# The Semi-Leptonic Weak Hamiltonian: Going Beyond Two-Loops

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This Ken is a 1st year PhD


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# Outline

# Introduction

## Motivations

- Measurements of $K_{\ell 3}$ decays [Estrada Tristan, 2019] provide a tool for the extraction of $|V_{us}|$ [Bazavov *et al.*, 2019]

$$\Gamma(K^0 \to \pi^- \ell^+ \nu_\ell(\gamma)) = \frac{G_F^2 m_K^5}{128\pi^3} |V_{us}|^2 \, S_{EW} \; |f_+^{K^0\pi^-}(0)|^2 \; I_{K^0\ell}^{(0)} \left(1 + \delta_{EM}^{K^0\ell} + \delta_{SU(2)}^{K^0\pi^-}\right).$$

Short-distance

Hadronic contribution

$$S_{EW} = 1 + \frac{\alpha}{\pi} \ln\left(\frac{M_Z}{M_W}\right)$$

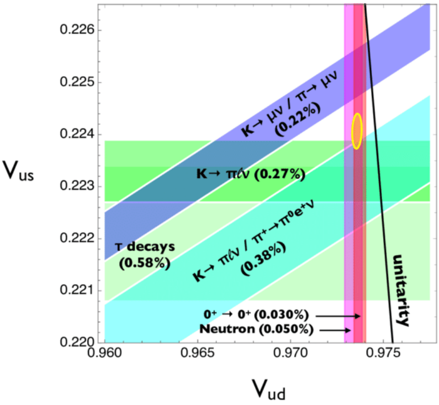
- The combination of $|V_{us}|$ and $|V_{ud}|$, extracted from nuclear $\beta$ decays [Hardy & Towner, 2020], test CKM unitarity in the first row $\Delta_{CKM} \equiv 1 - |V_{ud}|^2 - |V_{us}|^2 - \mathcal{O}(|V_{ub}|^2) = 0$.

# Introduction

## State of the art

### $V_{ud}$ from neutron decay

- $V_{ud} = 0.97430[88]_{total}$ [Particle Date Group, 2022]
- $V_{ud} = 0.97402[42]_{total}$ [Cirigliano *et al.*, 2023]




[Bryman *et al.*, 2021]

# Introduction

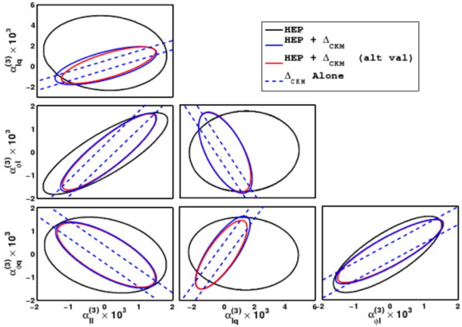
## BSM/SMEFT

### BSM Constraints from CKM Unitarity

- $SU(2)_W \times U(1)_Y$ invariant effective operators in Minimal Flavour Violation [Cirigliano *et al.*, 2010]

$$O^{(3)}_{\ell\ell} = \frac{1}{2}(\bar{\ell}\gamma^\mu\sigma^a\ell)(\bar{\ell}\gamma_\mu\sigma^a\ell), \quad O^{(3)}_{\ell q} = \frac{1}{2}(\bar{\ell}\gamma^\mu\sigma^a\ell)(\bar{q}\gamma_\mu\sigma^a q).$$

- $\Delta_{CKM} = 4\left(\hat{\alpha}^{(3)}_{\ell\ell} - \hat{\alpha}^{(3)}_{\ell q} - \hat{\alpha}^{(3)}_{\phi\ell} + \hat{\alpha}^{(3)}_{\phi q}\right)$



# Introduction

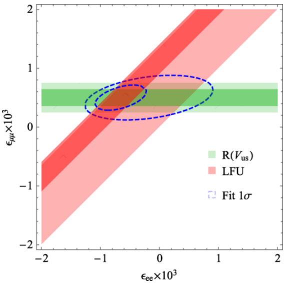
## BSM/SMEFT

### BSM Contribution to CKM

- In a recent work [Crivellin & Hoferichter, 2020], a modified $W - \ell - \nu$ coupling has been studied to explain LFUV;
- Modified $W - \ell - \nu$ coupling has been studied to explain LFUV [Crivellin & Hoferichter, 2020]

$$\mathcal{L} \supset -i\frac{g_2}{\sqrt{2}}\bar{\ell}_i\gamma^\mu P_L \nu_j W_\mu(\delta_{ij} + \epsilon_{ij}).$$

- $G_F^{BSM} = G_F^{SM}(1 + \epsilon_{ee} + \epsilon_{\mu\mu})$ and $V_{ud}^{\beta,BSM} = V_{ud}^{\beta,SM}(1 - \epsilon_{\mu\mu})$.



