Courant Algebroid Relations and T-duality

Tom De Fraja

Heriot-Watt University
Supervisor: Richard Szabo

Outline

Wess-Zumino functional

Consider maps f from a worldsheet surface Σ to a target space M. Take (topological) action functional

$$S_H[f] = \int_V f^* H, \qquad H \in \Omega^3_{\text{cl}}(M), \quad \partial V = \Sigma.$$

The variation $\delta S_H = 0$ is given by **generalised vectors** $X + \xi \in \Gamma(TM \oplus T^* M)$ such that

$$\iota_X H = d\xi.$$

Wess-Zumino functional

Consider maps f from a worldsheet surface Σ to a target space M. Take (topological) action functional

$$S_H[f] = \int_V f^* H, \qquad H \in \Omega^3_{\mathrm{cl}}(M), \quad \partial V = \Sigma.$$

The variation $\delta S_H = 0$ is given by **generalised vectors** $X + \xi \in \Gamma(TM \oplus T^* M)$ such that

$$\iota_X H = d\xi.$$

Courant algebroids

The **generalised tangent bundle**, $TM \oplus T^*M$, can be equipped with a pairing and a bracket:

$$\langle X+\xi, Y+\eta\rangle = \iota_X\eta + \iota_Y\xi,$$
$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H.$$

In general, a **(exact) Courant algebroid** (CA) is a vector bundle E over M with a pairing and a bracket of sections fitting the sequence

$$0 \to T^*M \to E \to TM \to 0. \qquad (E \cong TM \oplus T^*M)$$

One recovers $H \in \Omega^3_{\mathrm{cl}}(M)$ by considering a splitting $s\colon TM \to E$ and setting

$$H(X,Y,Z) = \langle [s(X), s(Y)]_E, s(Z)\rangle_E.$$

Courant algebroids

The **generalised tangent bundle**, $TM \oplus T^*M$, can be equipped with a pairing and a bracket:

$$\langle X+\xi, Y+\eta\rangle = \iota_X\eta + \iota_Y\xi,$$
$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H.$$

In general, a **(exact) Courant algebroid** (CA) is a vector bundle E over M with a pairing and a bracket of sections fitting the sequence

$$0 \to T^*M \to E \to TM \to 0. \qquad (E \cong TM \oplus T^*M)$$

One recovers $H \in \Omega^3_{\mathrm{cl}}(M)$ by considering a splitting $s\colon TM \to E$ and setting

$$H(X,Y,Z) = \langle [s(X), s(Y)]_E, s(Z)\rangle_E.$$

Polyakov term and Isomorphisms

We can also add dynamical data to our sigma model: the Polyakov functional, for a Riemannian metric g on M is

$$S_g[f] = \int_\Sigma \|df\|_g^2 d\mu_g.$$

The data of g can be captured in a **generalised metric**; a positive definite subbundle $V \subset E$, i.e. $\langle V, V\rangle > 0$. For example,

$$V = \mathrm{gr}(g) = \{X + g(X) : X \in TM\} \subset TM \oplus T^*M$$

The data of the full action $S = S_H + S_g$ is embedded in a CA equipped with a positive definite subbundle V.
Thus, an isomorphism $\Phi\colon E \to E'$ covering a diffeomorphism $\phi\colon M \to M'$ and preserving $\langle,\rangle_E, [,]_E$ and V will yield Sigma-modles over M and M' with the same equations of motion.

Polyakov term and Isomorphisms

We can also add dynamical data to our sigma model: the Polyakov functional, for a Riemannian metric g on M is

$$S_g[f] = \int_\Sigma \|df\|_g^2 d\mu_g.$$

The data of g can be captured in a **generalised metric**; a positive definite subbundle $V \subset E$, i.e. $\langle V, V \rangle > 0$. For example,

$$V = \mathrm{gr}(g) = \{X + g(X) : X \in TM\} \subset TM \oplus T^*M$$

The data of the full action $S = S_H + S_g$ is embedded in a CA equipped with a positive definite subbundle V.
Thus, an isomorphism $\Phi \colon E \to E'$ covering a diffeomorphism $\phi \colon M \to M'$ and preserving $\langle , \rangle_E, [,]_E$ and V will yield Sigma-modles over M and M' with the same equations of motion.

