Crewther's Relation in Gauge-Parameter Dependent Schemes

Based on work completed in collaboration with John Gracey

arXiv:2306.11416 [hep-ph] and arXiv:2309.16554 [hep-ph]

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December 15, 2023

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Motivation

- Crewther's relation equates the β-function to the Adler D function, the Bjorken sum rule and a new perturbative series.
- The relation as originally stated fails for gauge-parameter dependent schemes, except at certain values of the gauge parameter at low loop order, including $\alpha = -3$.
- Using the methodology we developed we consider this issue in a variety of different schemes and gauge-fixing terms in order to address this issue.

Crewther's Relation

- Crewther's original relation was found using non-perturbative methods in a conformal system

$$\mathrm{C(a)D(a) = d_R}, \tag{1}$$

C(a) is the Bjorken sum rule, D(a) is the Adler D function with $\mathrm{a} = \frac{\mathrm{g}^2}{(4\pi)^2}$.

- Broadhurst and Kataev modified this relation for perturbative systems

$$\mathrm{CD} = \mathrm{d_R}(1 + \Delta_{\mathrm{CSB}}) \qquad \text{where} \qquad \Delta_{\mathrm{CSB}} = \mathrm{K}\frac{\beta}{\mathrm{a}}$$

which holds to all available orders in $\overline{\mathrm{MS}}$.

Crewther's Relation

- Rewrite the series as

$$K(a) = \left[\frac{C(a)D(a) - d_R}{d_R \beta(a)}\right] a. \tag{2}$$

We can always define expansion coefficients for $K(a) = \sum_i K_i a^i$ such that this relation holds.

- $K(a)$ has the structure of other perturbative series. In particular, for $\rho(a) = C(a),\ D(a)$ or $K(a)$:

$$\begin{aligned} \rho(a) &= \rho^{(0)} + \rho^{(1)}_{C_F} C_F a + a^2 C_F \left[C_A \rho^{(2)}_{C_F C_A} + C_F \rho^{(2)}_{C_F^2} + T_f N_f \rho^{(2)}_{C_F T_f} \right] \\ &\quad + a^3 C_F \left[C_F^2 \rho^{(3)}_{C_F^3} + C_F C_A \rho^{(3)}_{C_F^2 C_A} + C_A^2 \rho^{(3)}_{C_F C_A^2} + C_F T_f N_f \rho^{(3)}_{C_F T_f} \right. \\ &\quad \left. + C_A T_f N_f \rho^{(3)}_{C_F C_A T_f} + T_f^2 N_f^2 \rho^{(3)}_{C_F T_f^2} \right] + \mathcal{O}(a^4) \end{aligned} \tag{3}$$

$\overline{\text{MS}}$ scheme

- For example

$$
\begin{aligned}
\mathrm{K}^{\overline{\mathrm{MS}}}(\mathrm{a}) \quad = \quad & +\tfrac{3}{2}\left[8\zeta_3 - 7\right]\mathrm{C_F a} \\
& + \left[\left(\frac{326}{3} - \frac{304}{3}\zeta_3\right)\mathrm{N_f T_F} + \left(\frac{884}{3}\zeta_3 - \frac{629}{2}\right)\mathrm{C_A} \right. \\
& \quad \left. + \left(\frac{397}{6} - 240\zeta_5 + 136\zeta_3\right)\mathrm{C_F} \right]\mathrm{C_F a^2} \\
& + \mathcal{O}(\mathrm{a}^3). \qquad (4)
\end{aligned}
$$

mMOM

- The mMOM scheme is a gauge-parameter dependent scheme, meaning its β-function coefficients explicitly contain the gauge parameter α.
- It is defined so that its field two-point functions have no order a corrections at a characteristic scale.
- The vertex function is defined such that the ghost-gluon vertex is finite to all orders for a general gauge parameter, this is practically implemented through the relation

$$\mathrm{Z}_{\mathrm{g}}^{\overline{\mathrm{MS}}}\sqrt{\mathrm{Z}_{\mathrm{A}}^{\overline{\mathrm{MS}}}}\mathrm{Z}_{\mathrm{c}}^{\overline{\mathrm{MS}}} = \mathrm{Z}_{\mathrm{g}}^{\mathrm{mMOM}}\sqrt{\mathrm{Z}_{\mathrm{A}}^{\mathrm{mMOM}}}\mathrm{Z}_{\mathrm{c}}^{\mathrm{mMOM}} \tag{5}$$

where Z_ϕ are the renormalization constants of ϕ.

