Quantum Field Theory – Lecture 7

The Feynman propagator has the non-covariant form

$$D_F(x-y) = \theta(x^0-y^0)\,D(x-y) + \theta(y^0-x^0)\,D(y-x)$$

We can, however, find a nicer (covariant) form.

Two properties of $D_F(x-y)$

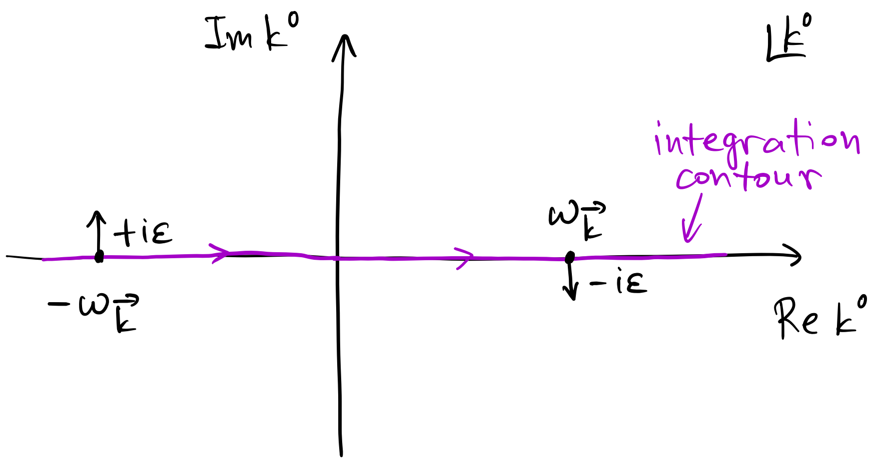
1) $$D_F(x-y) = \lim_{\varepsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,e^{-ik\cdot(x-y)}$$

- Here we integrate over $d^4k = d^3k\,dk^0$.
- Recall $\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$ (independent of k^0)
- But now we integrate over k^0, so we allow $(k^0)^2 \neq \vec{k}^2 + m^2$. Thus, we allow "off-shell" particles.

Proof

Take $x^0 > y^0$. Let us compute

$$\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\,\frac{i}{k^2-m^2+i\varepsilon}\,e^{-ik^0(x^0-y^0)}.$$



$$k^2 = (k^0)^2 - \vec{k}^2 \Rightarrow k^2 - m^2 = (k^0)^2 - \omega_{\vec{k}}^2$$

and so

$$k^2 - m^2 = 0 \quad \text{when} \quad k^0 = \pm\,\omega_{\vec{k}}.$$

With the $i\varepsilon$ prescription above we see that the poles are shifted to

$$k^0 = \pm(\omega_{\vec{k}} - i\varepsilon'), \quad \varepsilon' = \frac{\varepsilon}{2\omega_{\vec{k}}}.$$

This allows us to perform the integral over k^0. For $x^0 > y^0$, we can close the contour in the lower half plane, where $\mathrm{Im}\, k^0 < 0$ and we get zero contribution to the integral since

$$e^{-ik^0(x^0-y^0)} \to e^{-i\cdot \underbrace{i\cdot(\text{neg.}\infty)}_{k^0}\cdot\underbrace{(\text{pos})}_{x^0-y^0}} = e^{-\infty} = 0$$

By Cauchy's theorem, the result of the closed contour integration is given by the residue of the pole at $\omega_{\vec{k}} - i\varepsilon$:

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{i e^{-ik^0(x^0-y^0)}}{(k^0-\omega_{\vec{k}}+i\varepsilon')(k^0+\omega_{\vec{k}}-i\varepsilon')} \overset{\varepsilon'\to 0}{=} \underbrace{-}_{\text{clockwise contour}} 2\pi i \frac{1}{2\pi} \frac{i e^{-i\omega_{\vec{k}}(x^0-y^0)}}{2\omega_{\vec{k}}}$$

$$= \frac{e^{-i\omega_{\vec{k}}(x^0-y^0)}}{2\omega_{\vec{k}}}.$$

This is exactly what we needed so that

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-ik\cdot(x-y)} = \lim_{\varepsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik\cdot(x-y)}.$$

2) $D_F(x-y)$ is a Green's function of the Klein-Gordon operator, i.e.

$$(\partial^2+m^2)\, D_F(x-y) \sim \delta^{(4)}(x-y).$$

Proof

Indeed,

$$(\partial^2+m^2) \lim_{\varepsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik\cdot(x-y)} =$$

$$= \lim_{\varepsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \left[(-ik)\cdot(-ik) + m^2\right] e^{-ik\cdot(x-y)}$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, i\, e^{-ik\cdot(x-y)}$$

$$= -i\,\delta^{(4)}(x-y).$$

This wraps up our discussion of the free scalar field and its quantisation. Unfortunately, real-world computations are still a long way away, since the particles we discussed are free. The formalism we described, however, is very important as it provides the basis for the introduction of interactions through perturbation theory.

Particle number

Consider the operator

$$\hat{n} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\,\hat{a}^{+}_{\vec{p}}\,\hat{a}_{\vec{p}}\,.$$

It is

$\hat{a}^{+}_{\vec{p}_1}\hat{a}^{+}_{\vec{p}_2}|0\rangle$

$$\hat{n}\,|\vec{p}\rangle = 1\,|\vec{p}\rangle, \qquad \hat{n}\,|\vec{p}_1,\vec{p}_2\rangle = 2\,|\vec{p}_1,\vec{p}_2\rangle, \quad \dots$$

So $\hat{n}$ counts the number of particles in a state.
Now, one may verify that $[\hat{H},\hat{n}]=0$ and so

$$\frac{d\hat{n}}{dt} = i[\hat{H},\hat{n}] = 0.$$

Therefore, the number of particles is conserved under time evolution. There are of course situations in nature where the number of particles is not conserved. In order to allow for this we need to add interaction terms to our Hamiltonian.