Polyakov term and Isomorphisms

We can also add dynamical data to our sigma model: the Polyakov functional, for a Riemannian metric g on M is

$$S_g[f] = \int_\Sigma \|df\|_g^2 d\mu_g.$$

The data of g can be captured in a **generalised metric**; a positive definite subbundle $V \subset E$, i.e. $\langle V, V \rangle > 0$. For example,

$$V = \mathrm{gr}(g) = \{X + g(X) : X \in TM\} \subset TM \oplus T^*M$$

The data of the full action $S = S_H + S_g$ is embedded in a CA equipped with a positive definite subbundle V.

Thus, an isomorphism $\Phi\colon E \to E'$ covering a diffeomorphism $\phi\colon M \to M'$ and preserving $\langle , \rangle_E, [,]_E$ and V will yield Sigma-modles over M and M' with the same equations of motion.

Polyakov term and Isomorphisms

We can also add dynamical data to our sigma model: the Polyakov functional, for a Riemannian metric g on M is

$$S_g[f] = \int_\Sigma \|df\|_g^2 d\mu_g.$$

The data of g can be captured in a **generalised metric**; a positive definite subbundle $V \subset E$, i.e. $\langle V, V\rangle > 0$. For example,

$$V = \mathrm{gr}(g) = \{X + g(X) : X \in TM\} \subset TM \oplus T^*M$$

The data of the full action $S = S_H + S_g$ is embedded in a CA equipped with a positive definite subbundle V.
Thus, an isomorphism $\Phi\colon E \to E'$ covering a diffeomorphism $\phi\colon M \to M'$ and preserving $\langle,\rangle_E, [,]_E$ and V will yield Sigma-modles over M and M' with the same equations of motion.

Courant algebroid relations

T-duality is an equivalence of Sigma-models on different target manifolds M and M', though these are not necessarily diffeomorphic. A CA isomorphisms $\Phi\colon E \to E'$ must cover a diffeomorphism, so we seek to generalise the notion of isomorphism.

Consider the graph $\mathrm{gr}(\Phi) \subset E \times E'$. One sees that

$$[\Phi\cdot, \Phi\cdot]_{E'} = \Phi([\cdot,\cdot]_E) \iff \mathrm{gr}(\Phi) \text{ is involutive in } E \times E'$$
$$\langle\Phi\cdot, \Phi\cdot\rangle_{E'} = \langle\cdot,\cdot\rangle_E \iff \mathrm{gr}(\Phi) \text{ is isotropic in } E \times E'$$

A **CA relation** $R\colon E \dashrightarrow E'$ is an isotropic, involutive subbundle of $E \times E'$, supported on a submanifold $C \subset M \times M'$.

Courant algebroid relations

T-duality is an equivalence of Sigma-models on different target manifolds M and M', though these are not necessarily diffeomorphic. A CA isomorphisms $\Phi\colon E \to E'$ must cover a diffeomorphism, so we seek to generalise the notion of isomorphism.

Consider the graph $\mathrm{gr}(\Phi) \subset E \times E'$. One sees that

$$[\Phi\cdot, \Phi\cdot]_{E'} = \Phi([\cdot,\cdot]_E) \iff \mathrm{gr}(\Phi) \text{ is involutive in } E \times E'$$
$$\langle \Phi\cdot, \Phi\cdot \rangle_{E'} = \langle \cdot,\cdot \rangle_E \iff \mathrm{gr}(\Phi) \text{ is isotropic in } E \times E'$$

A **CA relation** $R\colon E \dashrightarrow E'$ is an isotropic, involutive subbundle of $E \times E'$, supported on a submanifold $C \subset M \times M'$.

