Aspects of Gravity and Entanglement

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Goal of this talk: to present recent (and not so recent) developments in semiclassical quantum gravity that were discovered by thinking about entanglement.

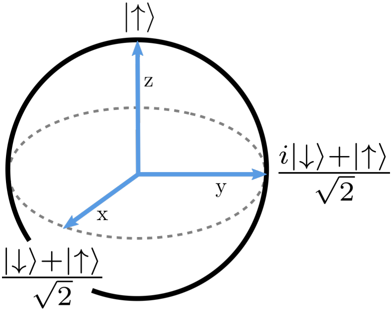
Part I: Why QFT is not "just QM"

I want to set up how entanglement can be a useful property in a quantum system.

A single spin

Consider such a quantum system built out of a single spin.

$\mathcal{H}$ is two-dimensional and spanned by $\{|\downarrow\rangle, |\uparrow\rangle\}$



The 3 vectors depicted here are those that map to themselves under the action of σ_x, σ_y, σ_z respectively

Two spins

You can't have entanglement with one d.o.f., so let's spice it up by considering a red spin and a blue spin.

$\mathcal{H}$ is now four dimensional:

$$\mathcal{H} = \mathrm{span}\{|\downarrow\rangle|\downarrow\rangle, |\downarrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\uparrow\rangle|\uparrow\rangle\}$$

Magic trick

Starting from the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\downarrow\rangle |\downarrow\rangle + |\uparrow\rangle |\uparrow\rangle \right)$$

but restricting to operators that only act on the red spin, e.g.:

$$\{ \mathbb{1} \otimes \mathbb{1},\ \sigma_x \otimes \mathbb{1},\ \sigma_y \otimes \mathbb{1},\ \sigma_z \otimes \mathbb{1} \}$$

we can generate all basis elements of $\mathcal{H}$!

Magic trick

Let's work it out explicitly:

$$\frac{\mathbb{1} - \sigma_z}{\sqrt{2}} \otimes \mathbb{1} |\psi\rangle = |\downarrow\rangle |\downarrow\rangle$$

$$\frac{\sigma_x - i\sigma_y}{\sqrt{2}} \otimes \mathbb{1} |\psi\rangle = |\downarrow\rangle |\uparrow\rangle$$

$$\frac{\sigma_x + i\sigma_y}{\sqrt{2}} \otimes \mathbb{1} |\psi\rangle = |\uparrow\rangle |\downarrow\rangle$$

$$\frac{\mathbb{1} + \sigma_z}{\sqrt{2}} \otimes \mathbb{1} |\psi\rangle = |\uparrow\rangle |\uparrow\rangle$$

Marvel that, starting from the state $|\psi\rangle$ on red and blue, but acting on *only red*, we can generate all states in $\mathcal{H}$

Important!

This would not have been possible had we started instead with the state:

$$|\chi\rangle = |\downarrow\rangle|\downarrow\rangle \ .$$

It is hopefully clear that, by acting only on the red spin in the state $|\chi\rangle$, we would only manage to generate a two-dimensional subspace of $\mathcal{H}$

What distinguishes between $|\psi\rangle$ and $|\chi\rangle$...

... is entanglement

Entanglement

To understand *how much* entanglement there is, let us construct the *reduced density matrix* on the red spin, in both states. We achieve this by tracing over the blue Hilbert space:

$$\rho_\psi = |\psi\rangle\langle\psi| \qquad\qquad \rho_\chi = |\chi\rangle\langle\chi|$$

$$\rho_\psi = \mathrm{Tr}_{\mathcal{H}}\rho_\psi \qquad\qquad \rho_\chi = \mathrm{Tr}_{\mathcal{H}}\rho_\chi$$

$$\rho_\psi = \frac{1}{2}(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|) \qquad\qquad \rho_\chi = |\downarrow\rangle\langle\downarrow|$$

From this exercise we see that ρ_ψ is mixed, while ρ_χ is pure.

Entanglement entropy

The question of *how much entanglement* there is, is measured by the entanglement entropy:

$$\begin{aligned} S_\psi &= -\mathrm{Tr}_{\mathcal{H}}\, \rho_\psi \log \rho_\psi \\ &= \log 2 \end{aligned}$$

whereas $S_\chi = 0$.

Lesson: In special entangled states,[1] we can generate the whole Hilbert space $\mathcal{H}$, even if we only have access to operators on a subset of the total degrees of freedom.

