Null energy bounds for non-minimally coupled scalar fields

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Pointwise restrictions imposed on the stress-energy tensor in order to encode physically reasonable constraints on the energy density

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Null energy condition (NEC):

- $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ with ℓ^μ null vector
- Perfect fluid: $\rho + P \geq 0$
- Minimal coupling to gravity for free massive scalar field ϕ, mass $m \geq 0$

$$
\begin{aligned}
S &= \int d^n x\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G_N} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right] \\
T_{\mu\nu} &= (\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}(m^2\phi^2 + (\nabla\phi)^2) \\
\rho_n &\equiv T_{\mu\nu}\ell^\mu\ell^\nu = (\ell^\mu\nabla_\mu\phi)(\ell^\nu\nabla_\nu\phi)
\end{aligned}
$$

  – NEC is obeyed.

Energy conditions (EC)

Pointwise restrictions imposed on the stress-energy tensor in order to encode physically reasonable constraints on the energy density

Null energy condition (NEC):

- Non-minimal coupling (NMC) to gravity for a free massive scalar field ϕ, mass $m \geq 0$

$$S = \int d^n x \sqrt{-g} \left[\frac{(R-2\Lambda)}{16\pi G_N} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\xi R\phi^2 - \frac{1}{2}m^2\phi^2 \right]$$

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}(m^2\phi^2 + (\nabla\phi)^2) + \xi(-g_{\mu\nu}\Box_g - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2$$

$$\rho_n \equiv T_{\mu\nu}\ell^\mu\ell^\nu = (1-2\xi)(\ell^\mu\nabla_\mu\phi)(\ell^\nu\nabla_\nu\phi) - 2\xi \left(\phi(\ell^\mu\ell^\nu\nabla_\mu\nabla_\nu\phi) + \frac{1}{2}R_{\mu\nu}\ell^\mu\ell^\nu\phi^2 \right)$$

  - ξ is a dimensionless coupling to gravity
  - NEC is violated even for $R_{\mu\nu} = 0$

Quantum energy inequalities (QEIs)

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- All pointwise energy conditions are violated in the context of quantum field theory [Epstein, Glaser, Jaffe, 1965]
- Quantum fields satisfy QEIs: lower bounds on weighted averages of components of the expectation value of the stress-energy tensor
- QEIs have been proved for free theories in Minkowski and curved spacetimes
- Timelike average energy density for massless scalar field minimally coupled in Minkowski spacetime

$$\int dt \langle : T_{\mu\nu} : \ell^\mu \ell^\nu \rangle_\omega f^2(t) \geq -\frac{1}{12\pi^2} \int dt f''(t)^2$$

For normalized Gaussian

$$\frac{1}{t_0} \int dt \langle : T_{\mu\nu} : \ell^\mu \ell^\nu \rangle_\omega f^2(t/t_0) \geq -\frac{1}{64\pi^2 t_0^4}$$

Goals

Can non-minimally coupled theories violate the usual laws of physics?

- Review of the classical theory. Can non-minimal coupling (NMC) lead to exotic spacetimes?
- Analisis in Jordan and Einstein frames
- QEIs for NMC theories
- Can we consider NMC as the first term in an effective field theory (EFT)?

Contents

ANEC and effective ANEC

Average null energy condition (ANEC)

$$\int \rho_n d\lambda = \int (\ell^\mu \nabla_\mu \phi)(\ell^\nu \nabla_\nu \phi) d\lambda - \xi \int \ell^\mu \ell^\nu \nabla_\mu \nabla_\nu (\phi^2) d\lambda$$

- Obeyed by minimally ($\xi = 0$) and NMC massive free scalar fields for $R_{\mu\nu} = 0$

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Effective ANEC

We define an effective stress-energy tensor by separating the curvature terms from the field terms in the Einstein equation

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}(\phi, G_{\mu\nu}(g_{\mu\nu}), g_{\mu\nu}) \rightarrow G_{\mu\nu} = 8\pi G_N T^{\text{eff}}_{\mu\nu}$$

where

$$\begin{aligned} T^{\text{eff}}_{\mu\nu} &= \frac{1}{1 - 8\pi G_N \xi \phi^2} \left((\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} \left[m^2 \phi^2 + (\nabla \phi)^2 \right.\right. \\ &+ \left.\left. \frac{\Lambda}{4\pi G_N} \right] + \xi(-g_{\mu\nu} \Box_g - \nabla_\mu \nabla_\nu) \phi^2 \right) \end{aligned}$$

ANEC and effective ANEC

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Effective ANEC

$$\int_\gamma \rho_n^{\text{eff}} d\lambda = \int_\gamma d\lambda \frac{1 - 8\pi\xi G_N (1-4\xi)\phi^2}{(1 - 8\pi G_N \xi \phi^2)^2} \left(\frac{d\phi}{d\lambda}\right)^2$$

- Non-negative for $\xi < 0$ and $\xi > 1/4$
- Negative for $0 < \xi < 1/4$ and large field $8\pi\xi(1–4\xi)G_N\phi^2>1$

Einstein and Jordan frames

- Transform the action of a non-minimally coupled scalar field (*Jordan frame*, JF) into the minimally coupled one (*Einstein frame*, EF) by a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and field redefinition $\tilde{\phi} = F(\phi)$.

