Systematic Parametrisation and Operator Product Expansion for the B_q-meson Light-Cone Distribution Amplitude

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IPPP Seminar, Durham

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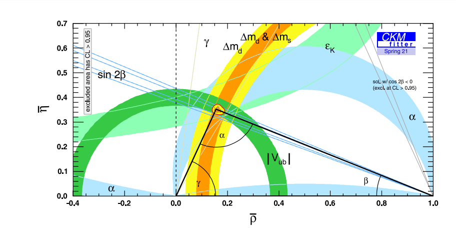
Motivation

Important goals of ***B*-physics phenomenology**:

- Constraining the CKM unitarity triangle in the SM
- Indirect probes of physics beyond the SM



Theoretical predictions need information on non-perturbative **hadronic matrix elements**, e.g.

$B \to \ell\nu$	$B \to \pi\ell\nu$	$B \to \gamma\ell\nu$
Need decay constant f_B	Need transision form factor	Need transition form factors
(Can extract V_{ub})	Have lattice prediction	Cannot use lattice
Have lattice prediction		Have QCD factorisation!

Role of the B-meson LCDA

$$\phi_+(\omega) = \int \frac{\mathrm{d}\tau}{2\pi}\, \mathrm{e}^{i\omega\tau}\, \tilde{\phi}_+(\tau)$$

Example for **factorisation theorem**

$$\mathcal{M}(B \to \gamma\ell\bar{\nu}) \sim m_B f_B \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\, T(\omega, \ldots)\, \phi_+(\omega)$$

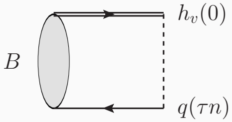
with perturbative process dependent kernel T

and non-perturbative process independent $\tilde{\phi}_+(\tau;\mu)$:

- Describes **momentum distribution** of partons in the B meson
- Formally, matrix element of non-local operator, [Grozin, Neubert (1997)]

$$\tilde{\phi}_+(\tau) \propto \langle 0 \,|\, \bar{q}(\tau n)\, [\tau n, 0]\, \not{n}\gamma_5\, h_v(0)\, |\bar{B}(v)\rangle$$

- Not available from first principles (yet?), only **encoded in data**!



Properties of the B-meson LCDA

What **information** about $\tilde{\phi}_+(\tau)$ is **available**?

E.g. "inverse moment" λ_B^{-1}

Measurable (pseudo-) observable in $B \to \gamma\ell\nu$

$$\lambda_B^{-1} = \int_0^\infty \frac{\mathrm{d}\tau}{2\pi}\,\tilde{\phi}_+(-i\tau) \quad \stackrel{!}{=} \text{finite}$$

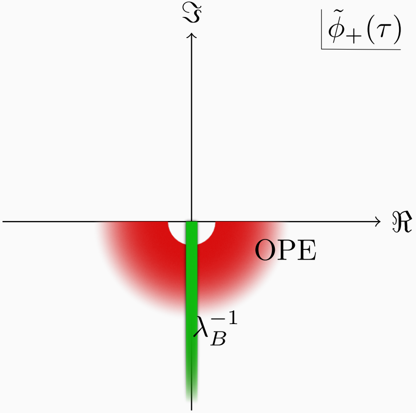
$\Rightarrow$ Constrains behaviour for **large** τ

"operator product expansion" (OPE)

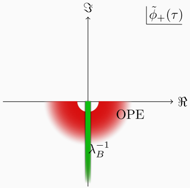
Theory/analyticity:

Lower half plane is analytic except for the origin

$\Rightarrow$ Constrains behaviour for **small** τ



Operator product expansion



Expand the bilocal operator using τ-dependent coefficients and local operators

Ultimate goal: connect information of OPE coefficients to the relevant region

$$\bar{q}(\tau n)\,[\tau n, 0]\,\not{n}\gamma_5\,h_v(0) = \sum_{n=3}^{\infty}\sum_{k=1}^{K_n} c_k^{(n)}(\tau)\,\mathcal{O}_k^{(n)}(0)$$

$$= c_1^{(3)}(\tau)\;\bar{q}(0)\,\not{n}\gamma_5\,h_v(0) \qquad \text{dim-3}$$

$$+\,c_1^{(4)}(\tau)\;\bar{q}(0)\,(in\cdot\overleftarrow{D})\,\not{n}\gamma_5\,h_v(0) + c_2^{(4)}(\tau)\;\bar{q}(0)\,(iv\cdot\overleftarrow{D})\,\not{n}\gamma_5\,h_v(0) \qquad \text{dim-4}$$

$$+\ldots \qquad \text{dim} \geq 5$$

- Determine coefficients in a **matching calculation**
- Dimensional grounds: $c_{3-n}^{(k)} \propto \tau^n$

$\Rightarrow$ How does this work on tree level?

OPE matching at tree level: partonic side

$$\bar{q}(\tau n)[\tau n, 0]\not{n}\gamma_5\, h_v(0) = c_1^{(3)}(\tau)\; \bar{q}\, \not{n}\gamma_5\, h_v \qquad \text{dim-3}$$

$$+\, c_1^{(4)}(\tau)\; \bar{q}\,(in \cdot \overleftarrow{D})\, \not{n}\gamma_5\, h_v + c_2^{(4)}(\tau)\; \bar{q}\,(iv \cdot \overleftarrow{D})\, \not{n}\gamma_5\, h_v \qquad \text{dim-4}$$

$$+ \ldots \qquad \text{dim} \geq 5$$

- Extract coefficients using **partonic** matrix element $\langle 0 | \ldots | q(k)\, h_v \rangle$
- Wilson line at LO: $[\tau n, 0] = 1 + \mathcal{O}(g_s)$

Result: [Grozin, Neubert (1997)]

$$c_1^{(3)}(\tau) = 1 + \mathcal{O}(\alpha_s)\,, \quad c_1^{(4)}(\tau) = -i\tau + \mathcal{O}(\alpha_s)\,, \quad c_2^{(4)}(\tau) = 0 + \mathcal{O}(\alpha_s)$$