# Outline

# Hadronic Matrix Elements

- $\mathrm{O_{sem}(x)} = \bar{\mathrm{d}}(\mathrm{x})\gamma^\mu \mathrm{P_L u(x)} \otimes \bar{\nu}_\ell(\mathrm{x})\gamma_\mu \mathrm{P_L}\ell(\mathrm{x}), \quad \mathrm{P_L} = (1-\gamma^5)/2$

$$\langle \pi(p)|\bar{\mathrm{d}}\gamma^\mu \mathrm{P_L u}|K(p')\rangle = f_+^{K\pi}(q^2)(p+p')^\mu + f_-^{K\pi}(q^2)(p-p')^\mu$$

## Evaluation of long-distance contribution

- $\chi PT$ [Cirigliano *et al.*, 2023],[Seng *et al.*, 2020]
- Lattice QCD [Di Carlo *et al.*, 2019],[Carrasco *et al.*, 2015]

## Lattice QCD

- QED corrections $\rightarrow$ Lattice renormalisation;
- We proposed a new scheme [Gorbahn, Jäger, Moretti, v. d. Merwe]

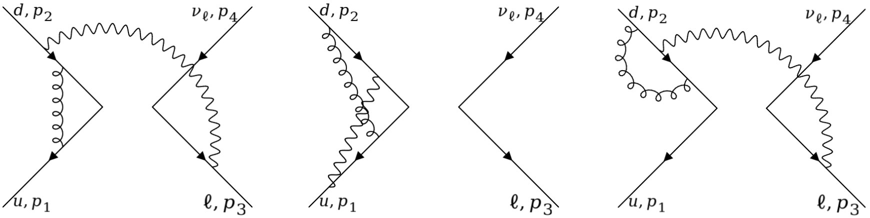
$$\Downarrow$$

Cancellation of extraneous pure QCD corrections;

- two-loop $\mathcal{O}(\alpha\alpha_s)$ scheme changing onto the $\overline{\mathrm{MS}}$;

# Hadronic Matrix Elements

## Lattice Renormalisation




### Kinematic Configurations

- MOM [Martinelli *et al.*, 1995]

$$p_1 = p_2 = p_3 = p_4 = p, \quad p^2 = -\mu^2;$$

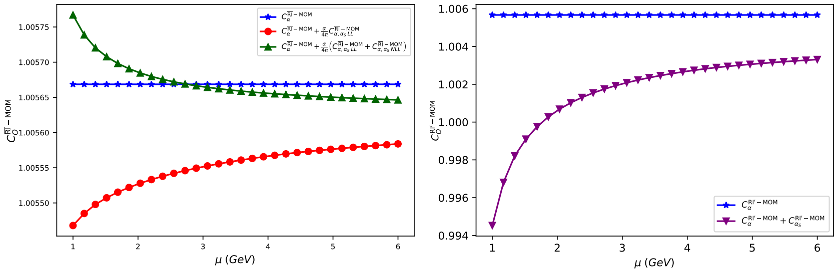
- SMOM [Sturm *et al.*, 2009]

$$p_1 = p_3, \quad p_2 = p_4, \quad p_1^2 = p_2^2 = -\mu^2, \quad p_1 \cdot p_2 = -\frac{1}{2}\mu^2.$$

# Hadronic Matrix Elements

## Low-Scale Matching onto $\overline{\mathrm{MS}}$

### Wilson Coefficient in RI schemes




## $\overline{\mathrm{RI}} - \mathrm{MOM}$ vs $\mathrm{RI}' - \mathrm{MOM}$

- Cancellation of artificial running at $\mathcal{O}(\alpha_s)$;
- Residual scale dependence suppressed by $\mathcal{O}(\alpha)$;
- Similar results for SMOM kinematics.

