Adiabatic inspirals under electromagnetic radiation reaction on Kerr spacetime.

YTF 23 Durham.

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Contents

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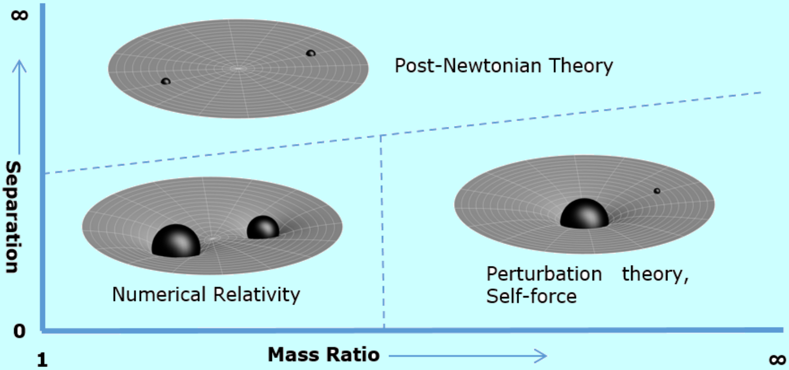
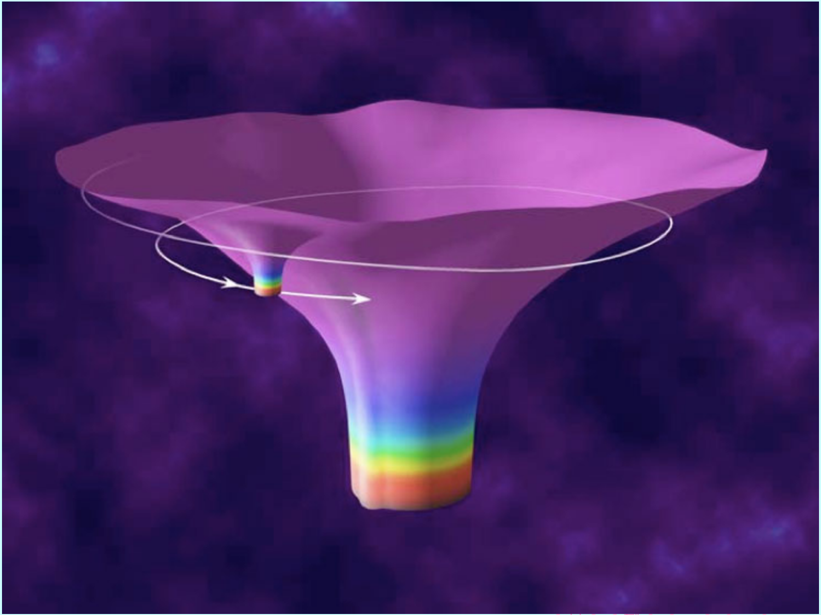
$$g_{\mu\nu} = g^{B}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \mathcal{O}(\varepsilon^3)$$

Binary inspiral modeling



Self-force



Newman Penrose (NP) Formalism of GR

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- For example consider the Maxwell tensor $F_{\mu\nu}$, we define the **Maxwell scalars**:

$$\Phi_{-1} = \phi_0 = F_{\mu\nu} l^\mu m^\nu \quad \Phi_0 = \phi_1 = \frac{1}{2} F_{\mu\nu} \left(l^\mu n^\nu + \bar{m}^\mu m^\nu \right)$$
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- We can also reconstruct the Maxwell tensor from these scalars:

$$F_{\mu v} = 2 \left[\phi_1 \left(n_{[\mu} l_{v]} + m_{[\mu} \bar{m}_{v1]} \right) + \phi_2 l_{[u} m_{v]} + \phi_0 \bar{m}_{[\mu} n_{v]} \right] + \text{c.c.}$$

where "c.c." denotes complex conjugates of previous terms.

Newman Penrose (NP) Formalism of GR

- Similarly useful quantities in the gravitational case are the Weyl Scalars $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$
- These are projections of the Weyl tensor onto the tetrad.
- In the Kerr spacetime the only non-zero Weyl scalar is:

$$\Psi_2 = -\frac{M}{(r - ia\cos\theta)^3}$$

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$$\Psi_2 = -\frac{M}{(r - ia\cos\theta)^3}$$

- We can also reconstruct the metric tensor using the 4 legs of the tetrad:

$$g_{ab} = -l_a n_b - n_a l_b + m_a \bar{m}_b + \bar{m}_a m_b$$

Perturbation theory in NP

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- We perturb each of the four legs of the complex null tetrad such that the inner products of the legs are preserved.
- In turn this means we perturb the relevant scalars (Weyl or Maxwell) directly $\phi_i = \phi_i^B + \phi_i^P$.