OPE matching at tree level: hadronic side

$$\begin{aligned}
\bar{q}(\tau n)\,[\tau n, 0]\,\not{n}\gamma_5\, h_v(0) = {} & c_1^{(3)}(\tau)\;\bar{q}\,\not{n}\gamma_5\, h_v & \text{dim-3}\\
& + c_1^{(4)}(\tau)\;\bar{q}\,(in\cdot\overleftarrow{D})\,\not{n}\gamma_5\, h_v + c_2^{(4)}(\tau)\;\bar{q}\,(iv\cdot\overleftarrow{D})\,\not{n}\gamma_5\, h_v & \text{dim-4}\\
& + \dots & \text{dim} \geq 5
\end{aligned}$$

Connect to **hadronic** matrix element $\langle 0|\,\dots\,|\bar{B}\rangle$:

- LHS yields LCDA by definition,

$$\langle 0|\,\bar{q}(\tau n)\,[\tau n, 0]\,\not{n}\gamma_5\, h_v(0)\,|\bar{B}\rangle \propto \tilde{\phi}_+(\tau)$$

- RHS yields (just a few) hadronic constants, e.g.

$$\langle 0|\,\bar{q}\,(in\cdot\overleftarrow{D})\,\not{n}\gamma_5\, h_v\,|\bar{B}\rangle \propto \frac{4}{3}\,\bar{\Lambda}$$

$$\langle 0|\,\bar{q}\,(iv\cdot\overleftarrow{D})\,\not{n}\gamma_5\, h_v\,|\bar{B}\rangle \propto \bar{\Lambda}$$

$\Rightarrow$ Insert coefficients, obtain expression for the LCDA!

OPE matching at tree level: hadronic side

$$
\begin{aligned}
\bar{q}(\tau n)\,[\tau n, 0]\,\not{n}\gamma_5\, h_v(0) &= c_1^{(3)}(\tau)\;\bar{q}\,\not{n}\gamma_5\, h_v && \text{dim-3} \\
&+ c_1^{(4)}(\tau)\;\bar{q}\,(in\cdot \overleftarrow{D}\,)\,\not{n}\gamma_5\, h_v + c_2^{(4)}(\tau)\;\bar{q}\,(iv\cdot \overleftarrow{D}\,)\,\not{n}\gamma_5\, h_v && \text{dim-4} \\
&+ \ldots && \text{dim} \geq 5
\end{aligned}
$$

The **OPE result** at tree-level (LO) up to mass dimension 4:

$$
\tilde{\phi}_+^{\mathrm{OPE}}(\tau)\Big|_{\text{LO, dim-4}} = 1 - i\tau\,\frac{4\bar{\Lambda}}{3}
$$

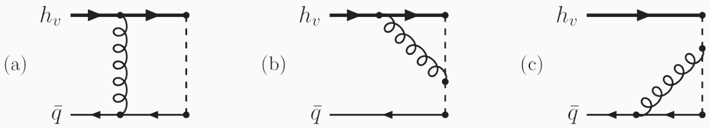
- Can easily extend this to higher mass dimensions – introduce new constants
- Limited range of validity (does not fall off)
- Cannot calculate inverse moment λ_B^{-1}
- Local limit $\tau \to 0$ is well-defined at tree-level (LO)

What about radiative corrections (NLO)?

OPE matching at NLO

Radiative corrections appear on the **partonic side** (in the coefficients)

For the bilocal operator (LHS):



For example,

[Lee, Neubert (2005)] [Kawamura, Tanaka (2009)]

$$c_1^{(3)}(\tau) = 1 - \frac{\alpha_s C_F}{4\pi}\left(2\log(i\tau\mu e^{\gamma_E})^2 + 2\log(i\tau\mu e^{\gamma_E}) + \frac{5\pi^2}{12}\right) + \mathcal{O}(\alpha_s^2)$$

- Manifestly scale dependent (must compensate the hard scattering kernel)
- Now even singular for $\tau \to 0$ ("renormalisation and local limit do not commute")
- Limit $\tau \to \infty$ remains incompatible with finite λ_B^{-1}

$\Rightarrow$ Need some means of extrapolation from OPE and other quantities!

Loose ends (for now)

Here briefly, later in detail:

Renormalisation group evolution

LCDA is scale dependent

Scale evolution is **convolution**:

$$\tilde{\phi}_+(\tau;\mu) = \int \mathrm{d}\tau'\,\gamma(\tau,\tau';\mu,\mu_0)\,\tilde{\phi}_+(\tau';\mu_0)$$

$\Rightarrow$ Difficult, esp. numerically

Finite spectator quark mass

BSM physics: e.g. $\bar{B}_s \to \gamma\mu^+\mu^-$

- Also uses LCDAs
- New dimensionful scale in the OPE
- What about radiative corrections?

Parametrisation

Model independence of an analysis

Model-based analysis

- Construct model that fulfills the general properties (to some extend)
- Popular **exponential model** (single parameter) [Grozin, Neubert (1997)]:

$$\tilde{\phi}_+(\tau) = \frac{1}{(1 + i\,\lambda_B\,\tau)^2}$$

Local limit like tree-level OPE and falls off to produce inverse moment

- **Problem**: there are more pseudo-observables than λ_B^{-1}

$\Rightarrow$ Those will be (highly) correlated!

Model-independent analysis

- Parametrisation has **infinite parameters**:

$$\tilde{\phi}_+(\tau;\mu) = \sum_{k=0}^{\infty} a_k(\mu)\, f_k(\tau;\mu)$$

$\Rightarrow$ Everything is decorrelated

- What about convergence? Need truncation!
- Estimate of the error?
- Must fulfill constraints order-by-order

Is this possible??

