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Towards Momentum Space Correlators for N=4 SYM

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YTF(2023)

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Introduction

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Conformal invariance imposes strong constraints on structure of correlators in any field theory.

For example, for scalar primary operators :

$$\langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = \frac{C_{\mathcal{O}}}{|\boldsymbol{x}|^{2\Delta}},$$

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\mathcal{O}_3(\boldsymbol{x}_3)\rangle = \frac{C_{123}}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\boldsymbol{x}_2-\boldsymbol{x}_3|^{\Delta_2+\Delta_3-\Delta_1}|\boldsymbol{x}_3-\boldsymbol{x}_1|^{\Delta_3+\Delta_1-\Delta_2}},$$

But recent works in Holographic Cosmology, Cosmological Bootstrap, Anomalies, etc have exemplified the need of CFT in momentum space!

All these analysis are in position space for separated points!

Simplest case: Two-Point Function

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Fourier Transforming the two-point correlator,

$$\int \mathrm{d}^d \boldsymbol{x}\, e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{x^{2\Delta}} = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d},$$

$$\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\rangle = (2\pi)^d \delta(\boldsymbol{p}_1+\boldsymbol{p}_2)\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(-\boldsymbol{p}_1)\rangle\!\rangle,$$

$$\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = \frac{C_{\mathcal{O}} \pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d}.$$

Poles → *Divergences*

Unitary theories

$\Delta > 0$

Upper-limit for convergence

$2\Delta < d$

Special cases

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, ..$$

short-distance singularities

Non-trivial regularization and renormalization necessary

↓

Conformal Anomalies!

[Osborn,Petkou,'93]

[Petkou,Skenderis,'99]

[Henninsgon,Skenderis,'98]

Does similar singularity structure and anomalies arise from three-point functions?

Two-Point Function -Regularization

Now instead of doing the F.T, one can try to directly get the two-point function from the Ward-identities (Special Conformal and Dilatation) :

$$\mathcal{K}\langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = \left[\frac{d^2}{dp^2} + \frac{d+1-2\Delta_1}{p}\frac{d}{dp}\right]\langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = 0.$$

$$D\langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = \left[d - \Delta_1 - \Delta_2 + p\frac{d}{dp}\right]\langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = 0.$$

$$\longrightarrow \quad \langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = c_\Delta p^{2\Delta-d},$$

For generic dimension, this is the correlator

$$\Delta = \frac{d}{2} + k,$$

$$\langle\langle\mathcal{O}_1(\boldsymbol{p})\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle = c_\Delta p^{2k}.$$

$$\downarrow$$

$$\langle\mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = c_\Delta(-\Box)^k\delta(\boldsymbol{x}). \longrightarrow \phi\,\Box^k\phi,$$

add new counterterm to renormalize

Regularization :

$$d \mapsto \tilde{d} = d + 2u\epsilon, \quad \Delta \mapsto \tilde{\Delta} = \Delta + (u+v)\epsilon,$$

$$\langle\langle\mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\rangle_{\text{reg}} = c_\Delta(\epsilon, u, v)p^{2\tilde{\Delta}-\tilde{d}} = c_\Delta(\epsilon, u, v)p^{2\Delta-d+2v\epsilon}.$$

$$= p^{2\Delta-d}\left[\frac{c_\Delta^{(-1)}}{\epsilon} + c_\Delta^{(-1)}v\ln p^2 + c_\Delta^{(0)} + O(\epsilon)\right]$$

[Bzowski, McFadden, Skenderis, et.al '15]

Two-Point Function -Renormalization

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Regulated action: $$S[\phi] = S_{\mathrm{CFT}} + \int \mathrm{d}^{d+2u\epsilon}\boldsymbol{x}\, \phi\mathcal{O}.$$

Generating Functional: $$Z[\phi] = \int \mathcal{D}\Phi\, e^{-S[\phi]},$$

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_{\mathrm{reg}} = \frac{\delta^2 Z}{\delta\phi(\boldsymbol{x}_1)\delta\phi(\boldsymbol{x}_2)}\bigg|_{\phi=0}.$$

Divergence of the correlator removed by adding $S_{\mathrm{ct}} = a_{\mathrm{ct}}(\epsilon, u, v)\int \mathrm{d}^{d+2u\epsilon}\boldsymbol{x}\, \mu^{2v\epsilon}\phi\,\Box^k\phi,$

[Bzowski, McFadden, Skenderis, et.al '15]

$$\langle\!\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle\!\rangle_{\mathrm{ct}} = -2a_{\mathrm{ct}}(\epsilon, u, v)(-p^2)^k\mu^{2v\epsilon}$$

with $$a_{\mathrm{ct}}(\epsilon, u, v) = \frac{(-1)^k}{2}\left[\frac{c_\Delta^{(-1)}(u, v)}{\epsilon} + a_0(u, v) + O(\epsilon)\right],$$

$$\mathcal{A}_2 = \mu\frac{\partial}{\partial\mu}\langle\!\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle\!\rangle = -2c_\Delta p^{2k}.$$