Crewther's Relation in the mMOM Scheme

- Garkusha, Kataev and Molokoedov calculated the Crewther series in this scheme. The $\mathcal{O}(\mathrm{a}^2)$ coefficient is

$$\begin{aligned}
\mathrm{K}_2(\alpha) &= \frac{\mathrm{C_F}}{12}\Big[(-2880\zeta_5 + 1632\zeta_3 + 794)\mathrm{C_F} \\
&\quad +(2184\zeta_3 - 2591)\mathrm{C_A} - (576\zeta_3 - 744)\mathrm{N_f T_f}\Big] \\
&\quad +3(7 - 8\zeta_3)\mathrm{C_F C_A}\left[\frac{(31 + 12\alpha + 3\alpha^2)\mathrm{C_A} - 8\mathrm{N_f T_f}}{88\mathrm{C_A} - 32\mathrm{N_f T_f}}\right]\alpha.
\end{aligned} \tag{6}$$

- Factorises as ordinary perturbative series if $\alpha = 0,\ -1$ or -3. But the latter two fail again at $\mathcal{O}(\mathrm{a}^3)$.

Problem

- Is there a way in which we can define a perturbative series for a general gauge parameter such that Crewther's relation holds in the mMOM scheme?

Fixed Points

- From Crewther's original argument, the product of the Adler D function and the Bjorken sum rule should be a constant in a conformally invariant system.
- We can enforce conformal invariance by considering the system at a fixed point, such that the running of the formal parameters are stationary.
- For gauge-parameter dependent schemes the formal parameters are the coupling constant a and the gauge parameter α.
- We find fixed points by solving

$$\frac{\mathrm{da}}{\mathrm{dl}} = \beta(\mathrm{a}_\infty, \alpha_\infty) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\alpha}{\mathrm{dl}} = \alpha_\infty \gamma_\alpha(\mathrm{a}_\infty, \alpha_\infty) = 0,$$

where $\mathrm{l} = \ln \frac{\mu^2}{\Lambda^2}$.

Loop Corrections

Define the α such that the original Crewther relation holds

$$\alpha = \alpha_0 + \alpha_1 \mathrm{a} \qquad \text{where} \qquad \alpha_1 = \alpha^{(1)}_{\mathrm{T_f}} \mathrm{T_f N_f} + \alpha^{(1)}_{\mathrm{C_F}} \mathrm{C_F} + \alpha^{(1)}_{\mathrm{C_A}} \mathrm{C_A} \quad (7)$$

For SU(3) mMOM we find

$$\alpha = -3 + \left[(2\mathrm{N_f} - 33)\alpha^{(1)}_{\mathrm{T_f}} - 36 \right] \frac{\mathrm{a}}{4}. \quad (8)$$

We can evaluate this at the fixed points a_∞ and compare to the gauge-parameter of the consistent infra-red stable fixed point

Loop Order	a_∞	α_∞	$\bar{\alpha}(\mathrm{a}_\infty)\Big\vert_{\alpha^{(1)}_{\mathrm{T_f}}=2}$
2	0.003200	-3.030182	-3.030402
3	0.003138	-3.027421	-3.029812
4	0.003143	-3.027354	-3.029859
5	0.003143	-3.027377	-3.029862

Fixed Point

N	a_∞	α_∞	$3(1+\Delta^{2L}_{csb})$	$3(1+\Delta^{3L}_{csb})$
2	0.0033112583	0.0000000000	2.9999991596	3.0000039877
	9.1803474173	2.4636080795	1271156.8083213258	17202735.3015072510
	0.0032001941	−3.0301823312	2.9999982468	3.0000012469
3	0.0031177883	0.0000000000	2.9999963264	3.0000001212
	0.1279084604	1.9051106246	6.2952539870	10.1893903424
	0.0031380724	−3.0274210489	2.9999973439	3.0000001217
4	0.0031213518	0.0000000000	2.9999963720	3.0000001843
	0.1902883419	0.0000000000	13.5399867931	66.1969134786
	0.1162651496	0.5286066929	5.3930704057	11.8942763573
	0.0031430130	−3.0273541344	2.9999974127	3.0000002080
5	0.0031220809	0.0000000000	2.9999963814	3.0000001972
	0.0577103776	0.0000000000	3.2818695828	3.7273436677
	0.0031434144	−3.0273765993	2.9999974183	3.0000002151
	0.0502252330	−3.8653031470	3.1912609578	3.2787374506

The Crewther product, $C(a,\alpha)D(a,\alpha)$ evaluated at fixed points in the mMOM where the anomalous dimensions are taken to the N loop level.