Construction of a parametrisation

$$\tilde{\phi}_+(\tau;\mu) = \sum_{k=0}^{K} a_k(\mu) f_k(\tau;\mu)$$

Important considerations:

- Control the truncation error ($K < \infty$) with a **bound**:

$$\chi \equiv \sum_k |a_k|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \ldots$$

- Construct this with weighted integral

$$\chi(\mu) \equiv \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \left|\tilde{\phi}_+(\tau;\mu)\right|^2 |r(\tau;\mu)|^2 , \quad 0 < \chi < \infty$$

- Choose the weight function according to known LCDA behaviour
- Keep series convergence in mind (low-order behavior)

Result

$$y(\tau) = \frac{i\omega_0\tau - 1}{i\omega_0\tau + 1}$$

$$\tilde{\phi}_+(\tau;\mu_0) = \frac{1}{(1+i\omega_0\tau)^2} \sum_{k=0}^{K} a_k(\mu_0)\, y(\tau)^k$$

$$\phi_+(\omega;\mu_0) = \frac{\omega}{\omega_0^2} e^{-\frac{\omega}{\omega_0}} \sum_{k=0}^{K} a_k(\mu_0)\, \frac{L_k^{(1)}(2\omega/\omega_0)}{1+k}$$

Additional features:

- Auxilliary dimensionful scale ω_0 to "measure" the scalar coefficients
- **Simple** functional form (and also for the Fourier transform)
- Generalises/extends the **exponential model**
- Numerically **efficient** RG evolution

$\Rightarrow$ Can we connect this to the OPE?

Test case: model for the radiative tail

$$\phi_+(\omega) = \int \frac{d\tau}{2\pi}\, e^{i\omega\tau}\, \tilde{\phi}_+(\tau)$$

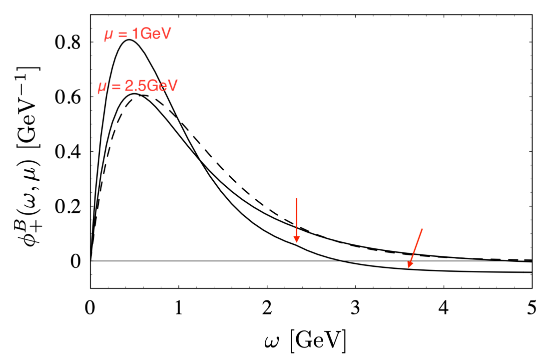
Model that reflects OPE-induced "radiative tail" [Lee, Neubert (2005)]:

$$\phi_+(\omega,\mu) = \mathcal{N}\,\frac{\omega\, e^{-\omega/\bar\omega}}{\bar\omega^2} \qquad \text{“exponential-like”}$$

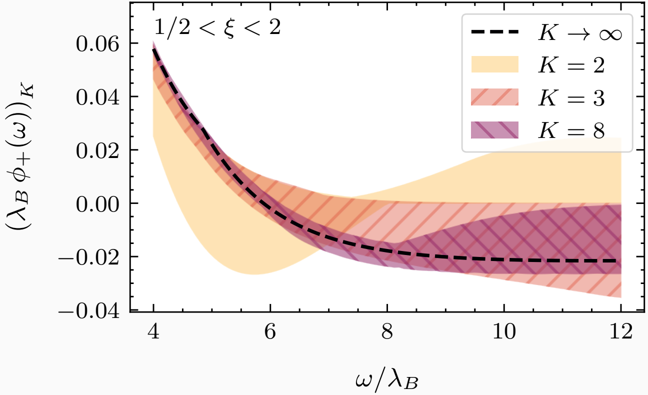
$$+ \frac{\alpha_s C_F}{\pi}\,\frac{\theta(\omega-\omega_t)}{\omega}\left\{\frac{1}{2} - \ln\frac{\omega}{\mu} + 4\,\frac{\bar\Lambda_{\rm DA}}{3\omega}\left(2 - \ln\frac{\omega}{\mu}\right)\right\} \qquad \text{“radiative tail”/OPE}$$

- For **small and intermediate** range of ω
- Scale-dependent parameter values are matched onto **partonic calculation**

⇒ Can we reproduce the tail?



Test case: model for the radiative tail



- $K = 3$ already consistent up to large ω
- Larger K increases "precise" range (slowly)

$\Rightarrow$ Even though the model is pathological, the **parametrisation is flexible enough!**

Renormalisation group evolution

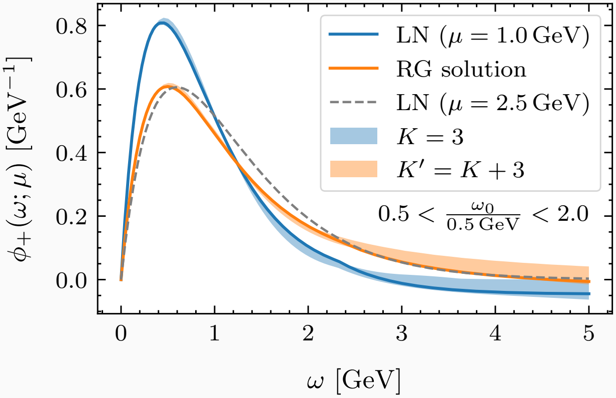
Exact RGE of the LCDA is **convolution**:

$$\tilde{\phi}_+(\tau;\mu) = \int \mathrm{d}\tau'\, \gamma(\tau,\tau';\mu,\mu_0)\, \tilde{\phi}_+(\tau';\mu_0)$$

Numerically **faster** for the parameters:

$$a_{k'}(\mu) = \mathrm{e}^V \left(\frac{\hat{\mu}_0}{2\omega_0}\right)^{-g} \sum_k \mathcal{R}_{k'k}(\mu,\mu_0)\, a_k(\mu_0)$$

- Compute expensive matrix only once!
- Excellent **agreement** with exact result
- Uncertainty band **remains consistent** (low K for illustration)



