

Majorana Phase and Matter Effects in Neutrino Chiral Oscillation

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1. Neutrino Chiral Oscillation

Equation of motion for a neutrino in free space

$$(i\cancel{\partial} - m)\psi = 0, \quad i\cancel{\partial}\psi_L - m\psi_R = 0, \quad i\cancel{\partial}\psi_R - m\psi_L = 0, \quad \psi(t, \mathbf{x}) = U(t)\psi(0)e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\psi_L = \frac{1-\gamma_5}{2}\psi, \quad \psi_R = \frac{1+\gamma_5}{2}\psi, \quad \psi = \psi_L + \psi_R. \quad U = e^{-iHt}, \quad H = \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0 = \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta$$

How left-handed and right-handed are entangled in free space?

In chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\psi^h(t, \mathbf{x}) = U(t)\psi^h(0)e^{i\mathbf{p}\cdot\mathbf{x}} = \psi^h(0)e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \quad \psi^h(0) = \frac{1}{\sqrt{2E}} \begin{pmatrix} \sqrt{E-h\cdot\mathbf{p}} u^h \\ \sqrt{E+h\cdot\mathbf{p}} u^h \end{pmatrix}$$

$$\psi(0)^\dagger\psi(0) = 1. \quad h = \pm 1 - \text{helicity}, \quad \mathbf{p}\cdot\boldsymbol{\sigma}u^h = (h\cdot p)u^h, \quad \mathbf{p} = (p_x, p_y, p_z) = p(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta).$$

$$u^{h=+1} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}, \quad u^{h=-1} = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix},$$

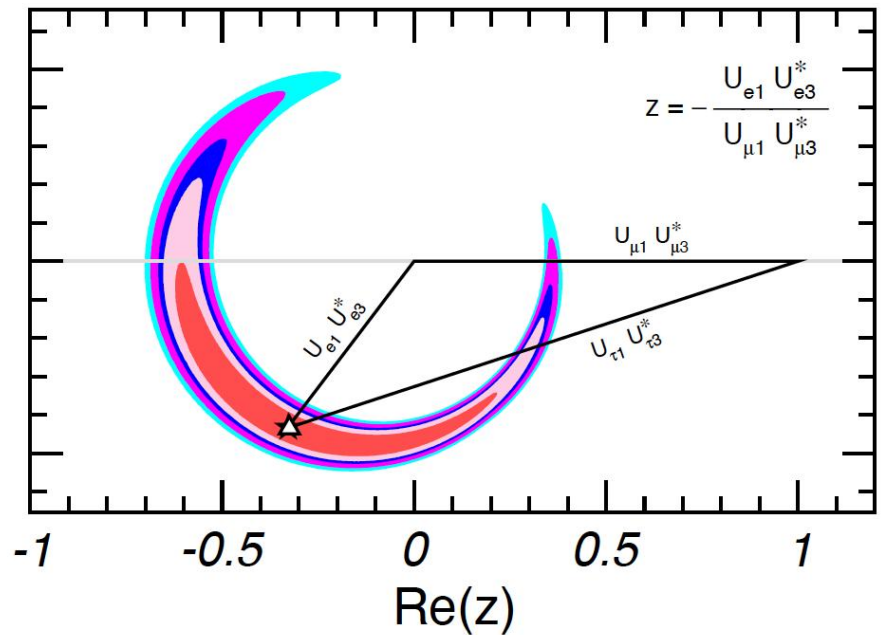
Oscillation probability from i to k for Dirac neutrinos:

$$P(\psi_i \rightarrow \psi_k) = |\langle \psi(0)_k | \psi(t)_i \rangle|^2 = \left| \sum V_{ij} V_{kj}^* e^{-i(E_j t - \mathbf{p}\cdot\mathbf{x})} \right|^2$$

Neutrino oscillations with 3 generations

$$\begin{pmatrix}
 c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{CP}} \\
 -s_{12} c_{23} - c_{12} s_{13} s_{23} e^{i\delta_{CP}} & c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta_{CP}} & c_{13} s_{23} \\
 s_{12} s_{23} - c_{12} s_{13} c_{23} e^{i\delta_{CP}} & -c_{12} s_{23} - s_{12} s_{13} c_{23} e^{i\delta_{CP}} & c_{13} c_{23}
 \end{pmatrix}$$

Table 14.7: 3ν oscillation parameters obtained from different global analyses of neutrino data. In all cases, the numbers labeled as NO (IO) are obtained assuming NO (IO), *i.e.*, relative to the respective local minimum. SK-ATM makes reference to the tabulated χ^2 map from the Super-Kamiokande analysis of their data in Ref. [97].