[1]We call such states *cyclic*.

Recap

Possible if $\mathcal{H}$ admits a tensor factorization

For $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are k dimensional, then

$$|\psi\rangle \equiv \sum_{i=1}^{k} \sqrt{p_i}|i\rangle|i\rangle$$

is *cyclic* w.r.t operators in 1 & 2 if none of the $p_i = 0$.

Local QFT

Now we move to continuum QFT on Minkowski space:

$$ds^2 = -dt^2 + d\vec{x}^2$$

Consider a real scalar field $\phi(x^\mu)$, with vacuum state $|0\rangle$.

Lorentz invariance implies:

$$P^\mu|0\rangle = 0 \ , \qquad [\phi(x^\mu), \phi(y^\mu)] = 0 \text{ if } (x^\mu - y^\mu) \text{ is spacelike} \ .$$

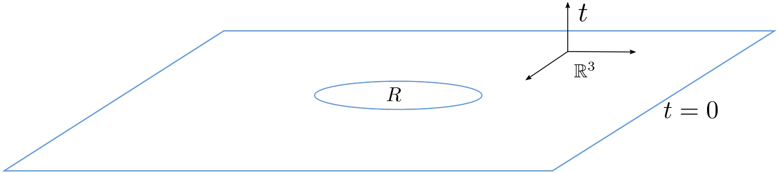
Local QFT fact

The Hilbert space $\mathcal{H}$ is generated by local operators on a Cauchy slice, e.g.:

$$\boxed{\mathcal{H} = \mathrm{span}\left\{ |0\rangle, \prod_{i=1}^{n} \phi(x_i^\mu)|0\rangle, \forall n \,\middle|\, \big(t_i = 0, \vec{x}_i \in \mathbb{R}^3\big) \right\}^*}$$

Reeh-Schlieder theorem

A surprising consequence of this construction is that the QFT vacuum $|0\rangle$ is *cyclic* w.r.t. subregions [Reeh-Schlieder '1961]



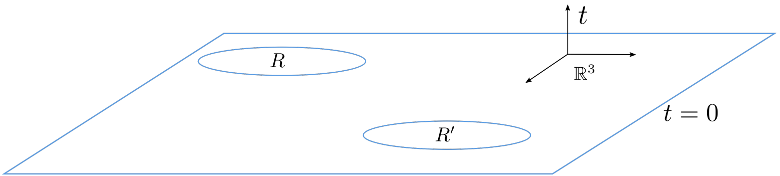
Meaning we can restrict to a region R and still generate $\mathcal{H}$:

$$\mathcal{H} = \mathrm{span}\left\{ |0\rangle, \prod_{i=1}^{n} \phi(x_i^\mu)|0\rangle, \forall n \,\middle|\, \left(t_i = 0, \boxed{\vec{x}_i \in R}\right) \right\}$$

The proof of the Reeh-Schlieder theorem is easy to follow (see e.g. [Witten's APS Lectures, 2018]), but is technical and not necessary for our purposes.

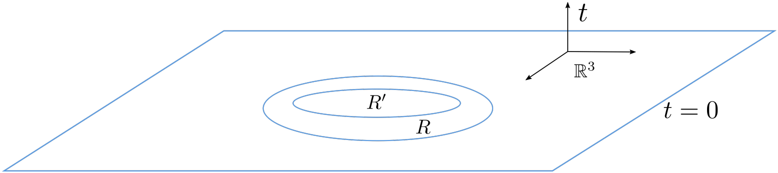
Reeh-Schlieder theorem

More surprisingly: if we consider two spacelike separated regions R and R', operators in each set individually generate $\mathcal{H}$



Reeh-Schlieder theorem

Similarly: if we consider two regions R and R', such that $R' \subset R$, both sets individually generate $\mathcal{H}$



Here the analogy with the finite dimensional QM breaks down. Saying $|0\rangle$ is cyclic for some region R leads us to write:

$$|0\rangle \equiv \sum_i \sqrt{p_i} |i\rangle_{\text{inside } R} |i\rangle_{\text{outside } R}$$

in analogy with the state $|\psi\rangle$ on two spins.