OPE with $m_q \neq 0$

Effect of the spectator quark mass

For $m \neq 0$, one new relevant operator in the OPE:

$$\bar{q}(\tau n)\,[\tau n, 0]\,\not{n}\gamma_5\, h_v(0) = \sum_{n=3}^{\infty}\sum_{k=1}^{K_n} c_k^{(n)}(\tau)\,\mathcal{O}_k^{(n)}(0)$$

$$= c_1^{(3)}(\tau)\ \bar{q}\,\not{n}\gamma_5\, h_v \qquad \text{dim-3}$$

$$+\, c_1^{(4)}(\tau)\ \bar{q}\,(in\cdot\overleftarrow{D}\,)\,\not{n}\gamma_5\, h_v + c_2^{(4)}(\tau)\ \bar{q}\,(iv\cdot\overleftarrow{D}\,)\,\not{n}\gamma_5\, h_v \qquad \text{massless dim-4}$$

$$+\, c_3^{(4)}(\tau)\ m\,\bar{q}\,\not{v}\not{n}\gamma_5\, h_v \qquad \text{massive dim-4}$$

$$+\ \text{mass dimension} \geq 5$$

On the hadronic side,

$$\langle 0|\mathcal{O}_1^{(4)}|\bar{B}(v)\rangle \propto \frac{4\bar{\Lambda}-m}{3}\,, \quad \langle 0|\mathcal{O}_2^{(4)}|\bar{B}(v)\rangle \propto \bar{\Lambda}\,, \quad \langle 0|\mathcal{O}_3^{(4)}|\bar{B}(v)\rangle \propto -m$$

$$\Rightarrow\ \tilde{\phi}_+(\tau) = c_1^{(3)}(\tau) + \bar{\Lambda}\left(\frac{4}{3}\,c_1^{(4)}(\tau) + c_2^{(4)}(\tau)\right) - m\left(c_3^{(4)}(\tau) + \frac{1}{3}\,c_1^{(4)}(\tau)\right) + \mathcal{O}(\tau^2)$$

Problem description

What is the problem with non-zero spectator mass?

- Introduces new scale into 1-loop integrals

$\Rightarrow$ Exact position-space result at 1-loop not easily obtainable in closed form

- But we only need $\mathcal{O}(m)$ for dimension 4?

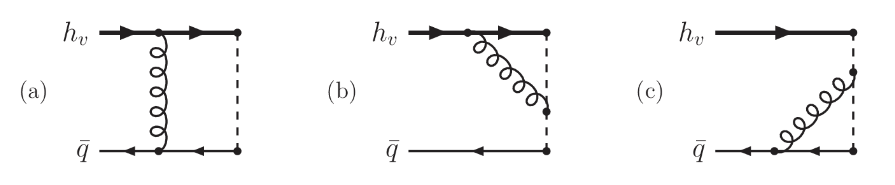
$\Rightarrow$ Yes, but needs careful calculation!

Dimensional regularisation can lead to non-analytical behaviour of intermediate results!

Partonic setup

Base the calculation on a simple setup available in literature

- Need **single new** Wilson coefficient ⇒ use not the most general but simple setup
- Extracts a sum of coefficients; further use result by Kawamura and Tanaka
- Our choice corresponds to the "non-relativistic setup": heavy-quark velocity v^μ, spectator-quark momentum mv^μ [Bell, Feldmann (2008) arXiv:0802.2221]
- They give momentum-space results for the three 1-loop contributions $I_i^+(\omega)$:



Matching relation (massive case)

Obtain **matching relation** from **partonic matrix element** $\langle 0| \ldots |\bar{q}\, h_v\rangle$ of the OPE

- Leads to:

$$1 - im\tau + \frac{\alpha_s C_F}{4\pi}\,\tilde{I}^{+}(\tau) + \mathcal{O}(\tau^2, \alpha_s^2) \qquad \text{bilocal operator}$$

$$= c_1^{(3)}(\tau)\left(1 + \frac{\alpha_s C_F}{4\pi}\,\tilde{l}_1^{(3)}\right) + m\, c_1^{(4)}(\tau)\left(1 + \frac{\alpha_s C_F}{4\pi}\,\tilde{l}_1^{(4)}\right) \qquad \text{local operators}$$

$$+\, m\, c_2^{(4)}(\tau) - m\, c_3^{(4)}(\tau) + \mathcal{O}(\tau^2, \alpha_s^2)$$

  - Expanded in m on the LHS
  - Local 1-loop contributions are taken into account on the RHS:

$$\tilde{l}_1^{(3)} = \int_0^\infty d\omega\, l^{+}(\omega) \quad \text{and} \quad \tilde{l}_1^{(4)} = \frac{1}{m}\int_0^\infty d\omega\, \omega\, l^{+}(\omega)$$

Matching relation (massive case)

$$1 - im\tau + \frac{\alpha_s C_F}{4\pi}\,\tilde{I}^{+}(\tau) + \mathcal{O}(\tau^2, \alpha_s^2)$$

bilocal operator

$$= c_1^{(3)}(\tau)\left(1 + \frac{\alpha_s C_F}{4\pi}\,\tilde{I}_1^{(3)}\right) + m\,c_1^{(4)}(\tau)\left(1 + \frac{\alpha_s C_F}{4\pi}\,\tilde{I}_1^{(4)}\right)$$
$$+ m\,c_2^{(4)}(\tau) - m\,c_3^{(4)}(\tau) + \mathcal{O}(\tau^2, \alpha_s^2)$$

local operators

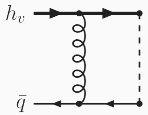
Re-arrange in terms of powers of α_s

$\Downarrow$

$$\frac{\alpha_s C_F}{4\pi}\left(\tilde{I}^{+}(\tau) - \tilde{I}_1^{(3)} + im\tau\,\tilde{I}_1^{(4)}\right)$$
$$= \left(c_1^{(3)}(\tau) - 1\right) + m\left(i\tau + c_1^{(4)}(\tau) + c_2^{(4)}(\tau) - c_3^{(4)}(\tau)\right) + \mathcal{O}(\tau^2, \alpha_s^2)$$

The **sum** of non-local and local terms on the LHS now allows **expansion** in m!