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- The minimally coupled action is

$$S = \int d^n x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} + \frac{1}{2}(\tilde{\nabla}\tilde{\phi})^2 - \tilde{V}(\tilde{\phi}) \right] , \qquad \text{with}$$

$$\begin{aligned} \Omega &= (1 - 8\pi G\xi\phi^2)^{1/(n-2)} \\ \tilde{V}(\tilde{\phi}) &= \Omega^{-n} \left(\frac{\Lambda}{8\pi G_N} + V(\phi) \right) , \quad \text{where} \quad \phi = F^{-1}(\tilde{\phi}) \end{aligned}$$

Einstein and Jordan frames

- The stress tensor in the Einstein frame

$$T_{\mu\nu} = (\tilde{\nabla}_\mu \tilde{\phi})(\tilde{\nabla}_\nu \tilde{\phi}) - \frac{1}{2}\tilde{g}_{\mu\nu}(2\tilde{V}(\tilde{\phi}) - (\tilde{\nabla}\tilde{\phi})^2)$$

  with null energy

$$\tilde{\rho}_n = (\ell^\mu \tilde{\nabla}_\mu \tilde{\phi})(\ell^\nu \tilde{\nabla}_\nu \tilde{\phi})$$

- NEC is obeyed in the EF but not in the JF.
- The two frames are not equivalent in terms of the classical EC.

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- NEC is obeyed in the EF but not in the JF.
- The two frames are not equivalent in terms of the classical EC.
- Physically relevant question: Can NMC lead to exotic spacetime geometries?
  - Transversable wormholes require ANEC violation. Only possible for large field values.Unphysical for NMC as EFT.
- JF and EF can be considered equivalent in this sense.

Contents

Null QEIs for minimal coupling

We would like to prove a QEIs over a null geodesic.

$$\int d\lambda \langle : T_{\mu\nu} : \ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -A \int d\lambda f'(\lambda)^2$$

[Fewster, Roman, 2002]: The null energy averaged over a null geodesic can become arbitrarily negative.

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[Fewster, Roman, 2002]: The null energy averaged over a null geodesic can become arbitrarily negative.
Idea Introduce an ultraviolet cutoff ℓ_{UV} which restricts the three-momenta

Smeared null energy condition (SNEC)

- SNEC for the minimally coupled scalar field in 4d Minkowski spacetime [Freivogel, Krommydas, 2018]

$$\int d\lambda \langle : T_{\mu\nu} : \ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -\frac{4B}{\ell_{UV}^2} ||f'||^2$$

Null QEIs for minimal coupling

Double smeared null energy condition (DSNEC)

- Smear over two null directions $x^{\pm}$: test function supported on $\delta^{\pm}$, [Fliss, Freivogel, Kontou, 2021].
- DSNEC for minimally coupled scalar field

$$\int d^2x^{\pm} f^2(x^{\pm}) \langle T_{--} \rangle_\omega \geq -\frac{\mathcal{N}}{(\delta^+)^{n/2-1}(\delta^-)^{n/2+1}}$$

DSNEC for non-minimal coupling

- Bound for a massless scalar field in n-dimensional Minkowski spacetime.

$$\int d^2x^{\pm} f(x^{\pm})^2 \langle :T_{--}: \rangle_\psi \geq -P_n \left(\int dx^+ (f_+^{(n/2)}(x^+))^2 \right)^{\frac{n-2}{2n}} \left(\int dx^- (f_-^{(n/2)}(x^-))^2 \right)^{\frac{n+2}{2n}} - |\xi| \int d^2x^{\pm} \langle : \phi^2 : \rangle_\psi \partial_-^2 (f(x^{\pm})^2)$$

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- ϕ_{max} is a finite constant such that $|\langle :\phi^2: \rangle_\psi| \leq \phi^2_{\text{max}}$
- State-dependent QEI
- Bound for general ξ
- Violation of the classical NEC results in state-dependent bound

DSNEC for non-minimal coupling

- ANEC from DSNEC: Take the limit $\delta^+ \to 0$ and $\delta^- \to \infty$ while $\delta^+\delta^- \equiv \alpha^2$ fixed

$$\int_{-\infty}^{\infty} dx^- \langle T_{--}(x^-)\rangle_\omega \geq 0$$

DSNEC for non-minimal coupling

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$$\int_{-\infty}^{\infty} dx^- \langle T_{--}(x^-)\rangle_\omega \geq 0$$