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| NO | Ref. [185] w/o SK-ATM | | Ref. [185] w SK-ATM | | Ref. [186] w SK-ATM | | Ref. [187] w SK-ATM | |
|--|----------------------------|-----------------------------|----------------------------|-----------------------------|----------------------------|-----------------------------|---------------------------|---------------------------|
| | Best Fit Ordering | | Best Fit Ordering | | Best Fit Ordering | | Best Fit Ordering | |
| Param | bfp $\pm 1\sigma$ | 3σ range | bfp $\pm 1\sigma$ | 3σ range | bfp $\pm 1\sigma$ | 3σ range | bfp $\pm 1\sigma$ | 3σ range |
| $\sin^2 \theta_{12}$ | $3.10^{+0.13}_{-0.12}$ | 2.75 \rightarrow 3.50 | $3.10^{+0.13}_{-0.12}$ | 2.75 \rightarrow 3.50 | $3.04^{+0.14}_{-0.13}$ | 2.65 \rightarrow 3.46 | $3.20^{+0.20}_{-0.16}$ | 2.73 \rightarrow 3.79 |
| $\theta_{12}/^\circ$ | $33.82^{+0.78}_{-0.76}$ | 31.61 \rightarrow 36.27 | $33.82^{+0.78}_{-0.76}$ | 31.61 \rightarrow 36.27 | $33.46^{+0.87}_{-0.88}$ | 30.98 \rightarrow 36.03 | $34.5^{+1.2}_{-1.0}$ | 31.5 \rightarrow 38.0 |
| $\sin^2 \theta_{23}$ | $5.58^{+0.20}_{-0.33}$ | 4.27 \rightarrow 6.09 | $5.63^{+0.18}_{-0.24}$ | 4.33 \rightarrow 6.09 | $5.51^{+0.19}_{-0.80}$ | 4.30 \rightarrow 6.02 | $5.47^{+0.20}_{-0.30}$ | 4.45 \rightarrow 5.99 |
| $\theta_{23}/^\circ$ | $48.3^{+1.2}_{-1.9}$ | 40.8 \rightarrow 51.3 | $48.6^{+1.0}_{-1.4}$ | 41.1 \rightarrow 51.3 | $47.9^{+1.1}_{-4.0}$ | 41.0 \rightarrow 50.9 | $47.7^{+1.2}_{-1.7}$ | 41.8 \rightarrow 50.7 |
| $\sin^2 \theta_{13}$ | $2.241^{+0.066}_{-0.065}$ | 2.046 \rightarrow 2.440 | $2.237^{+0.066}_{-0.065}$ | 2.044 \rightarrow 2.435 | $2.14^{+0.09}_{-0.07}$ | 1.90 \rightarrow 2.39 | $2.160^{+0.083}_{-0.069}$ | 1.96 \rightarrow 2.41 |
| $\theta_{13}/^\circ$ | $8.61^{+0.13}_{-0.13}$ | 8.22 \rightarrow 8.99 | $8.60^{+0.13}_{-0.13}$ | 8.22 \rightarrow 8.98 | $8.41^{+0.18}_{-0.14}$ | 7.9 \rightarrow 8.9 | $8.45^{+0.16}_{-0.14}$ | 8.0 \rightarrow 8.9 |
| $\delta_{CP}/^\circ$ | 222^{+38}_{-28} | 141 \rightarrow 370 | 221^{+39}_{-28} | 144 \rightarrow 357 | 238^{+41}_{-33} | 149 \rightarrow 358 | 218^{+38}_{-27} | 157 \rightarrow 349 |
| $\frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2}$ | $7.39^{+0.21}_{-0.20}$ | 6.79 \rightarrow 8.01 | $7.39^{+0.21}_{-0.20}$ | 6.79 \rightarrow 8.01 | $7.34^{+0.17}_{-0.14}$ | 6.92 \rightarrow 7.91 | $7.55^{+0.20}_{-0.16}$ | 7.05 \rightarrow 8.24 |
| $\frac{\Delta m_{32}^2}{10^{-3} \text{ eV}^2}$ | $2.449^{+0.032}_{-0.030}$ | 2.358 \rightarrow 2.544 | $2.454^{+0.029}_{-0.031}$ | 2.362 \rightarrow 2.544 | $2.419^{+0.035}_{-0.032}$ | 2.319 \rightarrow 2.521 | 2.424 ± 0.03 | 2.334 \rightarrow 2.524 |
| IO | $\Delta\chi^2 = 6.2$ | | $\Delta\chi^2 = 10.4$ | | $\Delta\chi^2 = 9.5$ | | $\Delta\chi^2 = 11.7$ | |
| $\sin^2 \theta_{12}$ | $3.10^{+0.13}_{-0.12}$ | 2.75 \rightarrow 3.50 | $3.10^{+0.13}_{-0.12}$ | 2.75 \rightarrow 3.50 | $3.03^{+0.14}_{-0.13}$ | 2.64 \rightarrow 3.45 | $3.20^{+0.20}_{-0.16}$ | 2.73 \rightarrow 3.79 |
| $\theta_{12}/^\circ$ | $33.82^{+0.78}_{-0.76}$ | 31.61 \rightarrow 36.27 | $33.82^{+0.78}_{-0.75}$ | 31.62 \rightarrow 36.27 | $33.40^{+0.87}_{-0.81}$ | 30.92 \rightarrow 35.97 | $34.5^{+1.2}_{-1.0}$ | 31.5 \rightarrow 38.0 |
| $\sin^2 \theta_{23}$ | $5.63^{+0.19}_{-0.26}$ | 4.30 \rightarrow 6.12 | $5.65^{+0.17}_{-0.22}$ | 4.36 \rightarrow 6.10 | $5.57^{+0.17}_{-0.24}$ | 4.44 \rightarrow 6.03 | $5.51^{+0.18}_{-0.30}$ | 4.53 \rightarrow 5.98 |
| $\theta_{23}/^\circ$ | $48.6^{+1.1}_{-1.5}$ | 41.0 \rightarrow 51.5 | $48.8^{+1.0}_{-1.2}$ | 41.4 \rightarrow 51.3 | $48.2^{+1.0}_{-1.4}$ | 41.8 \rightarrow 50.9 | $47.9^{+1.0}_{-1.7}$ | 42.3 \rightarrow 50.7 |
| $\sin^2 \theta_{13}$ | $2.261^{+0.067}_{-0.064}$ | 2.066 \rightarrow 2.461 | $2.259^{+0.065}_{-0.065}$ | 2.064 \rightarrow 2.457 | $2.18^{+0.08}_{-0.07}$ | 1.95 \rightarrow 2.43 | $2.220^{+0.074}_{-0.076}$ | 1.99 \rightarrow 2.44 |
| $\theta_{13}/^\circ$ | $8.65^{+0.13}_{-0.12}$ | 8.26 \rightarrow 9.02 | $8.64^{+0.12}_{-0.13}$ | 8.26 \rightarrow 9.02 | $8.49^{+0.15}_{-0.14}$ | 8.0 \rightarrow 9.0 | $8.53^{+0.14}_{-0.15}$ | 8.1 \rightarrow 9.0 |
| $\delta_{CP}/^\circ$ | 285^{+24}_{-26} | 205 \rightarrow 354 | 282^{+23}_{-25} | 205 \rightarrow 348 | 247^{+26}_{-27} | 193 \rightarrow 346 | 281^{+23}_{-27} | 202 \rightarrow 349 |
| $\frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2}$ | $7.39^{+0.21}_{-0.20}$ | 6.79 \rightarrow 8.01 | $7.39^{+0.21}_{-0.20}$ | 6.79 \rightarrow 8.01 | $7.34^{+0.17}_{-0.14}$ | 6.92 \rightarrow 7.91 | $7.55^{+0.20}_{-0.16}$ | 7.05 \rightarrow 8.24 |
| $\frac{\Delta m_{32}^2}{10^{-3} \text{ eV}^2}$ | $-2.509^{+0.032}_{-0.032}$ | -2.603 \rightarrow -2.416 | $-2.510^{+0.030}_{-0.031}$ | -2.601 \rightarrow -2.419 | $-2.478^{+0.035}_{-0.033}$ | -2.577 \rightarrow -2.375 | -2.50 ± 0.04 | -2.59 \rightarrow -2.39 |

Neutrino Chiral Oscillation

In the SM neutrinos are produced by W and/or Z interactions.

At production $t = 0$ point, they are left-handed and normalized, $\psi_L^h(0) = \sqrt{\frac{2E}{E-h\cdot p}} \frac{1-\gamma_5}{2} \psi^h(0)$.

$$\psi_L^h(t) = \sqrt{\frac{2E}{E-h\cdot p}} e^{-iHt} \frac{1-\gamma_5}{2} \psi^h(0) = \psi_L^h(t) = \sqrt{\frac{2E}{E-h\cdot p}} \left(e^{-iEt} \frac{1-\gamma_5}{2} \psi^h(0) - i \frac{m}{E} \sin(Et) \left[\beta, \frac{1-\gamma_5}{2} \right] \psi^h(0) \right).$$

Used
$$U(t) = e^{-iHt} = \cos(Et) - i \frac{\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta}{E} \sin(Et)$$

$$\psi_L^{h\dagger} \psi_L^h(t) = \left(\cos(Et) + i \frac{h \cdot p}{E} \sin(Et) \right) e^{i\mathbf{p} \cdot \mathbf{L}}, \quad \psi_R^{h\dagger} \psi_L^h(t) = \left(-i \frac{m}{E} \sin(Et) \right) e^{i\mathbf{p} \cdot \mathbf{L}}$$

$$P(\nu_L^h \rightarrow \nu_L^h) = |\psi_L^{h\dagger} \psi_L^h(t)|^2 = 1 - \frac{m^2}{E^2} \sin^2(Et), \quad P(\nu_L^h \rightarrow \nu_R^h) = |\psi_R^{h\dagger} \psi_L^h(t)|^2 = \frac{m^2}{E^2} \sin^2(Et)$$