Example: vertex-like piece



Take **vertex correction** as **illustration** how this works

- In momentum space:

$$I_a^+(\omega) = 2\omega\,\Gamma(1+\epsilon)\left(\frac{\mu^2 e^{\gamma_E}}{(m-\omega)^2}\right)^{\epsilon}\left\{\frac{2}{(m-\omega)^2} - \frac{\theta(m-\omega)}{m\,(m-\omega)} - \frac{\theta(\omega-m)}{\omega\,(\omega-m)}\right\}$$

- Calculate the Fourier transform and expand *after* in m

$$\tilde{I}_a^+(\tau) = \int_0^\infty d\omega\, e^{-i\omega\tau}\, I_a^+(\omega) = \frac{2}{\epsilon} - 2L + 3\,\ln\frac{\mu^2}{m^2} - 2 - i\tau m\left(\frac{2}{\epsilon} - 6L + 5\,\ln\frac{\mu^2}{m^2} + 7\right) + \mathcal{O}(m^2,\epsilon)$$

  with $L = \log i\tau\mu e^{\gamma_E}$, exponential integral function $\mathrm{Ei}(z)$
- Local contributions added in the matching relation:

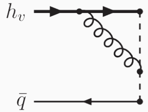
$$\tilde{I}_{1,a}^{(3)} = \int_0^\infty d\omega\, I_a^+(\omega) = \frac{3}{\epsilon} + 3\,\ln\frac{\mu^2}{m^2} - 2 + \mathcal{O}(\epsilon)$$

$$\tilde{I}_{1,a}^{(4)} = \frac{1}{m}\int_0^\infty d\omega\,\omega\, I_a^+(\omega) = \frac{5}{\epsilon} + 5\,\ln\frac{\mu^2}{m^2} + 3 + \mathcal{O}(\epsilon)$$

$\Rightarrow$ Result is free of IR logs; the *sum* can be linearized in m before FT

The other pieces

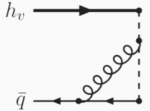
Pieces remain with gluon exchange from **Wilson line** and each of the quarks



- Gluon coupling to the **heavy quark** yields

$$\tilde{I}_b^+(\tau) = e^{-i\tau m}\left(-\frac{1}{\epsilon^2} - \frac{2L}{\epsilon} - 2L^2 - \frac{5\pi^2}{12}\right) + \mathcal{O}(\epsilon)$$

  - A *universal* contribution to all orders in mass dimension
  - No local subtractions (integrals in dim-reg are scaleless)



- Gluon coupling to the **light quark** vanishes:

$$I_c^+(\omega) = 2\Gamma(\epsilon)\int_0^m dk\,\frac{m-k}{m}\left(\frac{\mu^2 e^{\gamma_E}}{k^2}\right)^{\epsilon}\frac{\delta(k-m+\omega)-\delta(\omega-m)}{k}$$

  Only involves low-momentum region $\omega < m$

  $\Rightarrow$ Expansion of the Fourier integral and dim-reg commute and

$$\tilde{I}_c^+(\tau) - \tilde{I}_{1,c}^{(3)} + im\tau\,\tilde{I}_{1,c}^{(4)} = \int_0^\infty d\omega\left(e^{-i\omega\tau} - 1 + i\omega\tau\right) I_c^+(\omega) = 0$$

New primary result

The new **mass-induced** Wilson coefficient:

$$\boxed{c_3^{(4)}(\tau) = -i\tau\left[\frac{\alpha_s C_F}{4\pi}(L-1) + \mathcal{O}(\alpha_s^2)\right]}$$

Yields **OPE form** of the LCDA:

$$\tilde{\phi}_+(\tau) = \left[1 - i\tau\,\frac{4\bar{\Lambda}-m}{3}\right]\left[1 - \frac{\alpha_s C_F}{4\pi}\left(2L^2 + 2L + \frac{5\pi^2}{12}\right)\right] + i\tau\bar{\Lambda}\,\frac{\alpha_s C_F}{4\pi}\left(\frac{8}{3}L - 3\right) + i\tau m\,\frac{\alpha_s C_F}{4\pi}\left(\frac{4}{3}L - 1\right) + \mathcal{O}(\alpha_s^2, \tau^2)$$

$\Rightarrow$ What's the impact in a global analysis? Connect to the parametrisation

Connecting the OPE to the inverse moment

Important caveat:

- QCD factorization formulas and sum rules (primarily) probe the **low-ω region**

$\Rightarrow$ Radiative tail cannot predict this!

What *can* we do?

Make (strong) ad-hoc assumptions and check for consistency $\Rightarrow$ sanity check

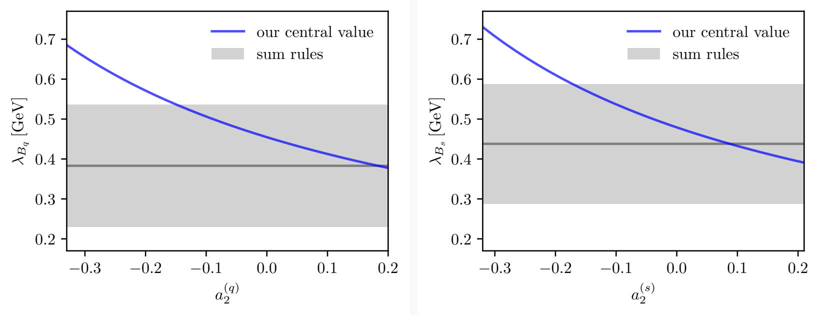
- Assume **truncation** $K = 2$ is sufficient $\Rightarrow$ Relate a_0, a_1, a_2 via **two** OPE constraints
- Assume that the **bound saturates** rapidly to constrain a_2:

$$\frac{|a_1^{(q)}|^2}{|a_0^{(q)}|^2 + |a_1^{(q)}|^2} < 0.25\,, \qquad \frac{|a_2^{(q)}|^2}{|a_0^{(q)}|^2 + |a_1^{(q)}|^2 + |a_2^{(q)}|^2} < 0.1\,, \quad \Rightarrow \quad \ldots < a_2^{(q)} < \ldots$$

- Fix ω_0 due to simultaneous convergence of OPE and parametrization

$\Rightarrow$ Check results!

