$$\langle 0|\hat{T}\phi(x)\phi(y)|0\rangle = D_F(x-y) =$$ 

$$\langle 0|\hat{T}(\phi_1\phi_2\phi_3\phi_4)|0\rangle =$$