Interactions

As an example we will discuss

$$\begin{aligned}\mathcal{L} &= \underbrace{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2} - \underbrace{\frac{1}{4!}\lambda\phi^4} \\ &= \mathcal{L}_0 + \mathcal{L}_{int}\end{aligned}$$

We will treat the coupling λ as small, i.e. we will treat $\mathcal{L}_{int}$ as a perturbation of $\mathcal{L}_0$.

Since $\mathcal{L}_{int}$ does not depend on $\dot{\phi}$,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi},$$

which is the same as the one before $\mathcal{L}_{int}$ was introduced. Thus,

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} \\ &= (\pi\dot{\phi} - \mathcal{L}_0) - \mathcal{L}_{int} \\ &= \mathcal{H}_0 + \mathcal{H}_{int},\end{aligned}$$

where

$$\mathcal{H}_0 = \frac{1}{2}\left(\pi^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + m^2\phi^2\right), \quad \mathcal{H}_{int} = \frac{1}{4!}\lambda\phi^4.$$

Interaction picture

In the Schrödinger picture we put the time dependence into the states and in the Heisenberg picture we put it into the operators. When we introduce interactions we keep the operators in the Heisenberg picture with H_0 and evolve the states with H_{int}. Thus, both operators and states evolve in time.

For example,

$$\begin{aligned}\hat{H}_{int,I}(t) &= e^{i\hat{H}_0 t}\,\hat{H}_{int}(0)\,e^{-i\hat{H}_0 t}, \\ \hat{\phi}_I(x) &= e^{i\hat{H}_0 t}\,\hat{\phi}(0,\vec{x})\,e^{-i\hat{H}_0 t}.\end{aligned}$$

For states,

$$\langle \psi_s(t) | \hat{O}_s | \psi_s(t) \rangle = \langle \psi_I(t) | \hat{O}_I(t) | \psi_I(t) \rangle$$

Schrödinger

$$= \langle \psi_I(t) | e^{i\hat{H}_0 t} \hat{O}_s e^{-i\hat{H}_0 t} | \psi_I(t) \rangle$$

For these to be equal we must have

$$|\psi_I(t)\rangle = e^{i\hat{H}_0 t} |\psi_s(t)\rangle$$

interaction Schrodinger

Now differentiate with respect to t to find

$$i\frac{\partial}{\partial t} |\psi_I(t)\rangle = e^{i\hat{H}_0 t}\left(-\hat{H}_0 + i\frac{\partial}{\partial t}\right)|\psi_s(t)\rangle$$

$$= e^{i\hat{H}_0 t}\left(-\hat{H}_0 + \hat{H}_0 + \hat{H}_{int}\right)|\psi_s(t)\rangle$$

$$= e^{i\hat{H}_0 t}\hat{H}_{int}|\psi_s(t)\rangle$$

$$= e^{i\hat{H}_0 t}\hat{H}_{int}e^{-i\hat{H}_0 t}|\psi_I(t)\rangle$$

$$= \hat{H}_{int,I}(t)|\psi_I(t)\rangle$$

To solve this equation we look for solutions of the form

$$|\psi_I(t)\rangle = \hat{U}(t,0)|\psi_I(t=0)\rangle$$

We find

$$i\frac{\partial}{\partial t}\hat{U}(t,0) = \hat{H}_{int,I}(t)\,\hat{U}(t,0)$$

for the time evolution operator $\hat{U}$. Naively it seems that the solution of this equation is

$$\hat{U}_{naive}(t,0) = e^{-i\int_0^t \hat{H}_{int,I}(t')dt'}$$

But this is not correct. The correct answer is

$$\hat{U}(t,0) = \hat{T}\, e^{-i\int_0^t \hat{H}_{int,I}(t')dt'}$$

This is Dyson's formula.

The naive guess fails because of the following reason. Write

$$\hat{U}_{naive}(t,0) = e^{-i\int_0^t \hat{H}_{int,I}(t')dt'}$$
$$= 1 - i\int_0^t \hat{H}_{int,I}(t')\,dt' + \frac{(-i)^2}{2!}\int_0^t dt'\,\hat{H}_{int,I}(t')\int_0^t dt''\,\hat{H}_{int,I}(t'') + \dots$$

and take $\partial/\partial t$ on the quadratic term:

$$-\frac{1}{2}\hat{H}_{int,I}(t)\int_0^t \hat{H}_{int,I}(t')\,dt' - \frac{1}{2}\int_0^t \hat{H}_{int,I}(t')\,dt'\;\hat{H}_{int,I}(t).$$

On the other side acting on the exponential we get

$$\hat{H}_{int,I}(t)\int_0^t \hat{H}_{int,I}(t')\,dt'.$$

Things would be okay if

$$[\hat{H}_{int,I}(t), \hat{H}_{int,I}(t')] = 0 \quad \text{for } t \neq t'.$$

But this is not true in general. Therefore, we introduce the $\hat{T}$ operator that commutes things appropriately. It guarantees that Hamiltonians act in the right order according to the time they involve. This is needed for causality of the theory.

To summarise, in interaction picture

$$\hat{O}_I(t) = e^{i\hat{H}_0 t}\,\hat{O}_S\,e^{-i\hat{H}_0 t},$$
$$|\psi_I(t_2)\rangle = \hat{U}(t_2,t_1)\,|\psi_I(t_1)\rangle,$$
$$\hat{U}(t_2,t_1) = \hat{T}\,e^{-i\int_{t_1}^{t_2}\hat{H}_{int,I}(t)\,dt}.$$