Range for λ_{B_q} and λ_{B_s}

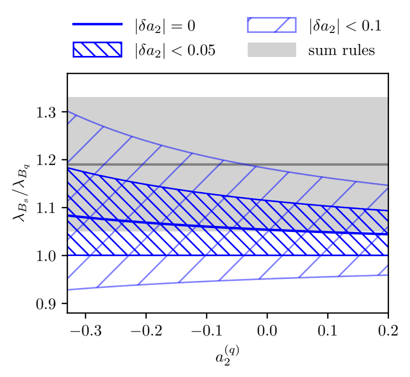


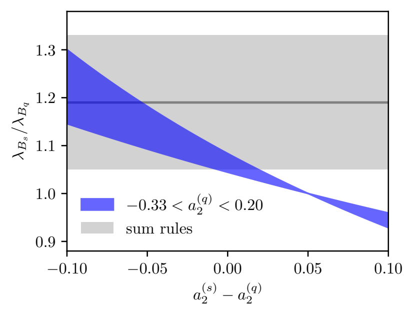
- Comparison to recent sum rule calculation [Khodjamirian et al. (2020)]

⇒ Large overlap, no “finetuning” required

- Not shown: difference of individual $a_i^{(q,s)}$ is about 10% to 15%

Ratio $\lambda_{B_s}/\lambda_{B_q}$





- Comparison to recent sum rule calculation [Khodjamirian et al. (2020)]
- Now two varied parameters $a_2^{(q,s)}$
- Vary $a_2^{(q)}$ as before and examine difference $\delta a_2 = a_2^{(s)} - a_2^{(q)}$

$\Rightarrow$ Again, large overlap consistent with $\delta a_2 = 0$, but favoring $\delta a_2 < 0$

Summary of numerical analysis

- Find a consistent picture, even with aggressive assumptions

⇒ OPE is **well suited** for global analysis using the parametrization

- Not shown: dominant source is the binding energy $\bar{\Lambda}_{B_{q,s}}$, not $m_{q,s}$

⇒ Lends support for the application even to charmed B_c

⇒ Find numerically similar coefficients, but for **drastically different scales** μ_0, ω_0:

	B_q	B_s	B_c
μ_0	1 GeV	1 GeV	2 GeV
ω_0	594 MeV	594 MeV	1.18 GeV
$\bar{\Lambda}_a$	367 MeV	437 MeV	1.24 GeV
m_a	0	106 MeV	1.00 GeV
$a_0 - a_2$	1.31	1.24	1.22
$a_1 - 2a_2$	0.47	0.42	0.35
λ_B	$(380, 690)$ MeV	$(390, 730)$ MeV	$(0.76, 1.49)$ GeV

Generalized calculation

Move from simple setup to **fully general setup**

- Generic Dirac structure in the operator allows to extract **subleading** $\tilde{\phi}_-(\tau;\mu)$

$$\text{operators} \quad \ldots \not{n}\gamma_5 \ldots \quad \rightarrow \quad \text{operators} \quad \ldots \Gamma \ldots$$

$\Rightarrow$ Requires more terms in the OPE

- General kinematic allows to extract **individual** Wilson coefficients

$$\text{mass } m \quad \rightarrow \quad \text{mass } m \quad \text{and} \quad \text{momentum components } n \cdot k,\, v \cdot k$$

$\Rightarrow$ Profit from Taylor expansion before Fourier transform!

Additional benefits

- Cross-check because
- Independent from external input (massless Wilson coefficients)
- Extendable to NNLO
- Renders operator basis transparent

General dim-4 operator basis

Generic **operator basis** to mass dimension 4:

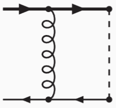
$$
\begin{aligned}
\mathcal{O}_\Gamma(\tau) &= \bar{q}(\tau n)\,[\tau n, 0]\,\Gamma\, h_v(0) \\
&= c_1^{(3)}(\tau)\,\bar{q}(0)\,\frac{\not{n}\not{v}}{2}\,\Gamma\, h_v(0) + d_1^{(3)}(\tau)\,\bar{q}(0)\,\frac{\not{v}\not{n}}{2}\,\Gamma\, h_v(0) \\
&\quad + c_1^{(4)}(\tau)\,\bar{q}(0)\,(in\cdot \overleftarrow{D}\,)\,\frac{\not{n}\not{v}}{2}\,\Gamma\, h_v(0) + d_1^{(4)}(\tau)\,\bar{q}(0)\,(in\cdot \overleftarrow{D}\,)\,\frac{\not{v}\not{n}}{2}\,\Gamma\, h_v(0) \\
&\quad + c_2^{(4)}(\tau)\,\bar{q}(0)\,(iv\cdot \overleftarrow{D}\,)\,\frac{\not{n}\not{v}}{2}\,\Gamma\, h_v(0) + d_2^{(4)}(\tau)\,\bar{q}(0)\,(iv\cdot \overleftarrow{D}\,)\,\frac{\not{v}\not{n}}{2}\,\Gamma\, h_v(0) \\
&\quad + c_3^{(4)}(\tau)\,m\,\bar{q}(0)\,\frac{2\not{v}-\not{n}}{2}\,\Gamma\, h_v(0) + d_3^{(4)}(\tau)\,m\,\bar{q}(0)\,\frac{\not{n}}{2}\,\Gamma\, h_v(0) + \mathcal{O}(\tau^2)
\end{aligned}
$$

Using “light-cone projectors”

$$
P_+ = \frac{\not{n}\not{v}}{2} = \frac{\not{n}\not{\bar{n}}}{4}\,, \qquad P_- = \frac{\not{v}\not{n}}{2} = \frac{\not{\bar{n}}\not{n}}{4}
$$

- Massless terms come with even number of Dirac matrices
- Linear terms in m come with odd number of Dirac matrices
- $\Gamma = \not{n}\gamma_5$ recovers previous calculation since $n \cdot n = 0$, $v \cdot n = 1$

Vertex correction piece, generic



Vertex correction with generic operator inserted:

$$I_a^{\Gamma}(\omega, m, k) = -i \int [d\ell]\, \delta(\omega - n\cdot(k-\ell))\, \frac{\bar{v}(k)\, \not{v}(-\not{k} + \not{\ell} + m)\, \Gamma\, u(v)}{[(k-\ell)^2 - m^2 + i0][v\cdot\ell + i0][\ell^2 + i0]}$$