Left-handed neutrinos oscillated into right-handed ones!

$$P(\nu_{Li}^h \rightarrow \nu_{Lk}^h) = |V_{ij} V_{kj}^* (\cos(E_j t) + i \frac{h \cdot p}{E_j} \sin(E_j t))|^2, \quad P(\nu_{Li}^h \rightarrow \nu_{Rk}^h) = | -i V_{ij} V_{kj}^* \frac{m_j}{E_j} \sin(E_j t) |^2 |\psi_R^{h\dagger} \psi_L^h(t)|^2$$

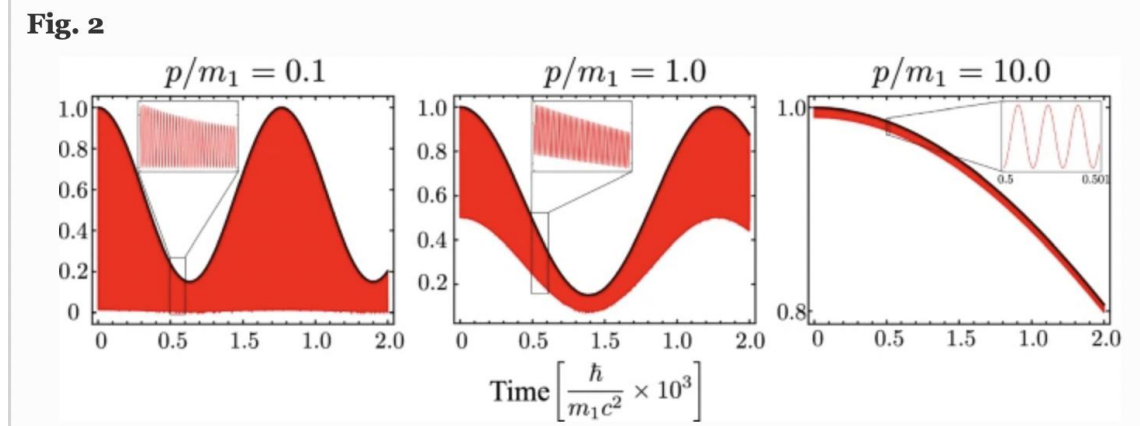
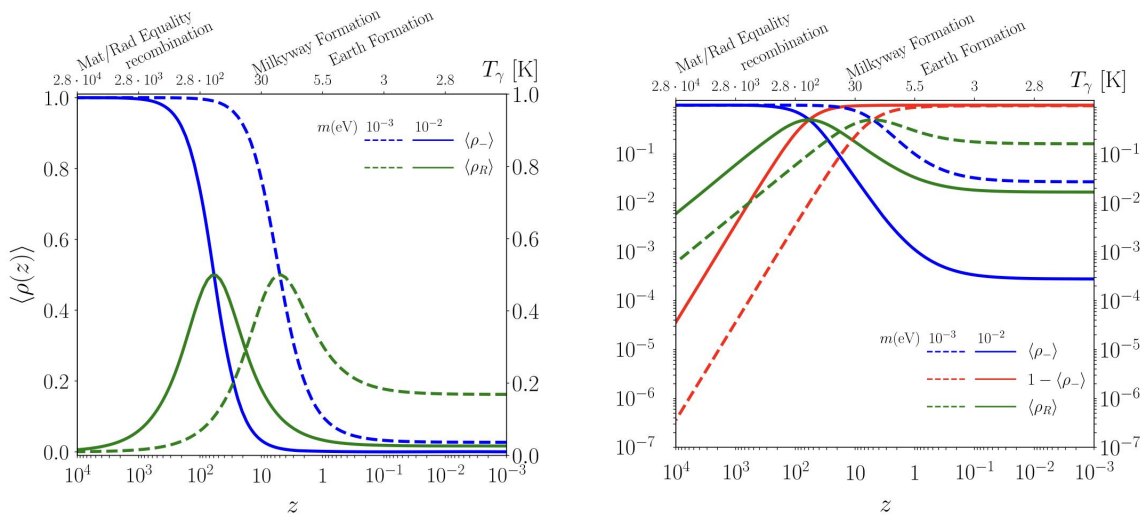
Chiral oscillation probability extremely small for relativistic neutrinos because the suppression factor: m^2/E^2 .

Numerically valid to use Dirac neutrinos to describe neutrino oscillations.

However the effect can be large for non-relativistic neutrinos, such as background cosmic neutrinos where p is small compared with m .

S-F Ge & P Pasquini, PLB811(2020)135961;

V Bittencourt, A. Bernardini & M. Blasone, EPJC81 (2021)411.



Survival probability $\mathcal{P}_{e \rightarrow e}(t)$ as a function of time. The black curves indicate the standard survival probability formula (16) and the red curves depict the full formula including the fast chiral oscillations (depicted in the insets). Parameters: $\sin^2 \theta = 0.306$, $m_2^2 = \Delta_{21}^2 + m_1^2$, with $\Delta_{21}^2/m_1^2 = 0.01$

Fig. 1. Evolution of the chiral density matrix elements $\langle \rho_- \rangle$ (blue) or $1 - \langle \rho_- \rangle$ (red), and $\rho_R = \mathbb{R}[c_L c_R^*]$ (green) as functions of the redshift z for $m = 10^{-2}$ eV (solid) and $m = 10^{-3}$ eV (dashed) with $|\mathbf{p}_d| = 1$ MeV. The redshift at neutrino decoupling is taken to be $z_d \sim 6 \times 10^9$.

2. Neutrino Oscillation in Matter

When neutrinos travel in matter, due to interaction of neutrinos with matter mediated by W and Z, the Lagrangian is modified

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi - j^\mu \bar{\psi} \gamma_\mu \frac{1 - \gamma_5}{2} \psi .$$

j^μ is the matter current which neutrino can interact.