Courant algebroid relations

T-duality is an equivalence of Sigma-models on different target manifolds M and M', though these are not necessarily diffeomorphic. A CA isomorphisms $\Phi\colon E \to E'$ must cover a diffeomorphism, so we seek to generalise the notion of isomorphism.
Consider the graph $\mathrm{gr}(\Phi) \subset E \times E'$. One sees that

$$[\Phi\cdot, \Phi\cdot]_{E'} = \Phi([\cdot,\cdot]_E) \iff \mathrm{gr}(\Phi) \text{ is involutive in } E \times E'$$
$$\langle \Phi\cdot, \Phi\cdot\rangle_{E'} = \langle\cdot,\cdot\rangle_E \iff \mathrm{gr}(\Phi) \text{ is isotropic in } E \times E'$$

A **CA relation** $R\colon E \dashrightarrow E'$ is an isotropic, involutive subbundle of $E \times E'$, supported on a submanifold $C \subset M \times M'$.

Courant algebroid relations

T-duality is an equivalence of Sigma-models on different target manifolds M and M', though these are not necessarily diffeomorphic. A CA isomorphisms $\Phi\colon E \to E'$ must cover a diffeomorphism, so we seek to generalise the notion of isomorphism.
Consider the graph $\mathrm{gr}(\Phi) \subset E \times E'$. One sees that

$$[\Phi\cdot, \Phi\cdot]_{E'} = \Phi([\cdot,\cdot]_E) \iff \mathrm{gr}(\Phi) \text{ is involutive in } E \times E'$$
$$\langle\Phi\cdot, \Phi\cdot\rangle_{E'} = \langle\cdot,\cdot\rangle_E \iff \mathrm{gr}(\Phi) \text{ is isotropic in } E \times E'$$

A **CA relation** $R\colon E \dashrightarrow E'$ is an isotropic, involutive subbundle of $E \times E'$, supported on a submanifold $C \subset M \times M'$.

Isometry and Composition

To preserve the structure of a generalised metric $V \subset E$, one sees that V is the $+1$ eigenbundle of a $\tau \in \mathrm{Aut}(E)$. If $R\colon (E, V) \dashrightarrow (E', V')$, then R is a **generalised isometry** if

$$(\tau \times \tau')(R) = R.$$

Contingent on some smoothness conditions, one can compose two relations $R\colon E \dashrightarrow E'$, $\tilde{R}\colon E' \to E''$

$$\tilde{R} \circ R = \{(e, e'') : (e, e') \in R \text{ and } (e', e'') \in \tilde{R}\} \subset E \times \overline{E}'',$$

Isometry and Composition

To preserve the structure of a generalised metric $V \subset E$, one sees that V is the $+1$ eigenbundle of a $\tau \in \mathrm{Aut}(E)$. If $R\colon (E, V) \dashrightarrow (E', V')$, then R is a **generalised isometry** if

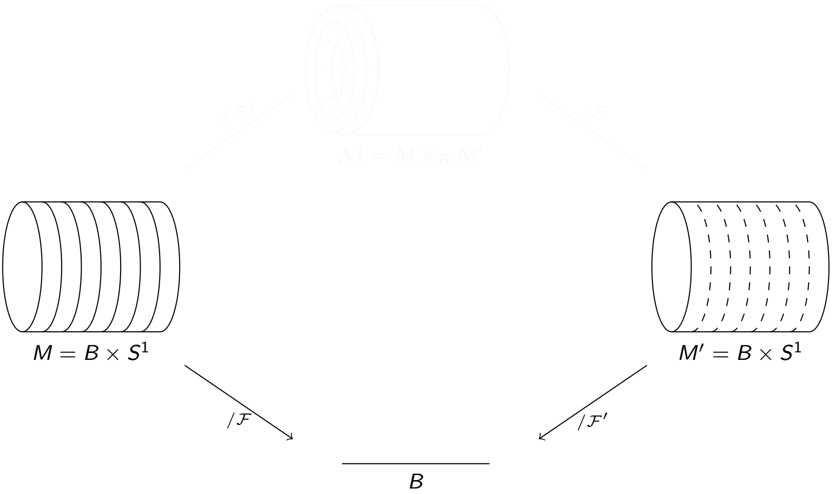
$$(\tau \times \tau')(R) = R.$$

Contingent on some smoothness conditions, one can compose two relations $R\colon E \dashrightarrow E'$, $\tilde{R}\colon E' \to E''$

$$\tilde{R} \circ R = \{(e, e'') : (e, e') \in R \text{ and } (e', e'') \in \tilde{R}\} \subset E \times \overline{E}'',$$