Renormalized correlator :

$$\langle\!\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle\!\rangle = p^{2k}\left[c_\Delta^{(-1)} v\ln\frac{p^2}{\mu^2} + c_\Delta^{(0)} - a_0\right] = p^{2k}\left[c_\Delta \ln\frac{p^2}{\mu^2} + c'_\Delta\right],$$

Conformal anomaly : $$\mu\frac{\partial}{\partial\mu}W = A,$$

with $W = \ln Z$

and $$A = \int \mathrm{d}^d\boldsymbol{x}\, \mathcal{A}_k\phi\Box^k\phi + \cdots$$

Three-Point Function

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(These results hold for all CFT's, perturbative or non-perturbative)

For three-point correlators,

$$\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle \propto \int_0^\infty \mathrm{d}x\, x^{d/2-1} \prod_{j=1}^{3} p_j^{\Delta_j - d/2} K_{\underbrace{\Delta_j - d/2}_{\beta}}(p_j x).$$

→ *Expand the Bessel functions with respect to ε, and extract the divergences*

This can also be obtained directly from the conformal Ward Identities in momentum space without any reference to Fourier Transform

Condition for divergence (obtained from the convergence of triple-K integral) :

$$\frac{d}{2} \pm (\Delta_1 - \frac{d}{2}) \pm (\Delta_2 - \frac{d}{2}) \pm (\Delta_3 - \frac{d}{2}) = -2k.$$

(four kind of divergences depending on the relative plus or minus signs)

each of these condition can hold separately or in combinations

Classification :

$(---)$-condition: $\square^{k_1}\phi_1\square^{k_2}\phi_2\square^{k_3}\phi_3,$ → *anomalies*

$(--+)$-condition: $\square^{k_1}\phi_1\square^{k_2}\phi_2\square^{k_3}\mathcal{O}_3,$ → *β- functions* → *double-log renormalized correlators*

Not true divergences of the correlator, rather the triple-K representation is singular

$(-++)$-condition

$(+++)$-condition

[Bzowski, McFadden, Skenderis, et.al '13]

[Bzowski, McFadden, Skenderis, et.al '15]

Four-Point Function

4-point function not completely fixed by conformal symmetry

$$\langle \mathcal{O}_{\Delta_1}(\boldsymbol{x}_1)\mathcal{O}_{\Delta_2}(\boldsymbol{x}_2)\mathcal{O}_{\Delta_3}(\boldsymbol{x}_3)\mathcal{O}_{\Delta_4}(\boldsymbol{x}_4)\rangle = f(u, v) \prod_{1\le i<j\le 4} x_{ij}^{2\delta_{ij}},$$

arbitrary function of cross-ratio

$$u = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.$$

Condition for divergence

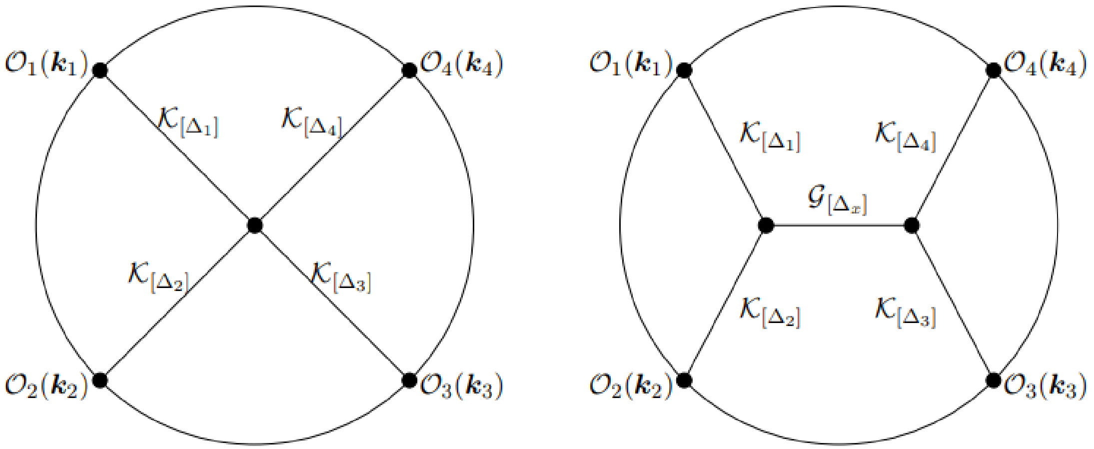
$$d + \sum_{j=1}^{4} \sigma_j(\Delta_j - d/2) = -2n$$

[Bzowski, McFadden, Skenderis, et.al '2020]

What is the structure of the divergence and counterterms that show up for the different possible cases here?