In the rest frame of the homogeneous, isotropic, unpolarized electrical neutrality medium, $j^\mu = (\rho, \vec{0})$
 $\rho = \sqrt{2}G_F (N_e \delta_{\alpha e} - \frac{1}{2}N_n)$. N_e, N_n number density of electron and neutron, $\delta_{\alpha e}$ are zero for ν_μ and ν_τ .

$$\left(i\cancel{\partial} - m - \rho \gamma_0 \frac{1 - \gamma_5}{2} \right) \psi = 0 \quad H = \mathbf{p} \cdot \boldsymbol{\alpha} + m\beta + \rho \frac{1 - \gamma_5}{2} = \begin{pmatrix} \rho - \mathbf{p} \cdot \boldsymbol{\sigma} & m \\ m & \mathbf{p} \cdot \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - h \cdot p & m \\ m & h \cdot p \end{pmatrix}$$

The eigenvalues of H are: $E_1 = \frac{\rho}{2} + E_h$, $E_2 = \frac{\rho}{2} - E_h$, $E_h = \sqrt{m^2 + (h \cdot p - \rho/2)^2}$,

$$\psi_1 = \frac{1}{\sqrt{2E_h}} \begin{pmatrix} \sqrt{E_h - (h \cdot p - \frac{\rho}{2})} u^h \\ \sqrt{E_h + (h \cdot p - \frac{\rho}{2})} u^h \end{pmatrix}, \quad \psi_2 = \frac{1}{\sqrt{2E_h}} \begin{pmatrix} \sqrt{E_h + (h \cdot p - \frac{\rho}{2})} u^h \\ -\sqrt{E_h - (h \cdot p - \frac{\rho}{2})} u^h \end{pmatrix}$$

The evolution for a given wave function entering the media with momentum \vec{p} at $t = 0$ is

$$\psi_L^h(t) = e^{-iHt}\psi_L^h(0) = e^{-\frac{i}{2}\rho t} \begin{pmatrix} \cos(E_h t) + i\frac{h\cdot p - \frac{\rho}{2}}{E_h} \sin(E_h t) & -i\frac{m}{E_h} \sin(E_h t) \\ -i\frac{m}{E_h} \sin(E_h t) & \cos(E_h t) - i\frac{h\cdot p - \frac{\rho}{2}}{E_h} \sin(E_h t) \end{pmatrix} \begin{pmatrix} u^h \\ 0 \end{pmatrix},$$

$$P(\psi_L^h \rightarrow \psi_L^h) = 1 - \frac{m^2}{E_h^2} \sin^2(E_h t), \quad P(\psi_L^h \rightarrow \psi_R^h) = \frac{m^2}{E_h^2} \sin^2(E_h t).$$

Similar as that for free space, but dependent on matter density via $E_h = \sqrt{m^2 + (h \cdot p - \rho/2)^2}$.

There is a resonant enhanced chiral oscillation at $h \cdot p - \rho/2 = 0$!

From the above, one can easily recover the usual matter oscillation formalism with $J_L^\mu = (\rho, 0)$ in the relativistic case $p \gg M > m \gg \rho$. Keeping the leading effect in this limit, one obtains,

$$H_{\text{eff}} = p + \frac{M^\dagger M}{2p} - h \cdot \rho. \tag{13}$$

Then we can find that matter effect would influence the contribution from mixing angle and mass square. Note that for $h = -1$ helicity, it is the usual leading order neutrino oscillation in matter effective Hamiltonian which can cause matter induced MSW resonant effect. But for $h = +1$ or $\rho < 0$, the matter effects are different.

3. Majorana and Seesaw Neutrinos in Free Space

Pure left-handed neutrinos, having Majorana mass like in Type II seesaw model

$$\mathcal{L} = \bar{\nu}_L i \not{\partial} \nu_L - \frac{1}{2} m (\bar{\nu}_L^c \nu_L + \text{h.c.}) - \bar{\nu}_L j_L^\mu \gamma_\mu \nu_L = \frac{1}{2} \bar{\psi}^m (i \not{\partial} - m) \psi^m - \bar{\psi}^m j^\mu \gamma_\mu \frac{1-\gamma_5}{2} \psi^m, \quad \psi^m = \nu_L + \nu_L^c.$$

$$(i \not{\partial} - \widehat{M}) \psi^m - j_L^\mu \gamma_\mu \frac{1-\gamma_5}{2} \psi^m + (j_L^\mu)^* \gamma_\mu \frac{1+\gamma_5}{2} \psi^m = 0,$$

$$\Rightarrow H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m - \beta j_L^\mu \gamma_\mu (1 - \gamma_5) = \begin{pmatrix} \rho - \mathbf{p} \cdot \boldsymbol{\sigma} & m \\ m & \mathbf{p} \cdot \boldsymbol{\sigma} - \rho \end{pmatrix},$$

ρ in the Dirac neutrino case is replaced by 2ρ .

$$U(t) = \begin{pmatrix} \cos(E_h^m t) + i \frac{h \cdot p - \rho}{E_h^m} \sin(E_h^m t) & -i \frac{m}{E_h^m} \sin(E_h^m t) \\ -i \frac{m}{E_h^m} \sin(E_h^m t) & \cos(E_h^m t) - i \frac{h \cdot p - \rho}{E_h^m} \sin(E_h^m t) \end{pmatrix}$$

where $E_h^m = \sqrt{m^2 + (h \cdot p - \rho)^2}$. Resonant point shifted to: $p - \rho = 0$.

$$P(\nu_L^h \rightarrow \nu_L^h) = \cos^2(E_h^m t) + \frac{(h \cdot p - \rho)^2}{(E_h^m)^2} \sin^2(E_h^m t), \quad P(\nu_L^h \rightarrow (\nu_L^h)^c) = \frac{m^2}{(E_h^m)^2} \sin^2(E_h^m t)$$

One also obtains the same $H_{eff} = p + \frac{M^\dagger M}{2p} - h \cdot \rho$.

Seesaw Neutrino Oscillation

$$\mathcal{L} = \bar{\nu}_L i \not{\partial} \nu_L + \bar{N}_R i \not{\partial} N_R - \frac{1}{2} \left((\bar{\nu}_L^c \quad \bar{N}_R) \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} + \text{h.c.} \right) = \bar{\psi}_L i \not{\partial} \psi_L - \frac{1}{2} (\bar{\psi}_L^c \mathcal{M} \psi_L + \text{h.c.}) ,$$

$$\psi_L = \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} , \quad \mathcal{M} = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix} ,$$

$$V^T \mathcal{M} V = \widehat{M} = \text{diag}\{m_1, m_2, m_3, M_1, M_2, M_3\} , \quad \psi_L^m = V^\dagger \psi_L , \quad \psi^m = \psi_L^m + (\psi_L^m)^c$$

$$\mathcal{L} = \bar{\psi}_L^m i \not{\partial} \psi_L^m - \frac{1}{2} \left((\bar{\psi}_L^m)^c \widehat{M} \psi_L^m + \text{h.c.} \right) = \frac{1}{2} \left(\bar{\psi}^m (i \not{\partial} - \widehat{M}) \psi^m \right) ,$$

$$H = \begin{pmatrix} \boldsymbol{\alpha} \cdot \mathbf{p} + \widehat{M}_l \beta & 0 \\ 0 & \boldsymbol{\alpha} \cdot \mathbf{p} + \widehat{M}_h \end{pmatrix} ,$$

$$\widehat{M}_l = \text{diag}\{m_1, m_2, m_3\} , \quad \widehat{M}_h = \text{diag}\{M_1, M_2, M_3\} .$$