Linearise with local subtractions first and then perform the **Fourier transform**:

$$\begin{aligned}
&\int_0^\infty d\omega \left(e^{-i\omega\tau} - 1 + i\omega\tau + \ldots\right) I_a^{\Gamma}(\omega, m, k) \\
= \;& \bar{v}(k) \left\{ \left(-\frac{1}{\epsilon} - 2L + \left(\frac{1}{2\epsilon} + L\right) i\tau(n\cdot k) + \left(\frac{2}{\epsilon} + 4L - 3\right) i\tau(v\cdot k)\right) \frac{\not{n}\not{v}}{2} \right. \\
& + \left(\frac{1}{\epsilon} + 2L - \left(\frac{1}{2\epsilon} + L\right) i\tau(n\cdot k) + \left(\frac{1}{\epsilon} + 2L - 3\right) i\tau(v\cdot k)\right) \frac{\not{v}\not{n}}{2} \\
& \left. - \left(\frac{1}{2\epsilon} + L - 1\right) i\tau m\,\not{v} + \mathcal{O}(\tau^2) \right\} \Gamma\, u(v)
\end{aligned}$$

⇒ Can **read off** contributions to Wilson coefficients

Towards dimension-5 OPE

New hadronic parameters at mass dimension 5

- For local dimension-5 operators,

$$\langle 0|\, \bar{q}\, i\overleftarrow{D}^{\mu} i\overleftarrow{D}^{\nu}\, \Gamma h_v \,|\bar{B}\rangle \sim \text{five tensor structures}$$

- Quark equations of motion pose three constraints $\Rightarrow$ two degrees of freedom remain
- (Canonically) define ME of gluon field strengh tensor $G^{\mu\nu} = [D^\mu, D^\nu]$

$$\frac{\langle 0|\bar{q}_\beta\, iG^{\mu\nu}\,(h_v)_\alpha|\bar{B}(v)\rangle}{\langle 0|\mathcal{O}_1^{(3)}|\bar{B}(v)\rangle} = \frac{1}{4}\left[(1+\slashed{v})\left(\frac{\lambda_H^2-\lambda_E^2}{3}\,(\gamma^\mu v^\nu - \gamma^\nu v^\mu) - \frac{\lambda_H^2}{3}\, i\sigma^{\mu\nu}\right)\gamma_5\right]_{\alpha\beta}$$

- But for symmetric terms, EOM introduce m as

$$\frac{\frac{1}{2}\langle 0|\bar{q}_\beta\,\{i\overleftarrow{D}^{\mu}, i\overleftarrow{D}^{\nu}\}\,(h_v)_\alpha|\bar{B}(v)\rangle}{\langle 0|\mathcal{O}_1^{(3)}|\bar{B}(v)\rangle} = -\frac{1}{4}\left[(1+\slashed{v})\left(\frac{6\bar{\Lambda}^2+2\lambda_E^2+\lambda_H^2-2m\bar{\Lambda}-m^2}{3}\, v^\mu v^\nu \right.\right.$$
$$\left.\left. -\frac{\bar{\Lambda}^2+\lambda_E^2+\lambda_H^2-m^2}{3}\, g^{\mu\nu} - \frac{2\bar{\Lambda}^2+\lambda_E^2-2m\bar{\Lambda}}{6}\,(\gamma^\mu v^\nu + \gamma^\nu v^\mu)\right)\gamma_5\right]_{\alpha\beta}$$

$\Rightarrow$ Yields leading-order Mellin moments $\langle \omega^{1,2}\rangle_\pm$

Estimate of dimension-5 OPE at NLO

For leading two-particle LCDA, we estimate

$$\begin{aligned}\tilde{\phi}_+(\tau) \quad = \quad & \left[1 - i\tau \langle\omega\rangle_+ - \tau^2 \frac{\langle\omega^2\rangle_+}{2}\right]\left[1 - \frac{\alpha_s C_F}{4\pi}\left(2L^2 + 2L + \frac{5\pi^2}{12}\right)\right] \\ & + i\tau\bar{\Lambda}\frac{\alpha_s C_F}{4\pi}\left(\frac{8}{3}L - 3\right) + i\tau m \frac{\alpha_s C_F}{4\pi}\left(\frac{4}{3}L - 1\right) \\ & + \tau^2\bar{\Lambda}^2\frac{\alpha_s C_F}{4\pi}\left(\frac{10}{3}L - \frac{35}{9} + \mathcal{O}\left(\frac{m}{\bar{\Lambda}}\right) + \mathcal{O}\left(\frac{\lambda_{E,H}^2}{\bar{\Lambda}^2}\right)\right) + \mathcal{O}(\alpha_s^2) + \mathcal{O}(\tau^3)\end{aligned}$$

- First line: tree-level to dimension 5, with universal 1-loop correction
- Third line: neglect mass and chromo-electric and -magnetic moments

Estimate of dimension-5 OPE at NLO

Numerical results:

$n_0 = 1/3$	tree-level, pole-scheme			1-loop$^{(*)}$, pole-scheme			1-loop, a-scheme
	dim-3	dim-4	dim-5	dim-3	dim-4	dim-5	dim-4
	$K=0$	$K=1$	$K=2$	$K=0$	$K=1$	$K=2$	$K=2$
$a_0^{(q)}$	1	1.44	1.54	0.78	1.07	1.07	$(0.98, 1.51)$
$a_1^{(q)}$	–	0.44	0.65	–	0.26	0.23	$(-0.19, 0.87)$
$a_2^{(q)}$	–	–	0.11	–	–	-0.03	$(-0.33, 0.20)$
$a_0^{(s)}$	1	1.37	1.43	0.78	0.99	0.96	$(0.92, 1.45)$
$a_1^{(s)}$	–	0.37	0.49	–	0.20	0.10	$(-0.22, 0.84)$
$a_2^{(s)}$	–	–	0.06	–	–	-0.06	$(-0.32, 0.21)$

- Completely consistent with ad-hoc procedure at dim-4
- Indicates similar amount of SU(3) breaking