Tree-level Holographic Four Point Function

Contact and Exchange Witten Diagrams, [Bzowski, McFadden, Skenderis, et.al '2022]



Bulk-to-Boundary : $\mathcal{K}_{d,\Delta}(z,k) = \dfrac{k^{\Delta-\frac{d}{2}} z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(kz)}{2^{\Delta-\frac{d}{2}-1}\Gamma\left(\Delta-\frac{d}{2}\right)}$

Bulk-to-Bulk :

$$\mathcal{G}_{d,\Delta}(z,k;\zeta) = \begin{cases} (z\zeta)^{\frac{d}{2}} I_{\Delta-\frac{d}{2}}(kz) K_{\Delta-\frac{d}{2}}(k\zeta) & \text{for } z<\zeta, \\ (z\zeta)^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(kz) I_{\Delta-\frac{d}{2}}(k\zeta) & \text{for } z>\zeta. \end{cases}$$

Contact correlator:

$$\hat{i}_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(k_1,k_2,k_3,k_4) = \int_0^\infty \mathrm{d}z\, z^{-\hat{d}-1} \hat{\mathcal{K}}_{[\Delta_1]}(z,k_1)\hat{\mathcal{K}}_{[\Delta_2]}(z,k_2)\hat{\mathcal{K}}_{[\Delta_3]}(z,k_3)\hat{\mathcal{K}}_{[\Delta_4]}(z,k_4).$$

Exchange correlator:

$$\hat{i}_{[\Delta_1\Delta_2,\Delta_3\Delta_4 x\Delta_x]}(k_1,k_2,k_3,k_4,s) = \int_0^\infty \mathrm{d}z\, z^{-\hat{d}-1} \hat{\mathcal{K}}_{[\Delta_1]}(z,k_1)\hat{\mathcal{K}}_{[\Delta_2]}(z,k_2) \times \\ \times \int_0^\infty \mathrm{d}\zeta\, \zeta^{-\hat{d}-1} \hat{\mathcal{G}}_{[\Delta_x]}(z,s;\zeta)\hat{\mathcal{K}}_{[\Delta_3]}(\zeta,k_3)\hat{\mathcal{K}}_{[\Delta_4]}(\zeta,k_4).$$

Renormalizing four-point holographic correlators

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For $d = 3$ and $\Delta = 2, 3.$ *(Bessel functions have half-integral value of β, making it easy to obtain closed form structure for the correlators)*

Amplitude	Singularity type	Counterterm	Type	Contributes to
2222	–	–	–	–
3222	ultralocal	$\phi^{[0]}\phi^{[1]}\phi^{[1]}\phi^{[1]}$	anomaly	4-pt
	3-pt	$\phi^{[0]}\phi^{[1]}\mathcal{O}^{[2]}$	beta for $\phi^{[1]}$	3, 4-pt
3322	3-pt	$\phi^{[0]}\phi^{[1]}\mathcal{O}^{[2]}$	beta for $\phi^{[1]}$	3, 4-pt
	3-pt	$\phi^{[0]}\phi^{[0]}\mathcal{O}^{[3]}$	beta for $\phi^{[0]}$	3, 4-pt
	2-pt	$\phi^{[0]}\phi^{[0]}\phi^{[1]}\mathcal{O}^{[2]}$	beta for $\phi^{[1]}$	4-pt
3332	ultralocal	$\phi^{[0]}\phi^{[0]}\phi^{[0]}\phi^{[1]}\partial^2$	anomaly	4-pt
	3-pt	$\phi^{[0]}\phi^{[1]}\mathcal{O}^{[2]}$	beta for $\phi^{[1]}$	3, 4-pt
	3-pt	$\phi^{[0]}\phi^{[0]}\mathcal{O}^{[3]}$	beta for $\phi^{[0]}$	3, 4-pt
3333	3-pt	$\phi^{[0]}\phi^{[0]}\mathcal{O}^{[3]}$	beta for $\phi^{[0]}$	3, 4-pt
	2-pt	$\phi^{[0]}\phi^{[0]}\phi^{[0]}\mathcal{O}^{[3]}$	beta for $\phi^{[0]}$	4-pt

(these kind of correlators relevant for Cosmological bootstrap)

Work in Progress: Four-Point Function in d=4

When d=4, analysis of these divergences become more complicated even at the level of four-point contact diagrams!

- *The values of β are no longer half-integer, thus the Bessel functions are not as simple as before and we need to use series expansion of the Bessel functions.*

- *We are currently looking into correlators of the form <2244>, <2255>, <2266>, etc. These are not only interesting to uncover the rich structure of singularities of four-point functions but are also important in the context of ½-BPS scalar operators in N=4 SYM in d=4.*

Goal: Correlators in N = 4 SYM

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$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle = \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle_{\text{free}} + \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle_{\text{int}} = R_2 R_p g_{12}^2 g_{34}^p G_p(u,v;\sigma,\tau)\,.$$

↓

At 1/N^2 we have the tree level correlator, then loops at higher order in 1/N.

Usually these correlators are bootstrapped using several constraints like crossing invariance and OPE expansions on the CFT side or through Witten diagrams using the AdS/CFT duality. But in momentum space there is no analogue of OPE. So we hope to constrain the momentum space correlators using this kind of a renormalization technique.

THANK YOU