Fixed Point

We could instead consider the same product at roots of the β function given a fixed non-zero gauge parameter, as in the table below for $\alpha = 1$.

mMOM $\alpha = 1$ β function zero			
FP Loop Order	a	$3(1 + \Delta^{2L}_{csb})$	$3(1 + \Delta^{3L}_{csb})$
2	0.0039840637	3.0000169021	3.0000244250
3	0.0037731278	3.0000104128	3.0000164646
4	0.0037925523	3.0000109619	3.0000171393
5	0.0037946540	3.0000110219	3.0000172130

Crewther Graphs

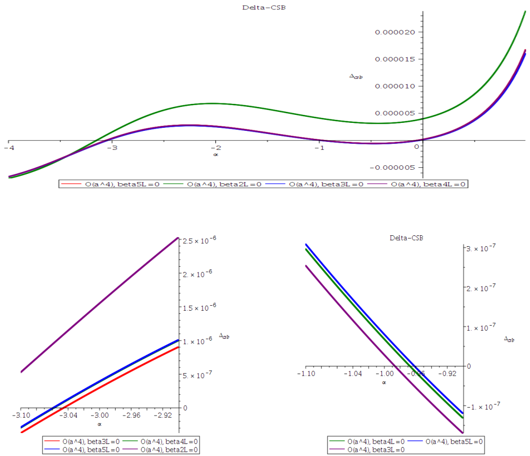

Figure 1: Δ_{CSB} to $\mathcal{O}(\mathrm{a}^4)$ in the mMOM scheme for different α values with a selected as the minimum real, positive value of the coupling constant such that the $\beta_{\mathrm{mMOM}} = 0$.

Scheme Change

- The product of the Adler and Bjorken sum rules is measurable and so its value is scheme independent up to order in truncation.
- The β function transforms under a scheme change according to

$$\beta(\mathrm{a}) \quad = \quad \beta_{\mathrm{S}}(\mathrm{a}_{\mathrm{S}}, \alpha_{\mathrm{S}}) \frac{\partial \mathrm{a}(\mathrm{a}_{\mathrm{S}}, \alpha_{\mathrm{S}})}{\partial \mathrm{a}_{\mathrm{S}}} + \alpha_{\mathrm{S}} \gamma_{\alpha}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}}, \alpha_{\mathrm{S}}) \frac{\partial \mathrm{a}(\mathrm{a}_{\mathrm{S}}, \alpha_{\mathrm{S}})}{\partial \alpha_{\mathrm{S}}}.$$

The Extended Crewther Relation

Taking Crewther's relation in $\overline{\mathrm{MS}}$ with $\Delta_{\mathrm{csb}} = \mathrm{K}_{\mathrm{a}}^{\beta} = \mathrm{K}_{\mathrm{a}}\beta$ we can consider how tis transforms under scheme change to give an extended new form of Crewther's relations The form of the conformal symmetry breaking term then transforms as

$$\begin{aligned}\Delta_{\mathrm{csb}}(\mathrm{a}) = \Delta^{\mathrm{S}}_{\mathrm{csb}}(\mathrm{a_S}, \alpha_{\mathrm{S}}) &= \mathrm{K}^{\mathrm{S}}_{\mathrm{a}}(\mathrm{a_S}, \alpha_{\mathrm{S}})\beta_{\mathrm{S}}(\mathrm{a_S}, \alpha_{\mathrm{S}}) \\ &+\mathrm{K}^{\mathrm{S}}_{\alpha}(\mathrm{a_S}, \alpha_{\mathrm{S}})\alpha_{\mathrm{S}}\gamma^{\mathrm{S}}_{\alpha}(\mathrm{a_S}, \alpha_{\mathrm{S}}).\end{aligned}$$

