Quantum Field Theory - Lecture 6

Last time we saw that $a^\dagger_{\vec{k}}|0\rangle$ can be interpreted as a one-particle state, of a particle that carries momentum $\vec{k}$ and energy $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$.

Statistics of our particles

Acting with more creation operators, with the same or different momenta, we can create multi-particle states, e.g. $\hat{a}^\dagger_{\vec{p}}\,\hat{a}^\dagger_{\vec{k}}|0\rangle$ is a two-particle state. Each multi-particle state lives in a Hilbert space $\mathbb{H}_n$, where n is the number of particles. We may think of the direct sum of these Hilbert spaces,

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \cdots,$$

which is called the Fock space. A state in the Fock space does not need to have a well-defined particle number, as it can involve states from different $\mathbb{H}_n$'s.

The state $\hat{a}^\dagger_{\vec{p}}\,\hat{a}^\dagger_{\vec{k}}|0\rangle$ and the state $\hat{a}^\dagger_{\vec{k}}\,\hat{a}^\dagger_{\vec{p}}|0\rangle$ both have momentum $\vec{k}+\vec{p}$ and energy $\omega_{\vec{k}} + \omega_{\vec{p}}$. Therefore, they are the same state:

$$\hat{a}^\dagger_{\vec{p}}\,\hat{a}^\dagger_{\vec{k}}\,|0\rangle = \hat{a}^\dagger_{\vec{k}}\,\hat{a}^\dagger_{\vec{p}}\,|0\rangle.$$

We see that these Klein-Gordon particles obey Bose-Einstein statistics.

Fields $\leftrightarrow$ particles

A particle with momentum $\vec{k}$ corresponds to an excited Fourier mode of the field. The field is a superposition of all possible modes. Therefore, the field contains all the ingredients necessary to describe all possible configurations of one or more particles in a given momentum state. So, in this sense

QFT unifies particles and fields!

You might recall learning at some point that forces are due to fields (e.g. electric field) while matter is made of particles. In QFT, however, everything is both a field and a particle.

Location of particles

Let us define, at $t=0$,

$$
\begin{aligned}
|\vec{x}\rangle &= \hat{\phi}(0,\vec{x})\,|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{+}_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}}\right)|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\,e^{-i\vec{k}\cdot\vec{x}}\,|\vec{k}\rangle\,, \quad |\vec{k}\rangle = \hat{a}^{+}_{\vec{k}}|0\rangle.
\end{aligned}
$$

The state $|\vec{x}\rangle$ is a linear superposition of single particle states that have well-defined momentum and energy. Except for the $\frac{1}{2\omega_{\vec{k}}}$, this is just like the nonrelativistic expression for the eigenstate of the position operator in quantum mechanics. The interpretation is that $\hat{\phi}(0,\vec{x})$ acts on the vacuum and creates a particle at position $\vec{x}$. That particle does not have a unique momentum, but the probability to find it with momentum $\vec{p}$ can be computed using

$$\langle 0|\,\hat{\phi}(0,\vec{x})\,|\vec{p}\rangle \sim e^{-i\vec{p}\cdot\vec{x}}.$$

Propagators

Let us define

$$|x\rangle = \hat{\phi}(x)|0\rangle \quad (\text{or } |t,\vec{x}\rangle = \hat{\phi}(t,\vec{x})|0\rangle).$$

We want to compute the amplitude of propagation from y to x:

$$\langle x|y\rangle = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \equiv D(x,y).$$

We have

$$D(x,y) = \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}2\omega_{\vec{p}}}\langle 0|(\hat{a}_{\vec{k}}e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k}}e^{ik\cdot x})(\hat{a}_{\vec{p}}e^{-ip\cdot y} + \hat{a}^\dagger_{\vec{p}}e^{ip\cdot y})|0\rangle$$

It is

$$\hat{a}_{\vec{p}}|0\rangle = 0 \quad \text{for all } \vec{p}$$

$$\langle 0|\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{\vec{p}}|0\rangle = \text{"}\langle 0|2\rangle\text{"} = 0$$

etc.

The only non-zero term is the cross term

$$\langle 0|\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{p}}|0\rangle = (2\pi)^3 2\omega_{\vec{k}}\,\delta^{(3)}(\vec{k}-\vec{p})\underbrace{\langle 0|0\rangle}_{1}.$$

Therefore,

$$D(x,y) = D(x-y) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot(x-y)}.$$

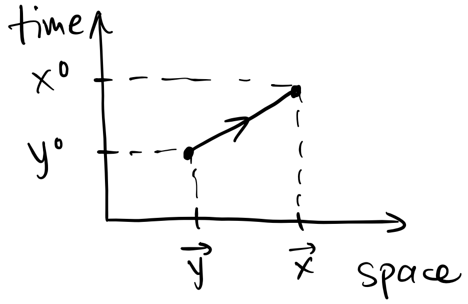
It makes sense that $D(x,y)$ only depends of $x-y$, since it should not matter what x, y are, but only how far apart they are.

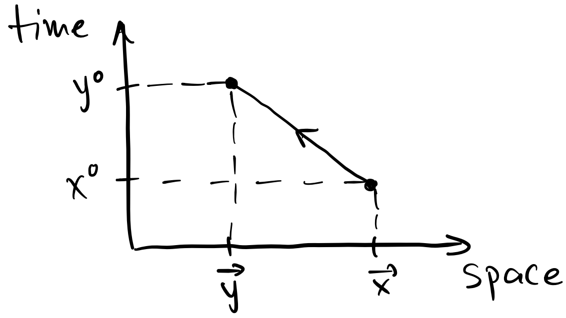
Feynman propagator

The Feynman propagator is defined by

$$D_F(x-y) = \begin{cases} D(x-y) & \text{if } x^0 > y^0 \\ D(y-x) & \text{if } y^0 > x^0 \end{cases}.$$

This makes more sense, as we want to go from y to x if x is at a later time $(x^0 > y^0)$ and from x to y if y is at a later time $(y^0 > x^0)$.





We may also use the step function to write
$$D_F(x-y) = \theta(x^0-y^0)\,D(x-y) + \theta(y^0-x^0)\,D(y-x).$$
A related operation is time ordering:
$$\hat{T}\left(\hat{\phi}(x)\,\hat{\phi}(y)\right) = \begin{cases} \hat{\phi}(x)\,\hat{\phi}(y) & \text{if } x^0 > y^0 \\ \hat{\phi}(y)\,\hat{\phi}(x) & \text{if } y^0 > x^0 \end{cases}.$$

Then,
$$D_F(x-y) = \langle 0 |\, \hat{T}\, \hat{\phi}(x)\, \hat{\phi}(y)\, | 0 \rangle.$$
The Feynman propagator will turn out to be part of the Feynman rules for Feyman diagrams.