- SNEC from DSNEC: We impose $\delta^+ \to 0$ while $\delta^+\delta^- \to \ell_{\text{UV}}^2$

$$\int dx^- f_-(x^-)^2 \langle T_{--}\rangle_\psi \geq -\frac{p_n}{\ell_{\text{UV}}^{n-2}} \int dx^- (\partial_- f(x^-))^2 - \frac{|\xi|\tilde{\phi}_{\max}^2}{\ell_{\text{UV}}^{n-2}} \int dx^- \left|(\partial_-^2(f(x^-)^2))\right|$$

- Application to prove singularity and area theorems with NMC theory, [Freivogel, Kontou, Krommydas, 2022], [Kontou, Sacchi, 2023]

Contents

The Einstein frame

- Reminder: classically, the Einstein frame stress tensor is

$$\tilde{T}_{\mu\nu}^{\text{classical}} = (\tilde{\nabla}_\mu \tilde{\phi})(\tilde{\nabla}_\nu \tilde{\phi}) - \frac{1}{2}\tilde{g}_{\mu\nu}(2\tilde{V}(\tilde{\phi}) + (\tilde{\nabla}\tilde{\phi})^2)$$

with an effective potential

$$\tilde{V}(\tilde{\phi}) = \left(1 - 8\pi G_N\,\xi\,\phi^2\right)^{\frac{n}{2-n}} \left(\frac{\Lambda}{8\pi G_N} + V(\phi)\right) \;, \quad \phi = F^{-1}(\tilde{\phi})$$

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- For small free scalar field ϕ, we do a power series expansion

$$\tilde{\phi} = \phi \left(1 + \frac{1}{6} \left(1 + \frac{\xi}{\xi_c} \right) (8\pi G_N \xi\, \phi^2) + \dots \right)$$

leading to an effective potential, for n=4

$$\tilde{V}(\tilde{\phi}) = \frac{\Lambda}{8\pi G_N} + \frac{1}{2}(m^2 + 4\xi\Lambda)\tilde{\phi}^2 + \frac{1}{6}\left(m^2 \left(5 - \frac{\xi}{\xi_c} \right) + 2\Lambda\xi \left(7 - 2\frac{\xi}{\xi_c} \right) \right) (8\pi G_N \xi)\tilde{\phi}^4 + \dots$$

The Einstein frame

Perturbative expansion in $8\pi G_N \xi \tilde{\phi}^2$

- Massive theory with quartic interaction $\frac{\lambda}{4!}\tilde{\phi}^4$:

$$\begin{aligned} m_{\text{eff}}^2 &= m^2 + 4\xi\Lambda \\ \lambda &= 4\left(m^2\left(5 - \frac{\xi}{\xi_c}\right) + 2\Lambda\xi\left(7 - 2\frac{\xi}{\xi_c}\right)\right)(8\pi G_N \xi) \end{aligned}$$

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$$

- [Bostelmann, Cadamuro, Fewster, 2013], [Kontou, Sanders, 2020] : Bosonic free theories that obey classcial EC$\rightarrow$ QEI with state-independent bound
  - Classical theory in Einstein frame obeys the NEC, but it is self-interacting. Not state-independent QEI expected

The Einstein frame. Quantum corrections

Euclidean path integral of matter coupled to gravity

$$Z_{\text{grav+matter}} = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\, e^{-I_\xi[\phi,g,V]}$$

where

$$I_\xi[\phi,g,V] = \int d^n x\sqrt{g}\left(-\frac{R-2\Lambda}{16\pi G_N} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\xi}{2}R\,\phi^2 + V(\phi)\right)$$

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- Change of path-integral variables $(g_{\mu\nu},\phi) \to (\tilde{g}_{\mu\nu},\tilde{\phi})$, such that $(\xi \to \tilde{\xi} = 0)$ and a new $\tilde{V}$:

$$Z_{\text{grav+matter}} = \int D\tilde{g}_{\mu\nu}\mathcal{D}\tilde{\phi}\, e^{-I_0[\tilde{\phi},\tilde{g},\tilde{V}]+\log J[\tilde{\phi},\tilde{g}]} \text{ where } J[\tilde{\phi},\tilde{g}] = \det\frac{\delta g_{\mu\nu}}{\delta\tilde{g}_{\mu\nu}}\det\frac{\delta\phi}{\delta\tilde{\phi}}$$

- In general, we expect J to introduce all possible irrelevant couplings, controlled by $M_{\text{cutoff}}^{-1} \sim \phi_{\text{max}}^{2/(2-n)}$ (dim. analysis).
- By assuming field values bounded by $|8\pi G_N\xi\phi^2| \ll 1$, we get EFT that allows mappings from JF to EF, with modified $\tilde{V}(\tilde{\phi})$

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- We have proved ANEC from DSNEC
- Transformation to EF leads to self-interacting fields, i.e. state-dependent bounds for QEIs.
- The EFT remains valid when the irrelevant interactions are suppressed by M_{cutoff}^{-1}.

Thank you for your attention!

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