The $\mathrm{K_a}$ and K_α series can be calculated using

$$\mathrm{K}^{\mathrm{S}}_{\mathrm{a}}(\mathrm{a_S}, \alpha_{\mathrm{S}}) = \frac{\partial \mathrm{a}}{\partial \mathrm{a_S}}\Big[\mathrm{K_a}(\mathrm{a})\Big]\Big|_{\overline{\mathrm{MS}}\to\mathrm{mMOM}} \tag{9}$$

$$\mathrm{K}^{\mathrm{S}}_{\alpha}(\mathrm{a_S}, \alpha_{\mathrm{S}}) = \frac{\partial \mathrm{a}}{\partial \alpha_{\mathrm{S}}}\Big[\mathrm{K_a}(\mathrm{a})\Big]\Big|_{\overline{\mathrm{MS}}\to\mathrm{mMOM}} \tag{10}$$

Crewther's Relation in the mMOM Scheme

We find the Crewther series for the mMOM scheme

$$
\begin{aligned}
&\mathrm{K}_{\mathrm{a}}^{\mathrm{mMOM}}(\mathrm{a},\alpha) = 2(8\zeta_3 - 7) \\
&+\Big[(-15552\zeta_3 + 13608)\alpha^2 + (-31104\zeta_3 + 27216)\alpha \\
&+(-20736\mathrm{N_f} + 628416)\zeta_3 + 26784\mathrm{N_f} - 276480\zeta_5 \\
&-483432\Big]\frac{\mathrm{a}}{648} + [...]\mathrm{a}^2,
\end{aligned}
$$

$$
\mathrm{K}_{\alpha}^{\mathrm{mMOM}}(\mathrm{a},\alpha) = -3(\alpha + 1)(8\zeta_3 - 7)\mathrm{a}^2 + [...]\mathrm{a}^3. \tag{11}
$$

Ambiguity

We can define $\bar{\mathrm{K}}_{\mathrm{a}}$ and $\bar{\mathrm{K}}_{\alpha}$ which obey the same relation when defined through the transformation

$$\begin{aligned} \bar{\mathrm{K}}_{\mathrm{a}}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}) &= \mathrm{K}_{\mathrm{a}}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}) - \mathrm{F}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}})\alpha_{\mathrm{S}}\gamma_{\alpha}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}), \quad (12)\\ \bar{\mathrm{K}}_{\alpha}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}) &= \mathrm{K}_{\alpha}^{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}) + \mathrm{F}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}})\beta_{\mathrm{S}}(\mathrm{a}_{\mathrm{S}},\alpha_{\mathrm{S}}). \quad (13)\end{aligned}$$

Consider F_0 such that $\bar{\mathrm{K}}_{\alpha}^{\mathrm{s}}(\mathrm{F}_0;\mathrm{a}_{\mathrm{s}},\alpha_{\mathrm{s}}) = 0$. In the mMOM scheme this would require a series such that

$$\mathrm{F}_0(\mathrm{a},\alpha) = -\frac{\mathrm{K}_{\alpha}^{\mathrm{mMOM}}(\mathrm{a},\alpha)}{\beta^{\mathrm{mMOM}}(\mathrm{a},\alpha)} \approx -\frac{3(\alpha+1)(8\zeta_3-7)}{11-\frac{2}{3}\mathrm{N}_{\mathrm{f}}} + \mathcal{O}(\mathrm{a}). \quad (14)$$

Connection to BZ twin

- The $\alpha = -3$ value identified can thus be understood as the point near the IRS stable FP

$$\gamma_1(-3) = \left[- \frac{1}{2}\alpha C_A + \frac{13}{6} C_A - \frac{4}{3} N_f T_F \right]\Bigg|_{\alpha=-3} = \beta_0,$$

- We can therefore write Crewther's relation in the mMOM scheme to $\mathcal{O}(a^3)$ as

$$\Delta_{csb}(a,-3) = -K_a^{(0)}\beta_0 a^2 - [K_a^{(0)}\beta_1(-3) + (K_a^{(1)} - 3K_\alpha^{(2)}(-3))\beta_0]a^3$$

- Therefore if we relabel the leading order K_a term

$$K_a^{(1)} + 3K_\alpha^{(2)}(-3) \rightarrow K_a^{(1)}. \qquad (15)$$

we arrive at the original Crewther relation to LO.