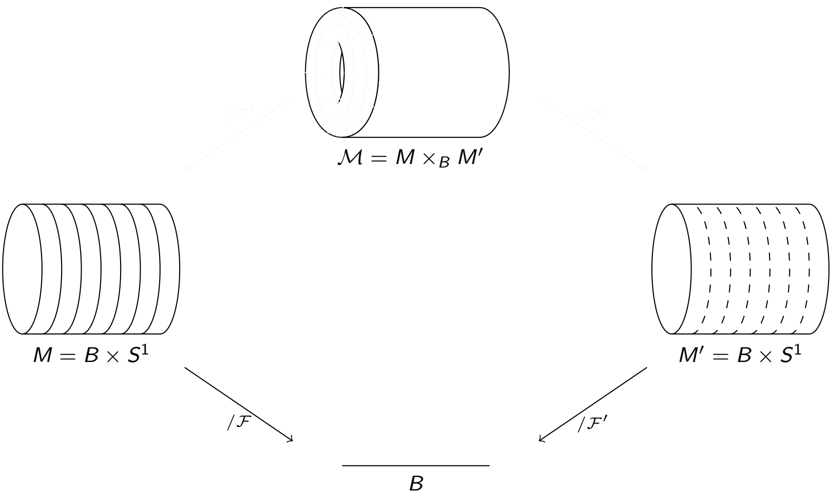
T-duality

T-dual spaces M, M' may be fibre bundles over a common base B



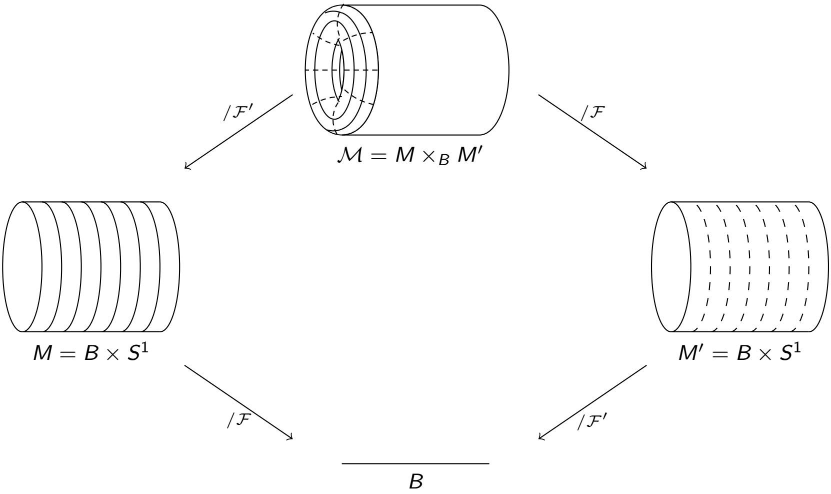
T-duality

T-dual spaces M, M' may be fibre bundles over a common base B



T-duality

T-dual spaces M, M' may be fibre bundles over a common base B



The T-duality relation I

Want to form a CA relation $R\colon E \dashrightarrow E'$ between CAs E, E' over M and M' respectively. Need to know how to form CAs on a quotient manifold:

$\mathcal{M}$ is foliated by $\mathcal{F}'$. If $\mathcal{E}$ is an exact CA over $\mathcal{M}$, then $\mathcal{E}/\mathcal{F}'$ will not be an exact CA over $M = \mathcal{M}/\mathcal{F}'$.

Take $K' = T\mathcal{F}' \subset \mathcal{E}$, then

$$E = \frac{K'^{\perp}}{K'} \Big/ \mathcal{F}'$$

is an exact CA over M.

Can form the CA relation $Q(K')$, supported on $\mathrm{gr}(q')$,

$$Q(K') = \{(e, \natural'(e))\colon e \in K^{\perp}\} \subset \mathcal{E} \times E$$

where $\natural\colon K^{\perp} \to E$ is the quotient map.