# Outline

# Short-Distance Contribution

$\overline{\text{MS}}$

## EFT approach in the $\overline{\text{MS}}$

- Clear scale separation;

$$\Downarrow$$

- High-scale matching onto the Standard Model $\to$ Wilson Coefficient $C_O$;
- $C_O = 1 + \frac{\alpha}{4\pi} C_O^e + \frac{\alpha}{4\pi}\frac{\alpha_s}{4\pi} \underbrace{C_O^{es}}_{\text{New result}}$
- Resummation of large logarithms via RGE solutions

$$\mu \frac{d}{d\mu} C_O = \gamma_{OO}\, C_O$$

- $\gamma_{OO} = \overbrace{\frac{\alpha}{4\pi}\gamma_{OO}^{e} + \frac{\alpha}{4\pi}\frac{\alpha_s}{4\pi}\gamma_{OO}^{es}}^{\text{[Gorbahn, Jäger, Moretti, v. d. Merwe]}} + \underbrace{\left(\frac{\alpha}{4\pi}\right)^2 \gamma_{OO}^{ee}}_{\text{[Cirigliano } et.\ al\text{, 2023]}} + \frac{\alpha}{4\pi}\left(\frac{\alpha_s}{4\pi}\right)^2 \underbrace{\gamma_{OO}^{ess}}_{\text{New result}}\ .$

# Short-Distance Contribution

## High-Scale Matching

### $\overline{\text{MS}}$ Wilson Coefficient at $\mathcal{O}(\alpha\alpha_s)$

- $C_O^{es} =$
$$\left(3C_F\left(\ln\left(\frac{\mu}{M_Z}\right) - \csc^2(\theta_W)\left(\csc^2(\theta_W)\ln\left(\frac{M_W}{M_Z}\right) - \ln\left(\frac{M_W}{M_Z}\right) + 1\right)\right) + \frac{95C_F}{24}\right)$$

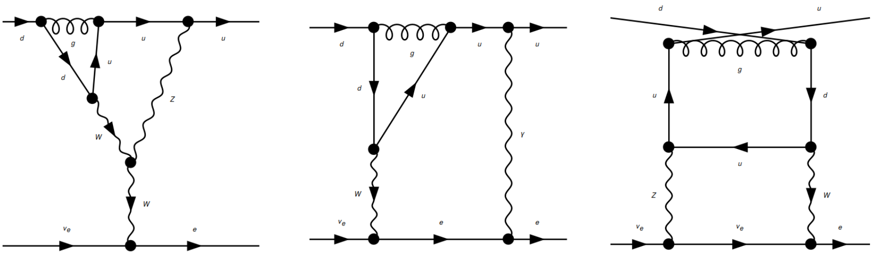



Figure: Some examples of the two loop diagrams calculated.

# Short-Distance Contribution

## 3-Loop Anomalous Dimension

### Low-scale Wilson Coefficient from RGE

- $\gamma_O^{ess} = \dfrac{33 - 26\, N_f}{27}$ (Preliminary).
- $C_O^{\overline{\text{MS}}}(m_c) = 1.00794$ (Preliminary);
- $C_O^{\overline{\text{MS}}}(m_c) = 1.00754$ [Cirigliano *et al.*, 2023];
- 0.04% impact on recent results;
- Compatible with current theoretical uncertainty.

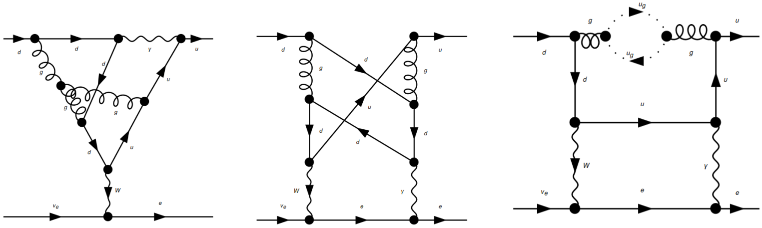


Figure: Some examples of the three loop diagrams calculated.

# Outline

# Summary & Future Outlooks

## Summary

- Improved scheme for Lattice evaluation of Hadronic Matrix Elements;
- Derivation of the $\mathcal{O}(\alpha\alpha_s)$ high-scale matching;
- Evaluation of the $\mathcal{O}\left(\alpha\alpha_s^2\right)$ ADM;
- 0.04% impact compatible with current uncertainty.

## Future Outlooks

- Evaluation of the two-loop $\mathcal{O}\left(\alpha^2\right)$ EW corrections to the Wilson Coefficient;
- New analysis & extraction of $V_{ud}$.