But this would imply there is a non-cyclic state, e.g.

$$|\chi\rangle = |1\rangle_{\text{inside } R} |1\rangle_{\text{outside } R} \ .$$

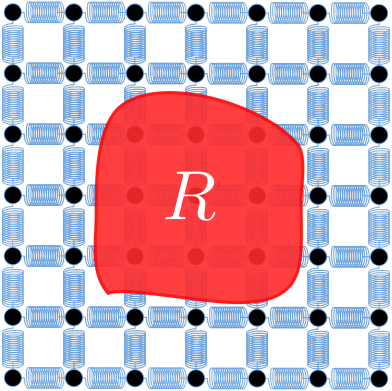
But, by construction, all states look like $|0\rangle$ at short distances, so such a $|\chi\rangle$ can't exist.

What has happened?

Again what's special in this case...

... is entanglement

Too much entanglement



If we try to quantify the entanglement of the region R in the continuum, we run into problems. Need to introduce a cutoff procedure, like a lattice spacing ϵ. Then:

$$S_R = \frac{\text{Area}(R)}{\epsilon^2} + \text{subleading}$$

Too much entanglement

$$S_R = \frac{\text{Area}(R)}{\epsilon^2} + \text{subleading}$$

But this means that the amount of entanglement between a region R and the rest of the Cauchy-slice is UV-divergent, a.k.a scheme-dependent, a.k.a ambiguous.

Even *entropy differences in the same state* are ambiguous, e.g. for two regions R and R', such that $R' \subset R$, we have

$$S_R - S_{R'} = \frac{\text{Area}(R) - \text{Area}(R')}{\epsilon^2} = \text{UV divergent}$$

However, because the divergence is geometric, it is universal. Therefore, for the same region R, but comparing two different states $\rho_{1,2}$:

$$S_R(\rho_1) - S_R(\rho_2) = \text{finite}$$

A classification

In [von Neumann '1930], von Neumann classified the operator algebras that act on Hilbert space:

- Type I: like our spin example. Entropies are well-defined.
- Type II: Like classical physics. Entropy is defined up to a state-independent additive constant. Entropy differences well-defined.
- Type III: like our QFT example. Entropies are UV divergent.

Recap

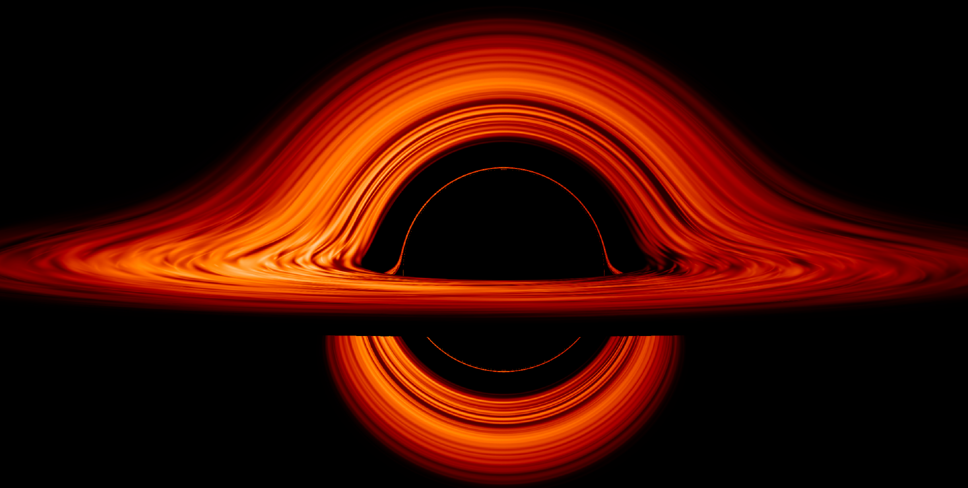
QFT states have enormous amounts of entanglement. So much that you can generate the whole Hilbert space by acting locally within a small region.

This is because the $\mathcal{H}$ factorizes into an $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$ for all subregions.

The amount of entanglement is so large, that entropy ceases to make sense as a calculable in the continuum.

Part II: Why QG is not "just QFT"

Goal of this part: to port some of the concepts from the previous section to *semi-classical gravity*



Black hole entropy

Brief timeline of events:

- [Hawking, '1971] showed $dA_{\mathrm{BH}} \geq 0$ in physical processes
- [Bekenstein '1972-1974] connected this to the thermodynamic second law $dS_{\mathrm{BH}} \geq 0$
- [Hawking, '1974] showed that $S_{\mathrm{BH}} = \frac{k_B c^3 A_{\mathrm{BH}}}{4G\hbar}$

$\frac{S_{\mathrm{BH}}}{A_{\mathrm{BH}}} = 1.4 \times 10^{69}$ bits per square meter.