Backup

arXiv:hep-ph/0509350

"Model-Independent Properties of the B-Meson Distribution Amplitude" (2005)

Seung J. Lee, Matthias Neubert

- OPE to dimension 3 and 4 for "Mellin moments" with cutoff Λ_{UV}:

$$\langle \omega^k \rangle_+ = \int_0^{\Lambda_{\text{UV}}} \mathrm{d}\omega\, \omega^k\, \phi_+(\omega)\,, \quad k = 0, 1$$

- Obtained from "partonic LCDA" in momentum space at 1-loop order ($\overline{\text{MS}}$):

$$\phi_+^B(\omega,\mu)_{\text{parton}} = \delta(\omega)\left(1 - \frac{C_F\alpha_s}{4\pi}\,\frac{\pi^2}{12}\right) + \frac{C_F\alpha_s}{4\pi}\left[-4\left(\frac{\ln\frac{\omega}{\mu}}{\omega}\right)_*^{[\mu]} + 2\left(\frac{1}{\omega}\right)_*^{[\mu]}\right]$$
$$+\,\delta'(\omega)\left\{-n\cdot p\left[1 - \frac{C_F\alpha_s}{4\pi}\left(1+\frac{\pi^2}{12}\right)\right] + v\cdot p\,\frac{C_F\alpha_s}{4\pi}\right\} + \ldots$$

$\Rightarrow$ Complicated distributions due to non-local operator with Wilson line

- Extract "radiative tail" from cut-off Mellin moment $\langle \omega^0 \rangle_+$

arXiv:0810.5628

"Operator product expansion for B-meson distribution amplitude and dimension-5 HQET operators" (2008)

Hiroyuki Kawamura, Kazuhiro Tanaka

- OPE calculation of the LCDA directly
- Up to mass dimension 5, i.e. $\propto \tau^2$ at 1-loop order ($\overline{\text{MS}}$)
- Involved framework in position space, few details, few intermediate results given
- For example, 1-loop 1PI diagrams ("1LDs") yield (before renormalization)

$$\text{1LDs} = \frac{\alpha_s C_F}{2\pi} \int_0^1 d\xi \Bigg[\Bigg\{ - \left(\frac{1}{2\varepsilon_{UV}^2} + \frac{L}{\varepsilon_{UV}} + L^2 + \frac{5\pi^2}{24} \right) \delta(1-\xi) + \left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}} \right) \left(\frac{\xi}{1-\xi} \right)_+ - \left(\frac{1}{2\varepsilon_{IR}} + L \right) \Bigg\} \langle \bar{q}(\xi t n) \not{n} \gamma_5 h_v(0) \rangle - t \left(\frac{1}{\varepsilon_{IR}} + 2L - 1 - \xi \right) \langle \bar{q}(\xi t n) v \cdot \overleftarrow{D} \not{n} \gamma_5 h_v(0) \rangle \Bigg] + \cdots$$

Here: (Simpler) distributions

arXiv:2306.14686

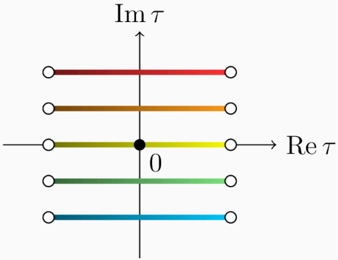
"Strange-quark mass effects in the B_s meson's light-cone distribution amplitude" (2023)

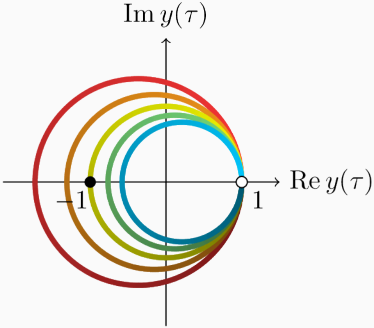
Thorsten Feldmann, PL, Nicolas Seitz – our new paper

In comparison:

- OPE calculation of the LCDA directly
- Up to mass dimension 4, i.e. $\propto \tau^1$ at 1-loop order ($\overline{\text{MS}}$)
- Including spectator quark mass, i.e. extend application from $B_{u,d}$ to B_s
- Systematic and explicit framework for calculation in momentum space
- Optimize for extraction of Wilson coefficients
    - Avoid intermediate calculation of distributions
    - Allow convenient Taylor expansion w.r.t. dimensionful scales
- Investigate suitability for global analysis
- Results for subleading two-particle LCDA $\phi_-(\tau;\mu)$

Mapping onto the unit circle





$$y(\tau) = \frac{i\omega_0\tau - 1}{i\omega_0\tau + 1}$$

(using auxilliary parameter)

$$0 \mapsto -1$$

$$\mathbb{R} \mapsto \mathrm{D}_1$$

$$\lim_{|\tau|\to\infty} \tau \mapsto +1$$

$$\int_{\mathbb{R}} \mathrm{d}\tau \,\cdot \;\rightsquigarrow\; \oint_{\partial \mathrm{D}_1} \mathrm{dy} \,\cdot$$

Bound Construction

- The bound (with weight function) takes the form

$$\chi = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left|\Phi_+(\tau(y))\right|^2 \left|r(\tau(y))\right|^2 \frac{1+\omega_0^2\tau(y)^2}{2\omega_0}\Bigg|_{y=e^{i\theta}}$$

- Conveniently factorise the LCDA:

$$\Phi_+(\tau) \equiv \frac{f_+(y(\tau))}{r(\tau)\,(1+i\omega_0\tau)}\,, \qquad \Phi_+^*(\tau) = \frac{f_+^*(y^*(\tau))}{r^*(\tau)\,(1-i\omega_0\tau)}$$

- Then, with parameters,

$$\chi = \frac{1}{2\omega_0} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left|f_+\left(e^{i\theta}\right)\right|^2 \overset{!}{\propto} \sum_{n=0}^{\infty} |a_n|^2 \,.$$

$\Rightarrow$ Use math of orthogonal polynomials on unit circle $\mathbf{D_1}$.

- The set of functionsspanning $f_+(y)$ is unique: monomials y^n, $n \in \mathbb{N}_0$