# Summary & Future Outlooks

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**Thank You!**

# Backup Slides

# Lattice Renormalisation

- The RI schemes are defined by imposing the off-shell renormalization conditions on the projected Green's functions

$$\sigma^A \equiv \frac{1}{4\,p^2}\mathrm{Tr}(S_A^{-1}(p)\not{p}) \overset{A=RI}{=} 1, \quad \lambda^A \equiv \Lambda^A_{\alpha\beta\gamma\delta}\mathcal{P}^{\alpha\beta\gamma\delta} \overset{A=RI}{=} 1.$$

$\mathcal{P}$ is a constant Dirac tensor satisfying $\Lambda^{(\mathrm{tree})}_{\alpha\beta\gamma\delta}\mathcal{P}^{\alpha\beta\gamma\delta} = 1$.

- We define the scheme conversion factors as

$$\mathcal{C}_f^{\overline{\mathrm{MS}}\to RI} = \left(\sigma^{\overline{\mathrm{MS}}}\right)^{-1/2}, \quad \mathcal{C}_O^{\overline{\mathrm{MS}}\to RI} = \lambda^{\overline{\mathrm{MS}}}\left(\sigma_u^{\overline{\mathrm{MS}}}\sigma_d^{\overline{\mathrm{MS}}}\sigma_\ell^{\overline{\mathrm{MS}}}\right)^{1/2}.$$

## Choice of Projector

- Crucial role of $\mathcal{P}$ $\to$What is a "good" projector?
- Conventionally [Garron, 2018], $\mathcal{P} = -\frac{1}{16}\left(\gamma^\mu P_R \otimes \gamma_\mu P_R\right)^{\alpha\beta\gamma\delta}$.
- Ward Identity "violation" $\to$ scale dependence of the conversion factor already in pure QCD.

# Lattice Renormalisation

## Statement Of The Problem

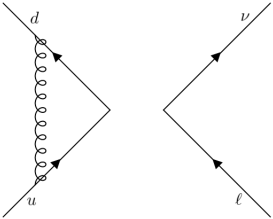



Figure: One-loop pure QCD correction.

- Neglecting QED $\rightarrow \Lambda^b = \Lambda^{b,\mu}(p) \otimes \gamma_\mu P_L + \mathcal{O}(\alpha)$, where

$$\Lambda^{b,\mu}(p) = F_1(p)\gamma^\mu P_L + F_2(p)\frac{p^\mu \not{p}}{p^2} P_L.$$

Scalar Form Factors

Lorentz Structures

- Conserved current $\rightarrow \underbrace{\Lambda^{b,\mu}(p) = \frac{\partial}{\partial p_\mu} S^b(\not{p})^{-1}}_{F_1(p) = S^{-1}(p^2)}$ (Ward Identity)

# Lattice Renormalisation

## Statement Of The Problem

### $\overline{\text{MS}}$

- Ward Identity holds in NDR after minimal subtraction;
- Cancellation of loop corrections against the field renormalisation;
- $Z_{OO}^{\overline{\text{MS}}} = 1 + \mathcal{O}(\alpha)$.

### RI

- Extraneous contribution not matched by field renormalisation;
- $Z_{OO}^{\text{RI}} = 1 + \mathcal{O}(\alpha_s)$;
- Artificial scale dependence dominant at low scales.

# Lattice Renormalisation

## Alternative Scheme MOM

- $\overline{\mathrm{RI}}$ scheme defined via Ward Identity;
- Imposing $\begin{cases} \mathcal{P}(\gamma^\mu P_L \otimes \gamma_\mu P_L) = 1 \\ \mathcal{P}(\frac{p^\mu \not{p}}{p^2} P_L \otimes \gamma_\mu P_L) = 0 \end{cases}$

$$\Downarrow$$

$$\mathcal{P}^{\overline{\mathrm{RI}}-\mathrm{MOM}} = -\frac{1}{12\, p^2}\left(\not{p} P_R \otimes \not{p} P_R + \frac{p^2}{2} \gamma^\nu P_R \otimes \gamma_\nu P_R\right).$$

**$\overline{\mathrm{RI}} - \mathrm{MOM}$**

- $Z_{OO}^{\overline{\mathrm{RI}}-\mathrm{MOM}} = 1 + \mathcal{O}(\alpha)$.