This is an enormous number, but certainly not divergent or ambiguous.

How do we reconcile that this number is finite, with the expectation from QFT that entropies of regions are UV divergent?

Moreover, the first law $dA_{\mathrm{BH}} \geq 0$—interpreted in terms of entropy—implies we can compute *entropy differences*.

To illustrate these confusions, let us discuss a different derived concept: *The Bekenstein Bound*

Bekenstein Bound

Black hole considerations led Bekenstein to propose a universal bound on the entropy in a region of size L and energy E [Bekenstein '1981]:

$$S_L \leq \frac{2\pi k_B}{\hbar c} L E$$

which is saturated for black holes.

Based on the complaints described earlier, rigorously proving such an inequality will requires taming the infinities to say something scheme-independent

Bekenstein Bound

Another issue in the Bekenstein bound:

$$S_L \leq \frac{2\pi k_B}{\hbar c} \boxed{LE} \ .$$

- QFT Hamiltonian is not local (acts on Cauchy slice)
- Restricting it to a region L introduces cutoff dependence.

To make sense of this, Casini [Casini '2008] replaced Bekenstein's formula with the UV finite:

$$\boxed{S_L(\rho_1) - S_L(\rho_0) \leq \mathrm{Tr}(K\rho_1) - \mathrm{Tr}(K\rho_0)}$$

where ρ_0 and ρ_1 density matrices reduced to the region L.

We have also defined:

$$K \equiv -\log \rho_0$$

is the *Modular Hamiltonian* of ρ_0. (Think of the thermal density matrix at $T = 1$: $\rho = e^{-H}$)

Casini's formula is equivalent to:

$$S_L^{\rho_1||\rho_0} \equiv -\mathrm{Tr}(\rho_1 \log \rho_1) + \mathrm{Tr}(\rho_1 \log \rho_0) \geq 0$$

which is a result known as *monotonicity of relative entropy*—proved in the 70s [Lindblad, '1975][Araki, '1976][Uhlmann '1977] (see also [Petz, '2002])

It's interesting to note that Bekenstein landed on a (incorrect) version of the monotonicity of relative entropy just by thinking about black holes.

Moreover relative entropy *rigorously* satisfies:

$$S_{L+\delta L}^{\rho_1||\rho_0} - S_{L}^{\rho_1||\rho_0} \geq 0$$

which is how we think a BH behaves as matter falls in and its area increases.

Following these developments, Wall [Wall, 2011] proved the Generalized Second Law of BH thermodynamics in semiclassical gravity.

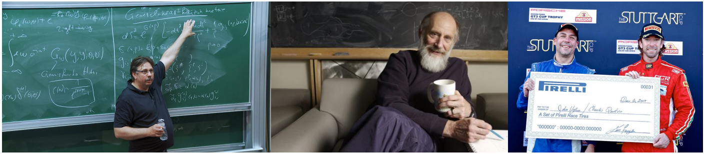
But these notions are cumbersome. Some thoughts:

- Area theorem and BH entropy never referenced a regulator, or needed regularization.
- In fact, no known lattice cutoff procedure resulting in a diff. inv. theory in the continuum.
- What is the significance of 1.4×10^{69} bits per square meter?

Alternatively: somtimes said that the difficulty in defining entropy in gravity comes from the failure of $\mathcal{H}$ to split into tensor factors due to the diff constraint.[Donnelly, Giddings '2016-18] [Raju '2021]

Controversial opinion: This can't be the full story. Bekenstein-Hawking formula does just fine.

It’s not an Xmas meeting without a picture of Santa



In a series of papers [Solodukhin ‘1994] [Susskind and Uglum, ‘1994] showed that a QFT coupled to gravity will change the entropy of a BH:

$$
\begin{aligned}
S_{\mathrm{BH+QFT}} &= \frac{A}{4G} + \frac{A}{\epsilon^2} + \text{subleading} \\
&= \frac{A}{4G(\epsilon)} + \dots
\end{aligned}
$$

which they interpreted as a running of the bare Newton’s constant G.