The initial state would be

$$\psi_{Li}^h = \begin{pmatrix} V_{i1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ \dots \\ V_{i6}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}, \quad (\psi_{Li}^h)^c = \begin{pmatrix} V_{i1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ \dots \\ V_{i6} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix},$$

$$U(t) = e^{-iHt}$$

$$= \begin{pmatrix} \begin{pmatrix} \cos(E_{m_1} t) + i \frac{h \cdot p}{E_{m_1}} \sin(E_{m_1} t) & -i \frac{m_1}{E_{m_1}} \sin(E_{m_1} t) \\ -i \frac{m_1}{E_{m_1}} \sin(E_{m_1} t) & \cos(E_{m_1} t) - i \frac{h \cdot p}{E_{m_1}} \sin(E_{m_1} t) \end{pmatrix} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \begin{pmatrix} \cos(E_{M_3} t) + i \frac{h \cdot p}{E_{M_3}} \sin(E_{M_3} t) & -i \frac{M_3}{E_{M_3}} \sin(E_{M_3} t) \\ -i \frac{M_3}{E_{M_3}} \sin(E_{M_3} t) & \cos(E_{M_3} t) - i \frac{h \cdot p}{E_{M_3}} \sin(E_{M_3} t) \end{pmatrix} \end{pmatrix}$$

$$E_{m_i} = \sqrt{p^2 + m_i^2}, \quad E_{M_i} = \sqrt{p^2 + M_i^2}$$

The oscillation probability

$$P(\psi_{Li}^h \rightarrow \psi_{Lj}^h) = \left| \psi_{Lj}^\dagger e^{-iHt} \psi_{Li} \right|^2 = \left| \sum_k V_{ik} V_{jk}^* \left(\cos(E_k t) + i \frac{h \cdot p}{E_k} \sin(E_k t) \right) \right|^2$$

$$P(\psi_{Li}^h \rightarrow (\psi_{Lj}^h)^c) = \left| (\psi_{Lj}^c)^\dagger e^{-iHt} \psi_{Li} \right|^2 = \left| \sum_k V_{ik}^* V_{jk}^* \left(-i \frac{m_k}{E_k} \sin(E_k t) \right) \right|^2$$

$$P((\psi_{Li}^h)^c \rightarrow (\psi_{Lj}^h)^c) = \left| (\psi_{Lj}^c)^\dagger e^{-iHt} \psi_{Lj}^c \right|^2 = \left| \sum_k V_{ik}^* V_{jk} \left(\cos(E_k t) - i \frac{h \cdot p}{E_k} \sin(E_k t) \right) \right|^2$$

$$P((\psi_{Li}^h)^c \rightarrow \psi_{Lj}^h) = \left| \psi_{Lj}^\dagger e^{-iHt} \psi_{Lj}^c \right|^2 = \left| \sum_k V_{ik} V_{jk} \left(-i \frac{m_k}{E_k} \sin(E_k t) \right) \right|^2$$

Majorana phase effects on chiral oscillation

One light and one heavy example: $\widehat{M} = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}$, $V = \begin{pmatrix} V_{a1} & V_{a2} \\ V_{s1} & V_{s2} \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{i\eta} \sin \theta \\ -\sin \theta & e^{i\eta} \cos \theta \end{pmatrix}$.

$$\nu_L^h = \begin{pmatrix} V_{a1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ V_{a2}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}, \quad N_R^h = \begin{pmatrix} V_{s1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ V_{s2} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix}, \quad (\nu_L^h)^c = \begin{pmatrix} V_{a1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ V_{a2} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix}, \quad (N_R^h)^c = \begin{pmatrix} V_{s1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ V_{s2}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}. \quad (2)$$

$$P(\nu_L^h \rightarrow \nu_L^h) = (\cos^2 \theta \cos(E_m t) + \sin^2 \theta \cos(E_M t))^2 + p^2 \left(\frac{\cos^2 \theta}{E_m} \sin(E_m t) + \frac{\sin^2 \theta}{E_M} \sin(E_M t) \right)^2$$

$$P(\nu_L^h \rightarrow (N_R^h)^c) = \frac{\sin^2 2\theta}{4} \left((\cos(E_m t) - \cos(E_M t))^2 + p^2 \left(\frac{\sin(E_m t)}{E_m} - \frac{\sin(E_M t)}{E_M} \right)^2 \right)$$

$$P(\nu_L^h \rightarrow (\nu_L^h)^c) = \frac{m^2}{(E_m)^2} \cos^4 \theta \sin^2(E_m t) + \frac{mM}{2E_m E_M} \sin^2 2\theta \cos 2\eta \sin(E_m t) \sin(E_M t) + \frac{M^2}{(E_M)^2} \sin^4 \theta \sin^2(E_M t)$$

$$P(\nu_L^h \rightarrow N_R^h) = \frac{\sin^2 2\theta}{4} \left(\frac{m^2}{(E_m)^2} \sin^2(E_m t) - \frac{2mM}{E_m E_M} \cos 2\eta \sin(E_m t) \sin(E_M t) + \frac{M^2}{(E_M)^2} \sin^2(E_M t) \right).$$

Majorana phase show up in the chiral oscillation explicitly!
This also applies to Type II seesaw with two generations!

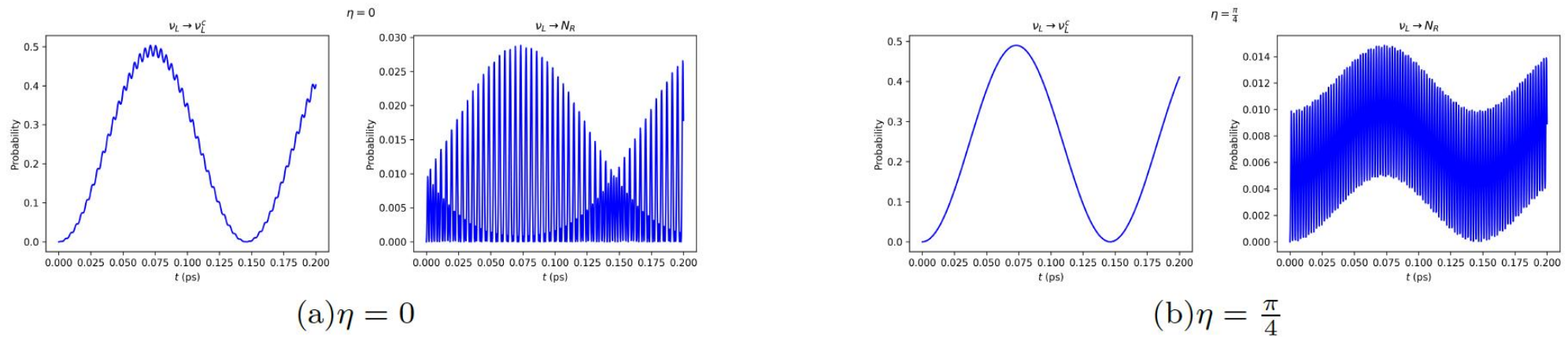


FIG. 1: Neutrino chiral rotation in last two equation in Eq.(23) in non-relativistic regime. The initial state is an active left-handed neutrino. The input parameters used are: $p = 0.01\text{eV}$, $m = 0.01\text{eV}$, $M = 1\text{eV}$, and $\sin\theta = \sqrt{m/M} = 0.1$, and the Majorana phase takes two different values $\eta = 0, \pi/4$.

