Quantum Field Theory – Lecture 5

Last time we solved the Klein-Gordon equation

$$(\partial^2 + m^2)\,\phi(x) = 0$$

and promoted the solution to a quantum operator in the Heisenberg picture:

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}\,e^{-ik\cdot x} + \hat{a}^{+}_{k}\,e^{ik\cdot x}\right)$$

where

$$k^0 = \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}\,.$$

The conjugate momentum to ϕ is

$$\hat{\pi}(x) = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left(\hat{a}_{\vec{k}}\,e^{-ik\cdot x} - \hat{a}^{+}_{\vec{k}}\,e^{ik\cdot x}\right).$$

We started to write the Hamiltonian $\hat{H}$ as a function of creation and annihilation operators and recognised the need to compute their commutators.

To write $\hat{a}_{\vec{k}}$ in terms of $\hat{\phi}$ and $\hat{\pi}$ we compute

$$\int d^3x\, e^{ip\cdot x}\,\hat{\phi}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\int d^3x\left(\hat{a}_{\vec{k}}\,e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\,e^{ik\cdot x}\right)e^{ip\cdot x}$$

$$= \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\Big(\hat{a}_{\vec{k}}\,e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\int d^3x\, e^{i(\vec{k}-\vec{p})\cdot\vec{x}} + \hat{a}^{+}_{\vec{k}}\,e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\int d^3x\, e^{-i(\vec{k}+\vec{p})\cdot\vec{x}}\Big)$$

$$= \int d^3k\,\frac{1}{2\omega_{\vec{k}}}\Big(\hat{a}_{\vec{k}}\,e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\,\delta^{(3)}(\vec{k}-\vec{p}) + \hat{a}^{+}_{\vec{k}}\,e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\,\delta^{(3)}(\vec{k}+\vec{p})\Big)$$

$$= \frac{1}{2\omega_{\vec{p}}}\left(\hat{a}_{\vec{p}} + \hat{a}^{+}_{-\vec{p}}\,e^{2i\omega_{\vec{p}}t}\right).$$

Similarly,

$$\int d^3x\, e^{ip\cdot x}\,\hat{\pi}(t,\vec{x}) = -\frac{i}{2}\left(\hat{a}_{\vec{p}} - \hat{a}^{+}_{-\vec{p}}\,e^{2i\omega_{\vec{p}}t}\right)$$

Therefore,

$$\underbrace{\hat{a}_{\vec{p}}}_{\text{t-independent}} = \int d^3x \, e^{ip\cdot x} \left(\omega_{\vec{p}} \hat{\phi}(t,\vec{x}) + i\hat{\pi}(t,\vec{x}) \right).$$

Similarly,

$$\hat{a}^\dagger_{\vec{p}} = \int d^3x \, e^{-ip\cdot x} \left(\omega_{\vec{p}} \hat{\phi}(t,\vec{x}) - i\hat{\pi}(t,\vec{x}) \right).$$

With these expressions we now find (at equal times)

$$\begin{aligned}
[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] &= \int d^3x \, d^3y \, e^{i(k\cdot x - p\cdot y)} [\omega_{\vec{k}} \hat{\phi}(x) + i\hat{\pi}(x), \omega_{\vec{p}} \hat{\phi}(y) - i\hat{\pi}(y)] \\
&= -i \int d^3x \, d^3y \, e^{i(k\cdot x - p\cdot y)} \big(\omega_{\vec{k}} [\hat{\phi}(x), \hat{\pi}(y)] \\
&\qquad + \omega_{\vec{p}} [\hat{\phi}(y), \hat{\pi}(x)] \big) \\
&= -i \int d^3x \, d^3y \, e^{i(k\cdot x - p\cdot y)} \big(\vec{\omega}_k \, i \delta^{(3)}(\vec{x}-\vec{y}) \\
&\qquad + \omega_{\vec{p}} \, i \delta^{(3)}(\vec{y}-\vec{x}) \big) \\
&= \int d^3x \, e^{i(\vec{k}-\vec{p})\cdot\vec{x}} \underbrace{e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t}}_{\text{equal time}} (\omega_{\vec{k}} + \omega_{\vec{p}}) \\
&= (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{p}) (\omega_{\vec{k}} + \omega_{\vec{p}}) e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t} \\
&= 2\omega_{\vec{k}} (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{p})
\end{aligned}$$

and also

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}] = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = 0.$$