Now there's a new interpretation of this idea in terms of Von Neumann Algebras [Leutheusser and Liu, '2021] [Witten '2021]:

Gravity turns a Type III operator algebra into a Type II algebra. This allows *entropy differences* to make sense, as needed for the second law.

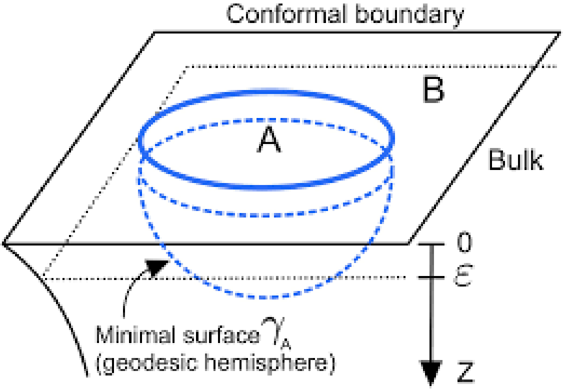
How gravity achieves this is the subject of the final part of the talk.

Part III: Some final comments about holography

You'll notice that in this entire talk I didn't mention holography (particularly in AdS).

But there have been many concerted efforts to understand entropy and entanglement in terms of semiclassical bulk gravity path integrals [Gibbons, Hawking '1977] [Hawking, Page '1983] [Ryu, Takayanagi '2006] [Hubeny, Rangamani, Takayanagi '2006] [Lewkowycz, Maldacena '2013] [Jafferis, Lewkowycz, Maldacena, Suh '2015]

Ryu-Takayanagi formula



[Ryu, Takayanagi ‘2006]

Ryu-Takayanagi proof

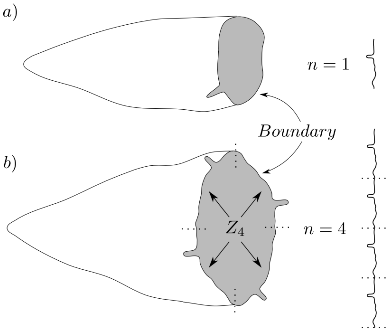


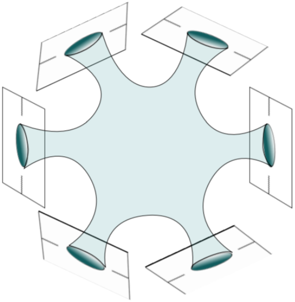
Fig. 2: Computing the entropy using the replica trick. (a) Euclidean solution for $n = 1$. (b) Solution for $n = 4$. At the boundary we go around the original circle n times before making the identification. We then find a smooth gravity solution with these boundary conditions. The curves in the right hand side are schematically giving the boundary conditions at infinity. We see that in (b), we simply repeat n times the boundary conditions we had in (a).

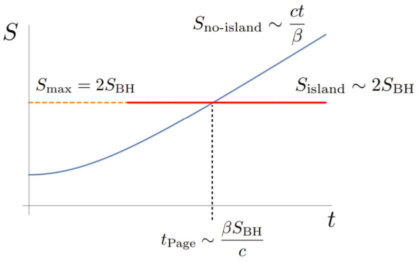
[Lewkowycz, Maldacena ‘2013]

In this example, *at leading order* the operator algebra remains type III because the volume of the RT surface is divergent at the boundary.

But it is possible to include quantum corrections, to get a formula closer to the results of Solodhukin and Susskind and Uglum. [Jafferis, Lewkowycz, Maldacena, Suh ‘2015]

Information Paradox





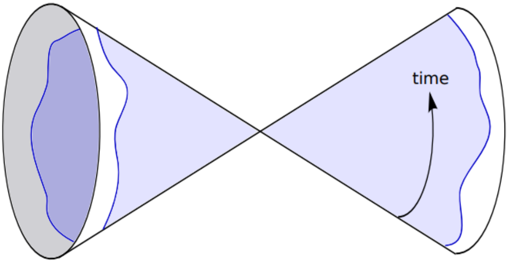
[Pennington '2019] [Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini '2019]

And just *thinking* about defining these entropies from the gravity path integral has led to progress on old problems.

Final comment:

The gravity path integral is obviously an ill-defined object....

Final comment:



[Saad, Shenker, Stanford '2018]

... but my hope for the future is that a better understanding of entanglement will teach us precisely how to compute the path integral.