# Lattice Renormalisation

## Alternative Scheme SMOM

- $\overline{\mathrm{RI}}$ scheme defined via Ward Identity;
- In SMOM, analogous conditions are imposed, now involving 6 Lorentz Structures

$$\Downarrow$$

- $$\mathcal{P}^{\overline{\mathrm{RI}}-\mathrm{SMOM}} = \frac{1}{4}\Big( -\frac{1}{2}\gamma^{\nu} P_R \otimes \gamma_{\nu} P_R + \frac{1}{p^2}\not{p}_1 P_R \otimes \not{p}_1 P_R + \frac{1}{p^2}\not{p}_2 P_R \otimes \not{p}_2 P_R - \frac{1}{p^2}\not{p}_1 P_R \otimes \not{p}_2 P_R - \frac{1}{p^2}\not{p}_2 P_R \otimes \not{p}_1 P_R \Big).$$

**$\overline{\mathrm{RI}} - \mathrm{SMOM}$**

- $Z_{OO}^{\overline{\mathrm{RI}}-\mathrm{SMOM}} = 1 + \mathcal{O}(\alpha)$.

# $\overline{\mathrm{MS}}$ Renormalisation

- Naive Dimensional Regularisation (NDR) $\Rightarrow d = 4 - 2\epsilon$;
- Presence of Evanescent Operators [Gorbahn & Haisch, 2005]

$$\mathrm{E} = (\bar{\mathrm{d}}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\mathrm{P_L u})(\bar{\nu}_{\ell}\gamma_{\mu}\gamma_{\nu}\gamma_{\lambda}\mathrm{P_L}\ell) - (16 - 4\epsilon - 4\epsilon^2)(\bar{\mathrm{d}}\gamma^{\mu}\mathrm{P_L u})(\bar{\nu}_{\ell}\gamma_{\mu}\mathrm{P_L}\ell);$$

- $\psi_f^b = \left(Z_{2,f}^{\overline{\mathrm{MS}}}\right)^{1/2} \psi_f^{\overline{\mathrm{MS}}}$, f = u, d, $\ell$;
- $\begin{pmatrix} \mathrm{O}_{\mathrm{sem}}^{\overline{\mathrm{MS}}} \\ \mathrm{E}^{\overline{\mathrm{MS}}} \end{pmatrix} = \begin{pmatrix} Z_{OO}^{\overline{\mathrm{MS}}} & Z_{OE}^{\overline{\mathrm{MS}}} \\ Z_{EO}^{\overline{\mathrm{MS}}} & Z_{EE}^{\overline{\mathrm{MS}}} \end{pmatrix} \begin{pmatrix} \mathrm{O}_{\mathrm{sem}}^{\mathrm{b}} \\ \mathrm{E}^{b} \end{pmatrix}$;

## Amputated Green's Function

- $\Lambda_{\mathrm{O_{sem}}}^{\overline{\mathrm{MS}}} = \left(Z_{2,\mathrm{u}}^{\overline{\mathrm{MS}}}\right)^{1/2}\left(Z_{2,\mathrm{d}}^{\overline{\mathrm{MS}}}\right)^{1/2}\left(Z_{2,\ell}^{\overline{\mathrm{MS}}}\right)^{1/2}\left(Z_{OO}^{\overline{\mathrm{MS}}}\,\Lambda_{\mathrm{O_{sem}}}^{b} + Z_{OE}^{\overline{\mathrm{MS}}}\,\Lambda_{\mathrm{E}}^{b}\right).$

# Hadronic Matrix Elements

## Low-Scale Matching onto $\overline{\mathrm{MS}}$

- The expression for the Wilson Coefficient in the RI schemes is given by

$$C_O^{\mathrm{RI}} = \overbrace{\mathcal{C}^{\overline{\mathrm{MS}}\to\mathrm{RI}}(\mu_L, p^2)}^{\text{low}-\text{scale}}\,\overbrace{\mathcal{U}^{\overline{\mathrm{MS}}}(\mu_W, \mu_L)\,C_O^{\overline{\mathrm{MS}}}(\mu_W)}^{\text{high}-\text{scale}}$$