4. Majorana and Seesaw Neutrinos in Matter

The general seesaw neutrino Lagrangian in matter propagation

$$\begin{aligned}\mathcal{L} &= \bar{\nu}_L i \not{\partial} \nu_L + \bar{N}_R i \not{\partial} N_R - \frac{1}{2} \left((\bar{\nu}_L^c \quad \bar{N}_R) \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} + \text{h.c.} \right) - (\bar{\nu}_L \quad \bar{N}_R^c) \begin{pmatrix} j_L^\mu & j_{RL}^\mu \\ j_{RL}^{\mu\dagger} & j_R^\mu \end{pmatrix} \gamma_\mu \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} \\ &= \bar{\psi}_L i \not{\partial} \psi_L - \frac{1}{2} (\bar{\psi}_L^c \mathcal{M} \psi_L + \text{h.c.}) - \bar{\psi}_L J^\mu \gamma_\mu \psi_L\end{aligned}$$

For homogeneous, isotropic, unpolarized electrical neutrality matter medium at rest, only j_L^0 is non-zero

$$j_L^0 = \begin{pmatrix} \rho_e & 0 & 0 \\ 0 & \rho_\mu & 0 \\ 0 & 0 & \rho_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{2} G_F (N_e - \frac{1}{2} N_n) & 0 & 0 \\ 0 & -\frac{G_F}{\sqrt{2}} N_n & 0 \\ 0 & 0 & -\frac{G_F}{\sqrt{2}} N_n \end{pmatrix}.$$

In terms of the mass eigenstate $\psi_L = V \psi_L^m$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left(\bar{\psi}^m (i \not{\partial} - \widehat{M}) \psi^m \right) - \bar{\psi}^m \tilde{J}^\mu \gamma_\mu \frac{1-\gamma_5}{2} \psi^m, \quad \psi^m = \psi_L^m + (\psi_L^m)^c, \quad \tilde{J}^\mu = V^\dagger J^\mu V.$$

$$(i \not{\partial} - \widehat{M}) \psi^m - \tilde{J}^\mu \gamma_\mu \frac{1-\gamma_5}{2} \psi^m + (\tilde{J}^\mu)^* \gamma_\mu \frac{1+\gamma_5}{2} \psi^m = 0.$$

$$H = \begin{pmatrix} \boldsymbol{\alpha} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\alpha} \cdot \mathbf{p} \end{pmatrix} + \begin{pmatrix} \beta \widehat{M}_l & 0 \\ 0 & \beta \widehat{M}_h \end{pmatrix} + V^\dagger \begin{pmatrix} j_L^\mu & j_{RL}^{\mu\dagger} \\ j_{RL}^\mu & -j_R^{\mu T} \end{pmatrix} V \gamma^0 \gamma_\mu \frac{1-\gamma^5}{2} - \left(V^\dagger \begin{pmatrix} j_L^\mu & j_{RL}^{\mu\dagger} \\ j_{RL}^\mu & -j_R^{\mu T} \end{pmatrix} V \right)^* \gamma^0 \gamma_\mu \frac{1+\gamma^5}{2}$$

Because the off-diagonal interaction, difficulty to get $U(t)$. For just one light and one heavy neutrinos, can get a closed analytic expression.

Let $\widehat{M}_l = m$ and $\widehat{M}_h = M$, similar to Dirac case $j^\mu = (\rho, \vec{0})$ and $j_{RL}^\mu = j_R^\mu = 0$,

$$\tilde{J}^\mu \gamma_\mu = V^\dagger J^\mu V \gamma_\mu = \begin{pmatrix} \frac{\rho}{2}(1 + \cos 2\theta) & \frac{\rho}{2}e^{i\eta} \sin 2\theta \\ \frac{\rho}{2}e^{-i\eta} \sin 2\theta & \frac{\rho}{2}(1 - \cos 2\theta) \end{pmatrix} \gamma_0.$$

$$H = \begin{pmatrix} \frac{\rho}{2}(1 + \cos 2\theta) - \mathbf{p} \cdot \boldsymbol{\sigma} & m & \frac{\rho}{2}e^{i\eta} \sin 2\theta & 0 \\ m & \mathbf{p} \cdot \boldsymbol{\sigma} - \frac{\rho}{2}(1 + \cos 2\theta) & 0 & -\frac{\rho}{2}e^{-i\eta} \sin 2\theta \\ \frac{\rho}{2}e^{-i\eta} \sin 2\theta & 0 & \frac{\rho}{2}(1 - \cos 2\theta) - \mathbf{p} \cdot \boldsymbol{\sigma} & M \\ 0 & -\frac{\rho}{2}e^{i\eta} \sin 2\theta & M & \mathbf{p} \cdot \boldsymbol{\sigma} - \frac{\rho}{2}(1 - \cos 2\theta) \end{pmatrix}$$

Time averaged oscillation

The eigenvalues of H are: $E_{1h} = -\sqrt{A_1 - A_2}$, $E_{2h} = \sqrt{A_1 - A_2}$,
 $E_{3h} = -\sqrt{A_1 + A_2}$, $E_{4h} = \sqrt{A_1 + A_2}$,

$$A_1 = \frac{m^2 + M^2}{2} + \left(h \cdot p - \frac{\rho}{2}\right)^2 + \frac{\rho^2}{4}$$

$$A_2 = \sqrt{\frac{2(m^2 - M^2)^2 + \rho^2 \left(2(m^2 + M^2 - 2mM \cos 2\eta) \sin^2 \theta + 8\left(h \cdot p - \frac{\rho}{2}\right)^2\right) - 8\rho(m^2 - M^2)\left(h \cdot p - \frac{\rho}{2}\right) \cos 2\theta}{8}}$$

$$U(t) = e^{-iHt} = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix}$$

We illustrate this with the limit $p \gg M > m \gg \rho$. In this case we have

$$\begin{aligned}
P(\nu_L^h \rightarrow (\nu_L^h)^c) &= \frac{(m^2 - M^2)^2}{8A_2^2} \left(\frac{(m^2 \cos^4 \theta + M^2 \sin^4 \theta)}{p^2} - \frac{\rho(4m^2 \cos^6 \theta - 4M^2 \sin^6 \theta + mM \cos 2\eta \cos 2\theta \sin^2 2\theta)}{p(m^2 - M^2)} \right. \\
&\quad \left. + \frac{2\rho^2(2m^2 \cos^4 \theta + 2M^2 \sin^4 \theta + mM \cos 2\eta \sin^2 2\theta)}{(m^2 - M^2)^2} \right), \\
P(\nu_L^h \rightarrow N_R^h) &= \frac{(m^2 - M^2)^2 \sin^2 2\theta}{32A_2^2} \left(\frac{m^2 + M^2}{p^2} - \frac{2\rho(m^2 + M^2 - 2mM \cos 2\eta) \cos 2\theta}{p(m^2 - M^2)} + \frac{4\rho^2 (m^2 + M^2 - 2mM \cos 2\eta)}{(m^2 - M^2)^2} \right)
\end{aligned} \tag{25}$$

