Quantum Field Theory – Lecture 4

Canonical quantisation

We will introduce a "recipe" to go from a classical theory to its associated quantum theory.

1. Start with $\mathcal{L}$
2. Calculate $\mathcal{H}(\pi, \phi)$, $\quad \pi = \partial\mathcal{L}/\partial\dot{\phi}$
3. Treat ϕ and π as operators and impose commutation relations
4. Expand the fields in terms of creation and annihilation operators.
5. Apply normal ordering (tomorrow).

Let's follow these steps for the free scalar field.

1. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$
2. $\pi = \partial\mathcal{L}/\partial\dot{\phi} \Rightarrow \pi = \dot{\phi}$

   $\mathcal{H} = \frac{1}{2}(\pi^2 + \vec{\nabla}\phi\cdot\nabla\phi + m^2\phi^2)$
3. $\phi \longrightarrow \hat{\phi}, \quad \pi \longrightarrow \hat{\pi}$

   Define equal-time commutation relations:

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$
$$[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = 0$$
$$[\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0$$

4. Recall:
   - for SHO, time-independent $\hat{x} = \frac{1}{\sqrt{2\omega m}}(\hat{a} + \hat{a}^\dagger)$
   - Classical K-G equation gave solution

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^*_{\vec{k}} e^{ik\cdot x} \right)$$

$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

To promote $\phi(x)$ to an operator we turn $a_{\vec{k}}$ to operators, first at $t=0$:

$$\hat{\phi}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

time-independent to agree with SHOs

This is an operator in the Schrödinger picture, since it is time-independent. The $\hat{a}_{\vec{k}}$'s are defined for each Fourier mode of the field. This is called the mode expansion of $\phi(x)$. We also need $\hat{\pi}$ (later).

5. Normal ordering (tomorrow).

Schrödinger vs. Heisenberg picture

Schrödinger:
- operators are time-independent
- states depend on time:

$$\hat{H} |\psi_s(t)\rangle = i \frac{\partial}{\partial t} |\psi_s(t)\rangle$$

In QFT it is more convenient to work in the Heisenberg picture:

- operators depend on time
- states are time-independent

How do we go to the Heisenberg picture? The t-dependence of states in the Schrödinger picture is given by

$$\hat{H} |\psi_s(t)\rangle = i \frac{\partial}{\partial t} |\psi_s(t)\rangle \Rightarrow |\psi_s(t)\rangle = e^{-i\hat{H}t} |\psi_s(0)\rangle.$$

Expectation values of operators are

$$\langle \hat{O} \rangle = \langle \psi_s(t) | \hat{O}_s | \psi_s(t) \rangle$$
$$= \langle \psi_s(0) e^{i\hat{H}t} | \hat{O}_s | e^{-i\hat{H}t} \psi_s(0) \rangle$$

$$= \langle \psi_H | \hat{O}_H | \psi_H \rangle$$

where

$$|\psi_H\rangle = |\psi_S(0)\rangle, \quad \hat{O}_H = e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}.$$

These are states and operators in the Heisenberg picture.

Since operators are time-dependent, they satisfy an equation of motion (Heisenberg equation of motion):

$$\frac{d\hat{O}_H}{dt} = i\hat{H} e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t} - i e^{i\hat{H}t} \hat{O}_S i\hat{H} e^{-i\hat{H}t}$$

$$= i\hat{H}\hat{O}_H - i O_H \hat{H}$$

$$= i[\hat{H}, O_H]$$

Scalar fields in Heisenberg picture

Starting from $\phi(\vec{x})$ we have

$$\hat{\phi}(x) = \hat{\phi}(t, \vec{x}) = e^{i\hat{H}t} \hat{\phi}(\vec{x}) e^{-i\hat{H}t}.$$

Using the mode expansion of $\hat{\phi}(\vec{x})$ we have

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(e^{i\hat{H}t} \hat{a}_{\vec{k}} e^{-i\hat{H}t} e^{i\vec{k}\cdot\vec{x}} + e^{i\hat{H}t} \hat{a}^{+}_{\vec{k}} e^{-i\hat{H}t} e^{-i\vec{k}\cdot\vec{x}} \right.$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(\hat{a}^H_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{+H}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

From the Heisenberg equation of motion,

$$\frac{d\hat{a}^H_{\vec{k}}}{dt} = i[\hat{H}, \hat{a}^H_{\vec{k}}]$$

$$= i[\hat{H}, e^{i\hat{H}t} \hat{a}_{\vec{k}} e^{-i\hat{H}t}]$$

$$= i e^{i\hat{H}t} [\hat{H}, \hat{a}_{\vec{k}}] e^{-i\hat{H}t}$$

$$= -i\omega_{\vec{k}} e^{i\hat{H}t} \hat{a}_{\vec{k}} e^{-i\hat{H}t}$$

$$= -i\omega_{\vec{k}} \hat{a}^H_{\vec{k}}.$$

Therefore,

$$\hat{a}^H_{\vec{k}} = \hat{a}_{\vec{k}}(0)\, e^{-i\omega_{\vec{k}} t}.$$

Similarly,

$$\hat{a}^{\dagger H}_{\vec{k}} = \hat{a}^\dagger_{\vec{k}}(0)\, e^{i\omega_{\vec{k}} t}.$$

Plugging these back into $\hat{\phi}(x)$ we get

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(\hat{a}_{\vec{k}}(0)\, e^{-i\omega_{\vec{k}} t} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{k}(0)\, e^{i\omega_{\vec{k}} t} e^{-i\vec{k}\cdot\vec{x}} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left(\hat{a}_{\vec{k}}(0)\, e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k}}(0)\, e^{ik\cdot x} \right)$$

time-independent

Now we can finally compute $\hat{\pi}(x)$:

$$\hat{\pi}(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$= \dot{\phi}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} (-i\omega_{\vec{k}}) \left(\hat{a}_{\vec{k}}(0)\, e^{-ik\cdot x} - \hat{a}^\dagger_{k}(0)\, e^{ik\cdot x} \right)$$

$$= -\frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left(\hat{a}_{\vec{k}}(0)\, e^{-ik\cdot x} - \hat{a}^\dagger_{\vec{k}}(0)\, e^{ik\cdot x} \right)$$

To derive the same result we could have used

$$\hat{\pi} = \hat{\dot{\phi}} = i[H, \phi].$$

Hamiltonian

For the real scalar field,

$$\hat{H} = \int d^3x\, \frac{1}{2} \left(\hat{\pi}^2 + \vec{\nabla}\hat{\phi} \cdot \vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^2 \right).$$

Our aim is to write $\hat{H}$ in terms of creation and annihilation operators. This is rather tedious, but it is clear that to simplify the resulting expressions we need to compute commutators of $\hat{a}_{\vec{k}}$'s and $\hat{a}^{+}_{\vec{k}}$'s. To do this computation we can use the already discussed commutators between $\hat{\phi}$ and $\hat{\pi}$.

Summary

- Introduced canonical quantisation recipe.
- Applied to scalar field.