- $C_O^{\mathrm{RI}}(\mu_L, p^2) = C_\alpha^{\mathrm{RI}} + C_{\alpha_s}^{\mathrm{RI}} + \dfrac{\alpha}{4\pi}\left(C^{\mathrm{RI}}_{\alpha,\alpha_s\,LL} + C^{\mathrm{RI}}_{\alpha,\alpha_s\,NLL}\right)$
- $C_\alpha^{\mathrm{RI}}$ and $C_{\alpha_s}^{\mathrm{RI}}$ are the resummed QED and leading QCD contributions. Neglecting threshold corrections

$$C^{\mathrm{RI}}_{\alpha,\alpha_s LL} = -\frac{\gamma^{(1)}_{OO}}{2\beta^{(5)}_{(0)}}\ln\left(\frac{\alpha_s(\mu_L)}{\alpha_s(\mu_W)}\right),\;\; C^{\mathrm{RI}}_{\alpha,\alpha_s NLL} = \frac{\alpha_s(\mu_L)}{4\pi}\left(\mathcal{C}_O^{es}(-p^2, \mu_L^2) + \bar{\gamma}^{(5)}\right)$$

$$+\frac{\alpha_s(\mu_W)}{4\pi}\left(C_O^{es}(\mu_W, M_Z) - \bar{\gamma}^{(5)}\right),\quad \bar{\gamma}^{(N_f)} = \frac{1}{2\beta_0^{(N_f)}}\left(\gamma^{(1)}_{OO}\,\frac{\beta_1^{(N_f)}}{\beta_0^{(N_f)}} - \gamma^{(2)}_{OO}\right)$$

# Short-Distance Contribution

## 3-Loop Anomalous Dimension

### ADM from Renormalisation Constants

- $\gamma_{\mathrm{ij}} = Z_{\mathrm{ik}} \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)} \left(Z^{-1}\right)_{\mathrm{kj}}$

$\Downarrow$ Mass-Independent Renormalisation Scheme

- $\gamma_{\mathrm{ij}} = 2\beta(\epsilon, \alpha, \alpha_s) Z_{\mathrm{ik}} \frac{\partial}{\partial\, \alpha_s} \left(Z^{-1}\right)_{\mathrm{kj}} + 2\beta_e(\epsilon, \alpha, \alpha_s) Z_{\mathrm{ik}} \frac{\partial}{\partial\, \alpha} \left(Z^{-1}\right)_{\mathrm{kj}};$

- $\beta(\epsilon, \alpha, \alpha_s) = \alpha_s(-\epsilon + \beta(\alpha, \alpha_s))$ $\qquad$ $\beta_e(\epsilon, \alpha, \alpha_s) = \alpha(-\epsilon + \beta_e(\alpha, \alpha_s)).$

# Short-Distance Contribution

## 3-Loop Anomalous Dimension/Setting Up The Calculation

- Feynman diagrams generated using the Mathematica package *FeynArts* [Hahn, 2010] $\sim$ 600 diagrams;
- *FeynArts* built-in routines used to create Feynman amplitudes;

$\Downarrow$ Conversion to personal notation

- Personal Mathematica libraries for the final evaluation of amplitudes.

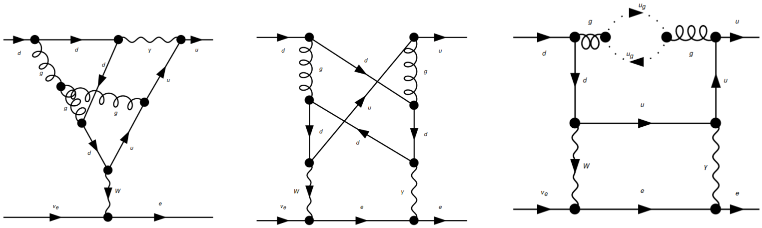



Figure: Some examples of the three loop diagrams calculated.

# Short-Distance Contribution

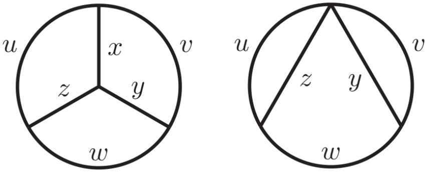
3-Loop Anomalous Dimension/Extracting The Divergences

## Infra-Red Rearrangement

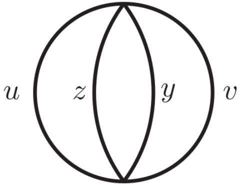
- Isolating the UV poles $\Rightarrow$ zero masses and external momenta;
- $$\frac{1}{(k+p)^2} = \frac{1}{k^2 - M^2} - \frac{p^2 + 2p\cdot k + M^2}{k^2 - M^2}\,\frac{1}{(k+p)^2}$$ [Chetyrkin *et al.*, 1998]

  IR regulator
- Gauge non-invariant counter-terms: $M^2 G^{\mu\, a} G^a_\mu$.

  "Gluon Mass"




[Broadhurst, 1999]




[Broadhurst, 1992]