The probabilities for the other two oscillation modes are

$$P(\nu_L^h \rightarrow \nu_L^h) = \frac{1}{2} + \frac{\cos^2 2\theta_{\text{eff}}}{2} - P(\nu_L^h \rightarrow (\nu_L^h)^c), \quad P(\nu_L^h \rightarrow (N_R^h)^c) = \frac{\sin^2 2\theta_{\text{eff}}}{2} - P(\nu_L^h \rightarrow N_R^h). \tag{26}$$

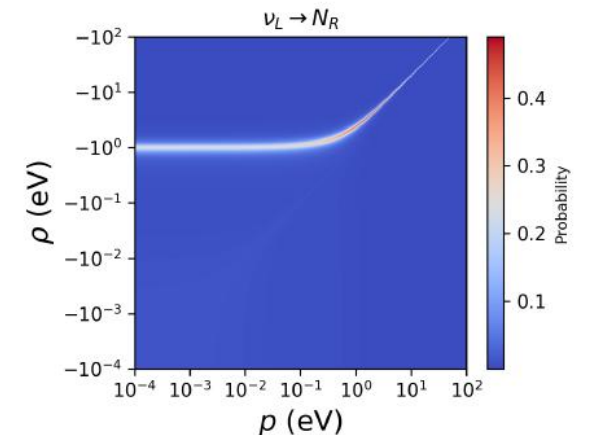
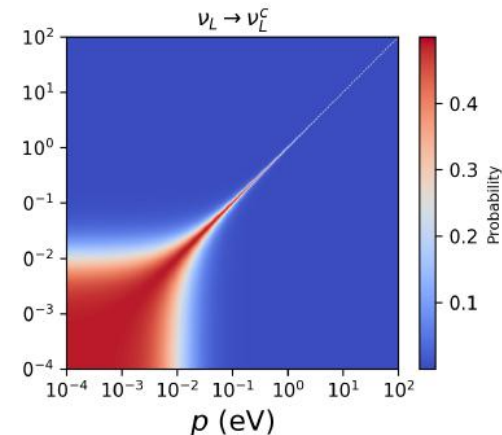
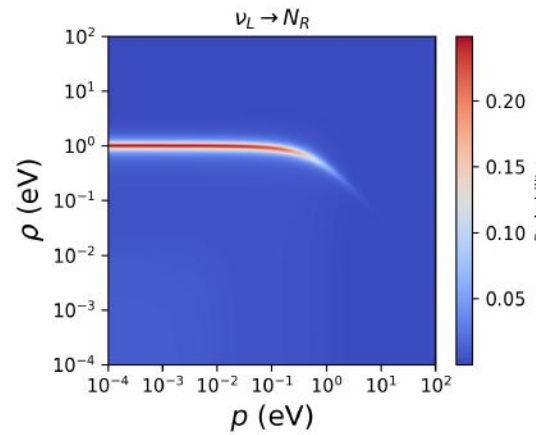
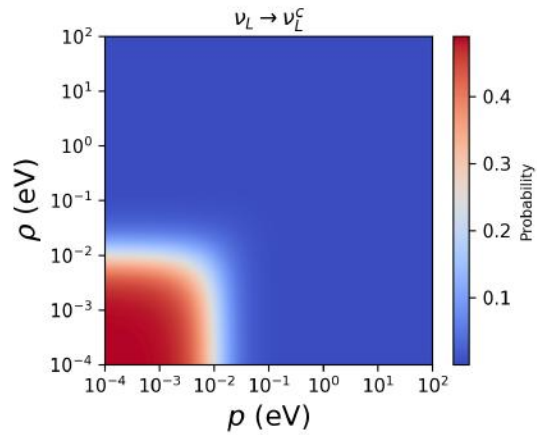
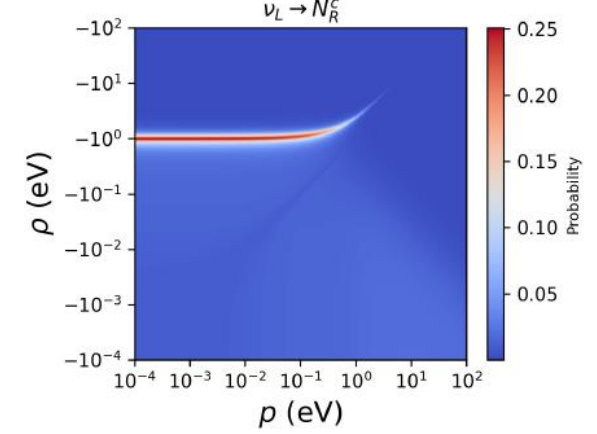
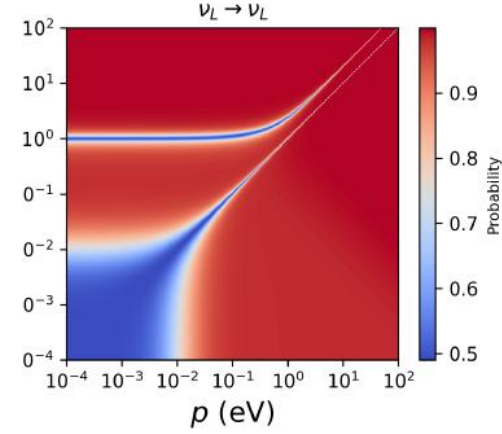
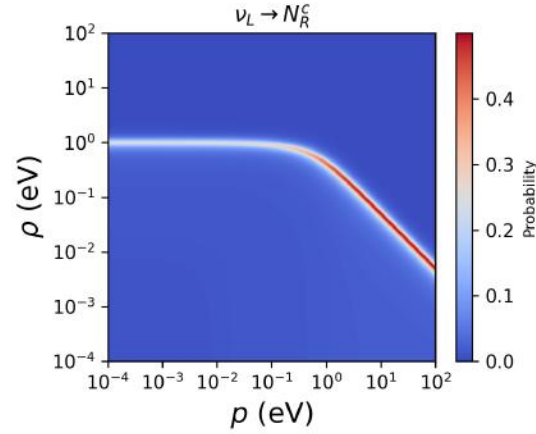
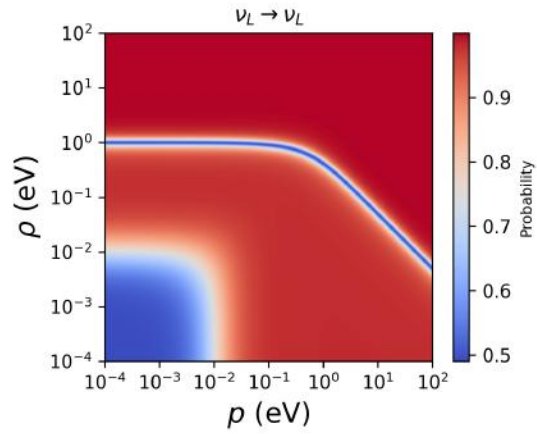
where $\cos 2\theta_{\text{eff}} = ((M^2 - m^2) \cos 2\theta + 2\rho(h \cdot p - \frac{\rho}{2})) / (2A_2)$. We do see the oscillation probabilities dependence on η even taking the time average. Note that the effect vanishes if any of the m , M and θ is zero.