The T-duality relation I

Want to form a CA relation $R\colon E \dashrightarrow E'$ between CAs E, E' over M and M' respectively. Need to know how to form CAs on a quotient manifold:

$\mathcal{M}$ is foliated by $\mathcal{F}'$. If $\mathcal{E}$ is an exact CA over $\mathcal{M}$, then $\mathcal{E}/\mathcal{F}'$ will not be an exact CA over $M = \mathcal{M}/\mathcal{F}'$.

Take $K' = T\mathcal{F}' \subset \mathcal{E}$, then

$$E = \frac{K'^{\perp}}{K'} \Big/ \mathcal{F}'$$

is an exact CA over M.

Can form the CA relation $Q(K')$, supported on $\mathrm{gr}(q')$,

$$Q(K') = \{(e, \natural'(e))\colon e \in K^{\perp}\} \subset \mathcal{E} \times E$$

where $\natural\colon K^{\perp} \to E$ is the quotient map.

The T-duality relation I

Want to form a CA relation $R\colon E \dashrightarrow E'$ between CAs E, E' over M and M' respectively. Need to know how to form CAs on a quotient manifold:

$\mathcal{M}$ is foliated by $\mathcal{F}'$. If $\mathcal{E}$ is an exact CA over $\mathcal{M}$, then $\mathcal{E}/\mathcal{F}'$ will not be an exact CA over $M = \mathcal{M}/\mathcal{F}'$.

Take $K' = T\mathcal{F}' \subset \mathcal{E}$, then

$$E = \frac{K'^{\perp}}{K'} \Big/ \mathcal{F}'$$

is an exact CA over M.

Can form the CA relation $Q(K')$, supported on $\mathrm{gr}(q')$,

$$Q(K') = \{(e, \natural'(e)) \colon e \in K^{\perp}\} \subset \mathcal{E} \times E$$

where $\natural\colon K^{\perp} \to E$ is the quotient map.

The T-duality relation I

Want to form a CA relation $R\colon E \dashrightarrow E'$ between CAs E, E' over M and M' respectively. Need to know how to form CAs on a quotient manifold:

$\mathcal{M}$ is foliated by $\mathcal{F}'$. If $\mathcal{E}$ is an exact CA over $\mathcal{M}$, then $\mathcal{E}/\mathcal{F}'$ will not be an exact CA over $M = \mathcal{M}/\mathcal{F}'$.

Take $K' = T\mathcal{F}' \subset \mathcal{E}$, then

$$E = \frac{K'^{\perp}}{K'} \Big/ \mathcal{F}'$$

is an exact CA over M.

Can form the CA relation $Q(K')$, supported on $\mathrm{gr}(q')$,

$$Q(K') = \{(e, \natural'(e))\colon e \in K^{\perp}\} \subset \mathcal{E} \times E$$

where $\natural\colon K^{\perp} \to E$ is the quotient map.

The T-duality relation II

The T-duality relation $R \colon E \to E'$ is then the composition

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad \Phi \quad} & \mathcal{E} \\ {\scriptstyle Q(K')^T} \uparrow & & \downarrow {\scriptstyle Q(K)} \\ E & \xrightarrow[\quad R \quad]{} & E' \end{array}$$

Theorem (DF, Marotta, Szabo [1])

Let V be a generalised metric on E. (Modulo invariance conditions) TFAE

1. $K^\perp \cap \Phi(K') \subseteq K$.
2. *There exists a unique generalised metric V' on E' such that R is a generalised isometry.*

The T-duality relation II

The T-duality relation $R\colon E \to E'$ is then the composition

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\Phi\quad} & \mathcal{E} \\ {\scriptstyle Q(K')^T}\uparrow & & \downarrow{\scriptstyle Q(K)} \\ E & \xrightarrow[\quad R\quad]{} & E' \end{array}$$

Theorem (DF, Marotta, Szabo [1])