The Kerr Metric

The Kerr metric is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\ \mathrm{d}\phi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta\mathrm{d}\phi^2$$

where $\Delta = r^2 - 2Ma^2$ and $\Sigma = r^2 + a^2\cos^2\theta$

Applying this to the Kerr Metric

- For the Kerr Metric we define the Kinnersley Tetrad:

$$l^\mu = [(r^2+a^2)/\Delta, 1, 0, a/\Delta] \quad n^\mu = [R^2+a^2, -\Delta, 0, a]/(2\Sigma)$$

$$m^\mu = \frac{1}{\sqrt{2}(r+ia\cos\theta)}[ia\sin\theta, 0, 1, i/\sin\theta]$$

- **l** and **n** are chosen so that they align with the principal null vectrors of the Kerr Spacetime.

Teukolsky's Master Equation (1974)

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- Dynamical equations for gravitational ($s = \pm 2$), electromagnetic ($s = \pm 1$), and scalar field ($s = 0$) perturbations in the Kerr metric.

$$\left[\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$-2s\left[\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + \left(s^2\cot^2\theta - s\right)\psi = 4\pi\Sigma T$$

Separability in vacuum

Ansatz: $\psi = e^{-i\omega t} e^{im\phi} S(\theta) R(r)$

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Teukolsky Radial Equation:

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0$$

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Spin-Weighted Spheroidal Harmonics:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) +$$
$$\left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right) S = 0$$

Where $K = (r^2 + a^2)\omega$, $\lambda = A + a^2\omega^2 - 2am\omega$ and A is the angular separation constant.

Teukolsky's equations in the Electromagnetic case

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In the electric case the dynamic equations come from Maxwell's equations. Projecting these onto our tetrad we can write the teukolsky master equation in the form:

$$[(\nabla_\mu \pm \Gamma_\mu)(\nabla^\mu \pm \Gamma^\mu) - 4\Psi_2]\Phi_{\pm 1} = 4\pi T_{\pm 1}$$

where:

$$\Gamma^\mu = \frac{1}{\Sigma}\left[\frac{M(r^2 - a^2)}{\Delta} - (r + ia\cos\theta), r - M, 0, \frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]$$

and $T_{\pm 1}$ are linear combinations of projections of the source current J^μ onto the tetrad.

Electrically charged particle in eccentric orbit

- Consider a particle of electric charge q orbiting a black hole in the Kerr Spacetime.
- Such a particle has 4-current:

$$J^\mu = q \int u^\mu \delta^4(x^\nu - x_0^\nu(\tau))\mathrm{d}\tau$$

where $x_0^\nu(\tau)$ is the worldline of the particle with tangent 4-velocity $u^\mu = \mathrm{d}x_0^\mu/\mathrm{d}\tau$

Solutions

- Using greens functions constructed from the two homogeneous solutions of the Teukolsky radial equation, one can solve the sourced equation giving:

$$\phi_0 = \Delta^{-1} \sum_{\ell mn} S_{+1}^{\ell m\gamma}(\theta) e^{-i\omega_{mn}t+im\phi} \begin{cases} \alpha_{+1}^{\infty} P_{+1}^{\infty,\ell m\omega_{mn}}(r), & r \geq r_{\max}, \\ \alpha_{+1}^{h} P_{+1}^{h,\ell m\omega_{mn}}(r), & r \leq r_{\min}, \end{cases}$$

$$2(r - ia\cos\theta)^2 \phi_2 = \Delta^{-1} \sum_{\ell mn} S_{-1}^{\ell m\gamma}(\theta) e^{-i\omega_{mn}t+im\phi} \begin{cases} \alpha_{-1}^{\infty} P_{-1}^{\infty,\ell m\omega_{mn}}(r), & r \geq r_{\max}, \\ \alpha_{-1}^{h} P_{-1}^{h,\ell m\omega_{mn}}(r), & r \leq r_{\min}. \end{cases}$$

where the α's are coefficients calculated by the integral over one orbit of linear combinations of the homogeneous solutions.