$$\begin{aligned}
P(\nu_L^h \rightarrow \nu_L^h) &= |(\nu_L^h)^\dagger e^{-iHt} \nu_L^h|^2 = \left| U_{11} \frac{1 + \cos 2\theta}{2} + U_{13} e^{-i\eta} \frac{\sin 2\theta}{2} + U_{31} e^{i\eta} \frac{\sin 2\theta}{2} + U_{33} \frac{1 - \cos 2\theta}{2} \right|^2 \\
P(\nu_L^h \rightarrow (N_R^h)^c) &= |((N_R^h)^c)^\dagger e^{-iHt} \nu_L^h|^2 = \left| -U_{11} \frac{\sin 2\theta}{2} - U_{13} e^{-i\eta} \frac{1 - \cos 2\theta}{2} + U_{31} e^{i\eta} \frac{1 + \cos 2\theta}{2} + U_{33} \frac{\sin 2\theta}{2} \right|^2 \\
P(\nu_L^h \rightarrow (\nu_L^h)^c) &= |((\nu_L^h)^c)^\dagger e^{-iHt} \nu_L^h|^2 = \left| U_{21} \frac{1 + \cos 2\theta}{2} + U_{23} e^{-i\eta} \frac{\sin 2\theta}{2} + U_{41} e^{-i\eta} \frac{\sin 2\theta}{2} + U_{43} e^{-2i\eta} \frac{1 - \cos 2\theta}{2} \right|^2 \\
P(\nu_L^h \rightarrow N_R^h) &= |(N_R^h)^\dagger e^{-iHt} \nu_L^h|^2 = \left| -U_{21} \frac{\sin 2\theta}{2} - U_{23} e^{-i\eta} \frac{1 - \cos 2\theta}{2} + U_{41} e^{-i\eta} \frac{1 + \cos 2\theta}{2} + U_{43} e^{-2i\eta} \frac{\sin 2\theta}{2} \right|^2
\end{aligned}$$

η enters the chiral oscillation, even when time averaged!

Detailed Numerical calculations

$$h = -1, \eta = 0, m = 0.01 \text{ eV}, M = 1.0 \text{ eV}, \sin\theta = \sqrt{\frac{m}{M}} = 0.1.$$



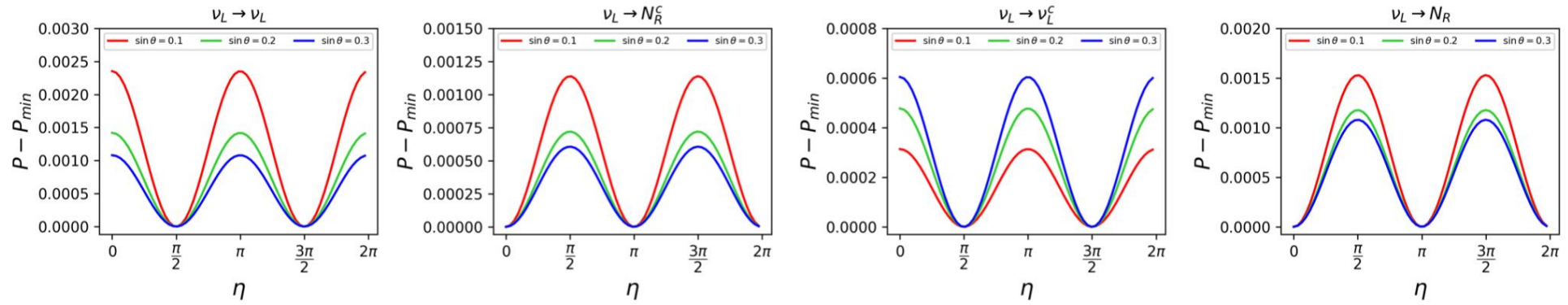


FIG. 3: The time averaged Majorana phase η effects on probabilities. The vertical axis is the difference between time averaged probabilities and their minimum values. The oscillating probabilities depend on the mixing angle θ . Other parameters are chosen from a point in matter effect significant region in Fig. 2(a).

$$h = -1, \rho_W = 0.0026\text{eV}, \rho_Z = -0.0013\text{eV}, \theta = 33.41^\circ, m = 10^{-3}\text{eV}, M^2 - m^2 = 7.41 \times 10^{-5}\text{eV}^2.$$

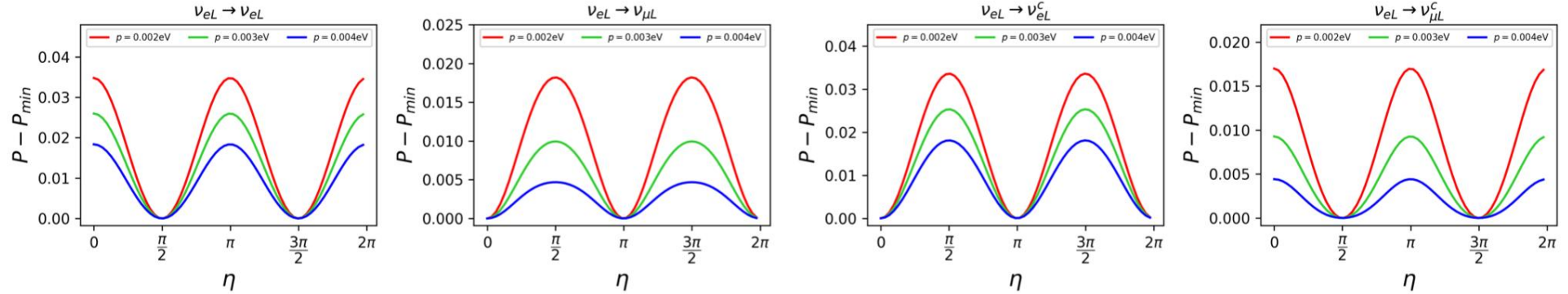


FIG. 4: The time averaged Majorana phase η effects on probabilities in two active Majorana neutrinos case. The vertical axis is the difference between time averaged probabilities and their minimum values. The red, green and blue lines are corresponding to $p = 0.002\text{eV}$, 0.003eV and 0.004eV , respectively. The mixing angle $\theta = \theta_{12}$ and mass square difference $M^2 - m^2 = \Delta m_{21}^2$, whose values are from [1]. $\rho_W = 0.0026\text{eV}$ and $\rho_Z = -0.0013\text{eV}$ are corresponding the mass density $a_p = 3.4 \times 10^{10}\text{g/cm}^3$ and $a_n = 3.4 \times 10^{10}\text{g/cm}^3$, respectively, so the total mass density $a \approx 6.8 \times 10^{10}\text{g/cm}^3$, which could be found in neutron star cluster [12].

Massive neutrinos can have chiral oscillation, suppressed by m^2/E^2 for relativistic neutrinos, but the suppression is lifted for non-relativistic neutrinos.

Majorana phases modify the oscillation pattern. In principle Majorana phases can be probed in chiral oscillation.

When matter effects are included, energies are split into two, there is a resonant point for chiral oscillation. Also even the oscillation is time averaged, Majorana phase still affects oscillation pattern.

Thank you for your attentions