Back to the Hamiltonian, we have

$$\begin{aligned}
H &= \int d^3x \, \frac{1}{2} \left(\hat{\pi}^2 + \vec{\nabla}\hat{\phi} \cdot \vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^2 \right) \\
&\vdots \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \frac{1}{2\omega_{\vec{k}}} \omega_{\vec{k}} \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}^\dagger_{\vec{k}} \right) \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{4} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] + \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} \right) \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} \underbrace{[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}]}_{2\omega_{\vec{k}} (2\pi)^3 \delta^{(3)}(0)} \right)
\end{aligned}$$

In the case of N SHOs :

$$H = \sum_{i=1}^{N} \omega \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right)$$

It looks like we have a continuous ∞ of SHOs.
Here we have a small issue:

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] = (2\pi)^3 \, 2\omega_{\vec{k}} \, \delta^{(3)}(0) = \infty$$

This implies that the ground-state energy is infinite. This is expected because it is like we have infinite SHOs, each contributing $\frac{\omega}{2}$ to the vacuum energy. However, this infinite ground state energy is unobservable, because we can only measure energy differences.

Normal ordering

If we compare

$$\hat{H}_1 = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

and

$$\hat{H}_2 = \omega \, \hat{a}^\dagger \hat{a} ,$$

we see that the only difference is that all energy levels are shifted by $\omega/2$; they are all lower by $\omega/2$ in $\hat{H}_2$ compared to $\hat{H}_1$. This is not a difference of any fundamental importance and so we should come up with a prescription to remove it. In field theory this prescription is called normal ordering and it amounts to simply moving all annihilation operators to the right without worrying about commutation relations. For example,

$$: \hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}} \, \hat{a}^\dagger_{\vec{k}} : = \hat{a}^\dagger_{\vec{k}} \, \hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}} .$$

Then,

$$:[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = :\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{p}}: - :\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{k}}:$$
$$= \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{k}} - \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{k}}$$
$$= 0$$

Therefore,

$$:\hat{H}: = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}.$$

The emergence of particles

In the SHO in quantum mechanics, states were constructed from $|0\rangle$ with applications of $\hat{a}^\dagger$. Here we have $\hat{a}^\dagger_{\vec{k}}$ to act with:

$$:\hat{H}:(\hat{a}^\dagger_{\vec{k}}|0\rangle) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{k}}|0\rangle$$

assume $|0\rangle$ is vacuum, $\hat{a}_{\vec{k}}|0\rangle = 0$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}}[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{k}}]|0\rangle$$
$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{p}}(2\pi)^3 2\omega_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{p})|0\rangle$$
$$= \omega_{\vec{k}}(\hat{a}^\dagger_{\vec{k}}|0\rangle)$$

Therefore, $\hat{a}^\dagger_{\vec{k}}|0\rangle$ is an eigenstate of the normal-ordered Hamiltonian with energy $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$.

What about momentum? We have

$$P^i = \int d^3x\, T^{0i}$$
$$= -\int d^3x\, \dot{\phi}\,\partial^i\phi$$
$$= -\int d^3x\, \pi\,\partial^i\phi$$

and (HW problem)

$$:\hat{P}^i: = \int \frac{d^3k}{(2\pi)^3}\frac{k^i}{2\omega_{\vec{k}}}\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}.$$

Therefore,

$$:\hat{P}^i:\left(\hat{a}^{+}_{\vec{p}}|0\rangle\right) = \int \frac{d^3k}{(2\pi)^3}\,\frac{k^i}{2\omega_{\vec{k}}}\,\hat{a}^{+}_{\vec{k}}\,\hat{a}_{\vec{k}}\,\hat{a}^{+}_{\vec{p}}\,|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\,\frac{k^i}{2\omega_{\vec{k}}}\,\hat{a}^{+}_{\vec{k}}\,2\omega_{\vec{k}}\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{p})\,|0\rangle$$

$$= p^i\left(\hat{a}^{+}_{\vec{p}}|0\rangle\right)$$

and so $\hat{a}^{+}_{\vec{p}}|0\rangle$ is a state with momentum $\vec{p}$.

We see that $\hat{a}^{+}_{\vec{k}}|0\rangle$:

- has momentum $\vec{k}$
- has energy $\omega_{\vec{k}}$
- $\omega^2_{\vec{k}} = \vec{k}^2 + m^2$

The last equation is the proper relativistic energy-momentum relation, and it is natural to interpret m as the mass. We also interpret $a^{+}_{\vec{k}}|0\rangle$ as a one-particle state, which is however not localised in real space but rather in momentum space. It is something that carries energy $\omega_{\vec{k}}$ and momentum $\vec{k}$.