Electromagnetic fluxes

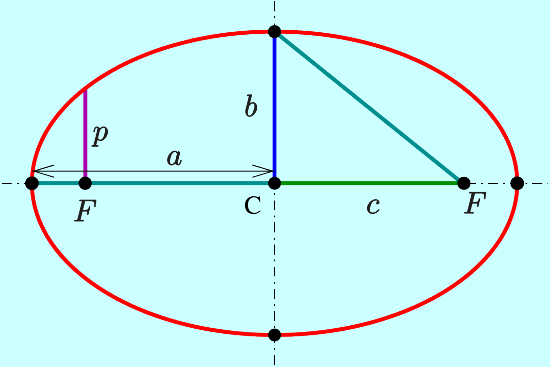
These α's are related to fluxes of energy E and angular momentum L, at infinity and the horizon, given as:

$$\Phi_{\infty}^{(E)} = \frac{1}{8\pi}\sum_{lmn}\left|\alpha_{-1}^{\infty}\right|^2, \qquad \Phi_{\infty}^{(L)} = \frac{1}{8\pi}\sum_{lmn}\frac{m}{\omega}\left|\alpha_{-1}^{\infty}\right|^2,$$

$$\Phi_{h}^{(E)} = \frac{1}{8\pi}\sum_{lmn}\frac{\omega}{2Mr_+\tilde{\omega}}\left|\alpha_{+1}^{h}\right|^2, \quad \Phi_{h}^{(L)} = \frac{1}{8\pi}\sum_{lmn}\frac{m}{2Mr_+\tilde{\omega}}\left|\alpha_{+1}^{h}\right|^2,$$

Orbital parameters for eccentric orbits

a →Semi-Major Axis
b →Semi-Minor Axis
p → Semi-Latus Rectum.
$c = ae$ → Linear eccentricity



Keplarian inspiral Approximations

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- Gravitational inspiral approximation (Peters and Matthews (1963)) :

$$p(e) = c_0 e^{12/19} \left(1 + \frac{121}{304} e^2\right)^{870/2299}$$

Relationships of constants of motion and orbital parameters

$$E = \left[1 - \left(\frac{M}{p}\right)\left(1-e^2\right)\left\{1 - \frac{x^2}{p^2}\left(1-e^2\right)\right\}\right]^{1/2}$$

$$L = x + aE$$

where $x = x(a, p, e)$ is the rather complicated function given as:

$$x = \left[\frac{-N - \text{sign}(a)\sqrt{N^2 - 4FC}}{2F}\right]^{1/2},$$

with

$$F(p,e) = \frac{1}{p^3}\left[p^3 - 2M\left(3+e^2\right)p^2 + M^2\left(3+e^2\right)^2 p - 4Ma^2\left(1-e^2\right)^2\right]$$

$$N(p,e) = \frac{2}{p}\left\{-Mp^2 + \left[M^2\left(3+e^2\right) - a^2\right]p - Ma^2\left(1+3e^2\right)\right\}$$

$$C(p) = \left(a^2 - Mp\right)^2$$

Adiabatic Inspiral Model

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- Thus by the chain rule we can write:

$$\dot{E} = \frac{\partial E}{\partial p}\dot{p} + \frac{\partial E}{\partial e}\dot{e}$$
$$\dot{L} = \frac{\partial L}{\partial p}\dot{p} + \frac{\partial L}{\partial e}\dot{e}$$

Flux Balancing

- As the particle moves it generates energy and angular momentum flux.
- This flux is lost from the system so we have that:

$$-\dot{E} = \Phi_E^{\infty}(p, e) + \Phi_E^{h}(p, e)$$
$$-\dot{L} = \Phi_L^{\infty}(p, e) + \Phi_L^{h}(p, e)$$

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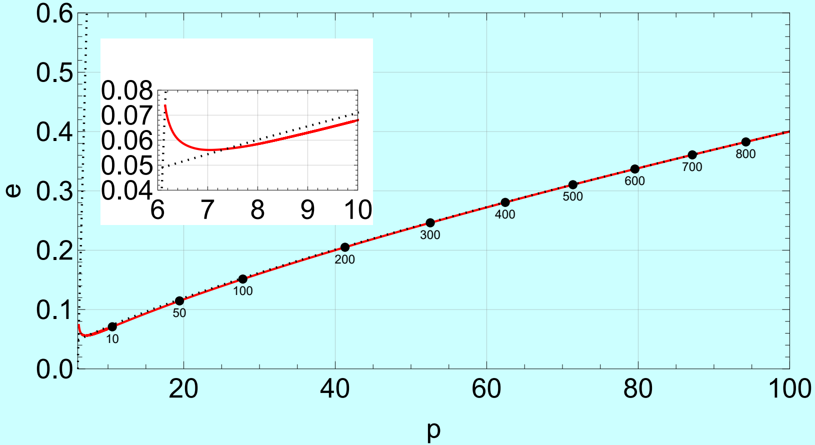
- Inverting:

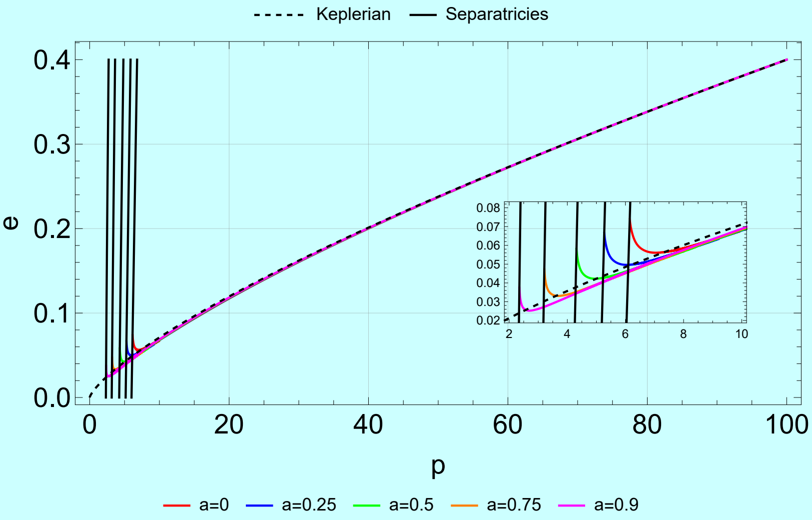
$$\begin{pmatrix} \dot{p} \\ \dot{e} \end{pmatrix} = \frac{1}{\frac{\partial E}{\partial p}\frac{\partial L}{\partial e} - \frac{\partial L}{\partial p}\frac{\partial E}{\partial e}} \begin{pmatrix} \frac{\partial L}{\partial e} & -\frac{\partial E}{\partial e} \\ -\frac{\partial L}{\partial p} & \frac{\partial E}{\partial p} \end{pmatrix} \begin{pmatrix} \dot{E} \\ \dot{L} \end{pmatrix}$$

Results



0.6
0.5
0.4
0.3
0.2
0.1
0.0
e
20
40
60
80
100
p
10
50
100
200
300
400
500
600
700
800
0.08
0.07
0.06
0.05
0.04
6
7
8
9
10

Results



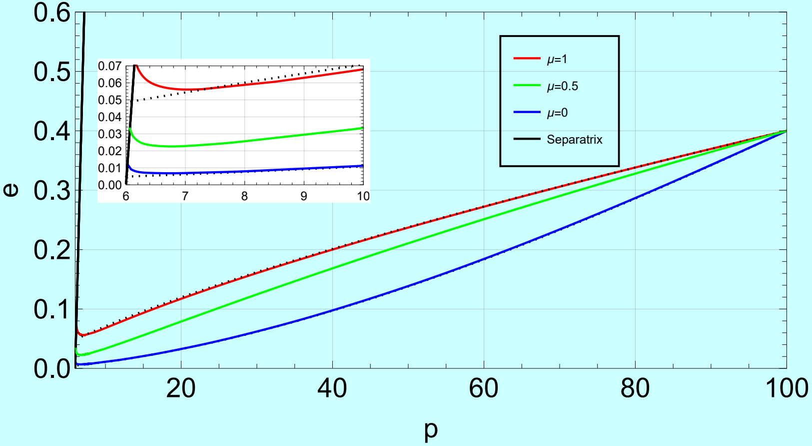
Keplerian
Separatricies
0.4
0.3
0.2
0.1
0.0
e
0
20
40
60
80
100
p
0.08
0.07
0.06
0.05
0.04
0.03
0.02
2
4
6
8
10
a=0
a=0.25
a=0.5
a=0.75
a=0.9

Mixed Flux Inspirals

$$\Phi = \mu \Phi_{EM} + (1 - \mu)\Phi_G$$

where $\mu \in [0, 1]$.

Results



0.6
0.5
0.4
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0.2
0.1
0.0
e
20
40
60
80
100
p
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
6
7
8
9
10
μ=1
μ=0.5
μ=0
Separatrix

Conclusions

- Graviationally driven inspirals circularize more efficiently than Electromagnetically inspirals
- The eccentricty of the orbit increases slightly before plunge.

Thank You!

Chebyshev Interpolation

$$f(x, y) \approx \sum_{i,j} f^{ij} T_i(u(x)) T_j(v(y))$$

where f^{ij} are the chebyshev coefficients, of the function, $u(x)$ and $v(y)$ map the domain of $f(x, y)$ to $[-1, 1]^2$, and $T_n(\cos\theta) = \cos(n\theta)$ are the Chebychev polynomials.

Chebychev Nodes