Outlook

- For gauge parameter dependent schemes the conformal properties should be described by the running of both couplings, a and α, and therefore the Crewther relation should then be modified to include this.
- Given the transformation equations in the series for any scheme can be found from the $\overline{\text{MS}}$, although the resultant series is not unique.
- While our exemplar has been mMOM, these relations hold true both for a variety of schemes and for several different gauge fixing terms and so for pQCD in general.

Generalisation

- An extension to this for systems of n dynamical variables $\mathrm{g_i}$ where $\mathrm{i} = 1, ..., \mathrm{n}$, would be:

$$\Delta^{\mathrm{s}}_{\mathrm{csb}}(\mathrm{g}^{\mathrm{s}}_{\mathrm{i}}) = \sum_{\mathrm{i}} \mathrm{K}^{\mathrm{s}}_{\mathrm{g_i}}(\mathrm{g}^{\mathrm{s}}_{\mathrm{i}})\left(\frac{\mathrm{d}\mathrm{g}^{\mathrm{s}}_{\mathrm{i}}}{\mathrm{dl}}\right). \tag{16}$$

- Which is suggestive of

$$\Delta^{\mathrm{s}}_{\mathrm{csb}}(\mathrm{g}^{\mathrm{s}}_{\mathrm{i}}) = \frac{\mathrm{d}}{\mathrm{dl}}\kappa^{\mathrm{s}}(\mathrm{g}^{\mathrm{s}}_{\mathrm{i}}) = \left[\sum_{\mathrm{j}} \partial^{\mathrm{s}}_{\mathrm{g_j}}\kappa(\mathrm{g}^{\mathrm{s}}_{\mathrm{i}})\frac{\mathrm{d}\mathrm{g}^{\mathrm{s}}_{\mathrm{j}}}{\mathrm{dl}}\right] \tag{17}$$

where $\partial^{\mathrm{s}}_{\mathrm{g_j}} = \frac{\partial}{\partial \mathrm{g}^{\mathrm{s}}_{\mathrm{i}}}$.

Generalisation

- For our two-coupling theory this reduces to

$$\Delta^{s}_{csb}(a_s, \alpha_s) = \Big(\partial^{s}_{a}\kappa^{s}(a_s, \alpha_s)\Big)\beta^{s}(a_s, \alpha_s) + \Big(\partial^{s}_{\alpha}\kappa^{s}(a_s, \alpha_s)\Big)\alpha_s\gamma^{s}_{\alpha}(a_s, \alpha_s). \quad (18)$$

- In $\overline{\mathrm{MS}}$ then

$$K^{\overline{MS}}_{a}(a_{\overline{MS}}) = \partial^{\overline{MS}}_{a}\kappa^{\overline{MS}}(a_{\overline{MS}}). \quad (19)$$

κ in $\overline{\mathrm{MS}}$

- κ is then a scheme-invariant quantity, in $\overline{\mathrm{MS}}$

$$
\begin{aligned}
\kappa_{\mathrm{a}}^{\overline{\mathrm{MS}}}(\mathrm{a}) \;=\; & \kappa_{(0)}^{\overline{\mathrm{MS}}} + \left[12\zeta_3 \mathrm{C_F} - \frac{21}{2}\mathrm{C_F}\right]\mathrm{a} \\
& + \left[\frac{326}{3}\mathrm{N_f T_F C_F} + \frac{397}{6}\mathrm{C_F^2} - 240\zeta_5 \mathrm{C_F^2} + 136\zeta_3 \mathrm{C_F^2} \right. \\
& \quad \left. + \frac{884}{3}\zeta_3 \mathrm{C_F C_A} - \frac{629}{2}\mathrm{C_F C_A} - \frac{304}{3}\zeta_3 \mathrm{N_f T_F C_F}\right]\frac{\mathrm{a}^2}{2} \\
& + \mathcal{O}(\mathrm{a}^3)
\end{aligned}
$$

Thank you for listening

What we've covered:

- The Crewther equation as originally formulated fails for gauge-parameter dependent schemes.
- To accommodate for this the relation needs to be extended to include running of the gauge-parameter.
- We argue for this using numerical data from fixed points.

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