Let V be a generalised metric on E. (Modulo invariance conditions) TFAE

1. $K^{\perp} \cap \Phi(K') \subseteq K$.
2. *There exists a unique generalised metric V' on E' such that R is a generalised isometry.*

The T-duality relation II

The T-duality relation $R\colon E \to E'$ is then the composition

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\Phi\quad} & \mathcal{E} \\ {\scriptstyle Q(K')^T}\uparrow & & \downarrow{\scriptstyle Q(K)} \\ E & \xrightarrow[\quad R\quad]{} & E' \end{array}$$

Theorem (DF, Marotta, Szabo [1])

Let V be a generalised metric on E. (Modulo invariance conditions) TFAE

1. $K^{\perp} \cap \Phi(K') \subseteq K$.
2. *There exists a unique generalised metric V' on E' such that R is a generalised isometry.*

The Hopf fibration and its T-dual

Consider $\mathcal{M} = S^3 \times S^1$, with a foliation $\mathcal{F}$ generated by the trivial fibres of S^1, and a foliation $\mathcal{F}'$ with whose leaves are the S^1 fibres of the Hopf fibration $S^3 \to S^2$. Thus $M = \mathcal{M}/\mathcal{F}' = S^2 \times S^1$ and $M' = S^3$. Let $\partial_\theta, \partial_{\theta'}$ generate $\mathcal{F}, \mathcal{F}'$ respectively.

Let $\mathcal{E} = T\mathcal{M} \oplus T^*\mathcal{M}$. To choose $\Phi \in \mathrm{Iso}(\mathcal{E})$, let θ, θ' be connections on $\mathcal{M} \to M'$ and $\mathcal{M} \to M$ respectively. Let $B = \theta \wedge \theta' \in \mathrm{Hom}(TP, T^*P)$. Then

$$\Phi(X + \xi) = X + B(X) + \xi.$$

Consider the element $w\partial_\theta + p\theta \in TM \oplus T^*M$, $w, p \in \mathbb{R}$. The relation R is then given by

$$R = \{(w\partial_\theta + p\theta, p\partial_{\theta'} + w\theta') \colon w, p \in \mathbb{R}\} \oplus TS^2 \oplus T^*S^2.$$

The Hopf fibration and its T-dual

Consider $\mathcal{M} = S^3 \times S^1$, with a foliation $\mathcal{F}$ generated by the trivial fibres of S^1, and a foliation $\mathcal{F}'$ with whose leaves are the S^1 fibres of the Hopf fibration $S^3 \to S^2$. Thus $M = \mathcal{M}/\mathcal{F}' = S^2 \times S^1$ and $M' = S^3$. Let $\partial_\theta, \partial_{\theta'}$ generate $\mathcal{F}, \mathcal{F}'$ respectively.
Let $\mathcal{E} = T\mathcal{M} \oplus T^*\mathcal{M}$. To choose $\Phi \in \mathrm{Iso}(\mathcal{E})$, let θ, θ' be connections on $\mathcal{M} \to M'$ and $\mathcal{M} \to M$ respectively. Let $B = \theta \wedge \theta' \in \mathrm{Hom}(TP, T^*P)$. Then

$$\Phi(X + \xi) = X + B(X) + \xi.$$

Consider the element $w\partial_\theta + p\theta \in TM \oplus T^*M$, $w, p \in \mathbb{R}$. The relation R is then given by

$$R = \{(w\partial_\theta + p\theta, p\partial_{\theta'} + w\theta') \colon w, p \in \mathbb{R}\} \oplus TS^2 \oplus T^*S^2.$$

The Hopf fibration and its T-dual

Consider $\mathcal{M} = S^3 \times S^1$, with a foliation $\mathcal{F}$ generated by the trivial fibres of S^1, and a foliation $\mathcal{F}'$ with whose leaves are the S^1 fibres of the Hopf fibration $S^3 \to S^2$. Thus $M = \mathcal{M}/\mathcal{F}' = S^2 \times S^1$ and $M' = S^3$. Let $\partial_\theta, \partial_{\theta'}$ generate $\mathcal{F}, \mathcal{F}'$ respectively.
Let $\mathcal{E} = T\mathcal{M} \oplus T^*\mathcal{M}$. To choose $\Phi \in \mathrm{Iso}(\mathcal{E})$, let θ, θ' be connections on $\mathcal{M} \to M'$ and $\mathcal{M} \to M$ respectively. Let $B = \theta \wedge \theta' \in \mathrm{Hom}(TP, T^*P)$. Then

$$\Phi(X + \xi) = X + B(X) + \xi.$$

Consider the element $w\partial_\theta + p\theta \in TM \oplus T^*M$, $w, p \in \mathbb{R}$. The relation R is then given by

$$R = \{(w\partial_\theta + p\theta, p\partial_{\theta'} + w\theta') \colon w, p \in \mathbb{R}\} \oplus TS^2 \oplus T^*S^2.$$

Buscher rules and topology

Take a Riemannian metric g on $M = S^2 \times S^1$, that has $g_0\theta\theta$ as the fibre component. Then $V = \mathrm{gr}(g) = \{w\partial_\theta + wg_0\theta\} \oplus \mathrm{gr}(g_{S^2})$.

Thus, the subbundle V' is

$$V' = \{wg_0\partial_{\theta'} + w\theta'\} \oplus \mathrm{gr}(g_{S^2}),$$

which is the graph of the metric $g' = \frac{1}{g_0}\theta'\theta' + g_{S^2}$.

One can also calculate the topological terms: If $H = \mathrm{vol}_{S^2} \wedge \mathrm{vol}_{S^1} \in \Omega^3_{cl}(M)$, then we have, in terms of the full action

$$S_H^{S^2\times S^1}[f] + S_g^{S^2\times S^1}[f] \xleftarrow{\text{T-duality}} S_{g'}^{S^3}[f']$$

Buscher rules and topology

Take a Riemannian metric g on $M = S^2 \times S^1$, that has $g_0 \theta\theta$ as the fibre component. Then $V = \mathrm{gr}(g) = \{w\partial_\theta + wg_0\theta\} \oplus \mathrm{gr}(g_{S^2})$. Thus, the subbundle V' is

$$V' = \{wg_0\partial_{\theta'} + w\theta'\} \oplus \mathrm{gr}(g_{S^2}),$$

which is the graph of the metric $g' = \frac{1}{g_0}\theta'\theta' + g_{S^2}$.

One can also calculate the topological terms: If $H = \mathrm{vol}_{S^2} \wedge \mathrm{vol}_{S^1} \in \Omega^3_{cl}(M)$, then we have, in terms of the full action

$$S_H^{S^2 \times S^1}[f] + S_g^{S^2 \times S^1}[f] \xleftrightarrow{\text{T-duality}} S_{g'}^{S^3}[f']$$

Buscher rules and topology

Take a Riemannian metric g on $M = S^2 \times S^1$, that has $g_0\theta\theta$ as the fibre component. Then $V = \mathrm{gr}(g) = \{w\partial_\theta + wg_0\theta\} \oplus \mathrm{gr}(g_{S^2})$. Thus, the subbundle V' is

$$V' = \{wg_0\partial_{\theta'} + w\theta'\} \oplus \mathrm{gr}(g_{S^2}),$$

which is the graph of the metric $g' = \frac{1}{g_0}\theta'\theta' + g_{S^2}$.
One can also calculate the topological terms: If $H = \mathrm{vol}_{S^2} \wedge \mathrm{vol}_{S^1} \in \Omega^3_{cl}(M)$, then we have, in terms of the full action

$$S_H^{S^2\times S^1}[f] + S_g^{S^2\times S^1}[f] \xleftrightarrow{\text{T-duality}} S_{g'}^{S^3}[f']$$

Outlook

1. Looking at Poisson-Lie T-duality, where fibres $\mathcal{F}$ are possibly non-abelian groups.
2. Add more structure: divergence (or CA connections), allowing to incorporate the dilaton, generalised Ricci tensors and Ricci flow; generalised complex structures, allowing incorporation of branes.

End

Thanks you for listening! Questions?

References: [1] https://arxiv.org/abs/2308.15147