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Bulk-Boundary bookkeeping of (Quasi)-Poisson structures: Chern-Simons and WZNW theories

Juan Carlos Morales Parra (Under supervision of Bernd Schroers)[1]

[1] *Maxwell Institute for Mathematical Sciences,*

Heriot-Watt University and University of Edinburgh, UK

I. HOLOGRAPHY IN 3 DIMENSIONS

The idea of **(3 dimensional) holography** in simple words consists of

> **If T_1 and T_2 are theories defined over M^3 and ∂M^3, How are $T_1|_{\partial M^3}$ and T_2 related?**

It has been effectively used and described at different **settings/levels**:

- **AdS$_3$/CFT$_2$ (Asymptotic symmetries)**: [1] (J.D.Brown and M.Henneaux 1986).
- **CS**(M^3)/**WZNW**(∂M^3) **equivalence:** [2] (S.Elitzur, G.Moore, A.Schwimmer, N.Seiberg 1989)
- **CS**(M^3)/**WZNW**(∂M^3) **(BRST) equivalence**: [3] (J.Fjelstad and S.Wang 1999)
- **AdS$_3$/CFT$_2$ (Partition functions):** [4] (S.Gukov, E.Martinec, G.W.Moore and A.Strominger 2004)
- **CS**(M^3)/**WZNW**(∂M^3) **functoriality:** [5] (D.S.Freed and C.Teleman 2021)

> **Today: CS(M^3)/WZNW(∂M^3) equivalence at the level of (Quasi)Poisson structures!**

II. 3D CHERN-SIMONS THEORY

A. Ingredients

The **Input** of a **3d Chern-Simons theory** consists of:

- Your favourite 3-manifold M^3 (Here we restrict to the $M^3 \cong \Sigma^m_{g,n} \times \mathbb{R}$ case) and Lie group G.
- A non-degenerate ad-invariant bilinear form $\mathrm{Tr} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

The **action** of the theory is

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right)$$

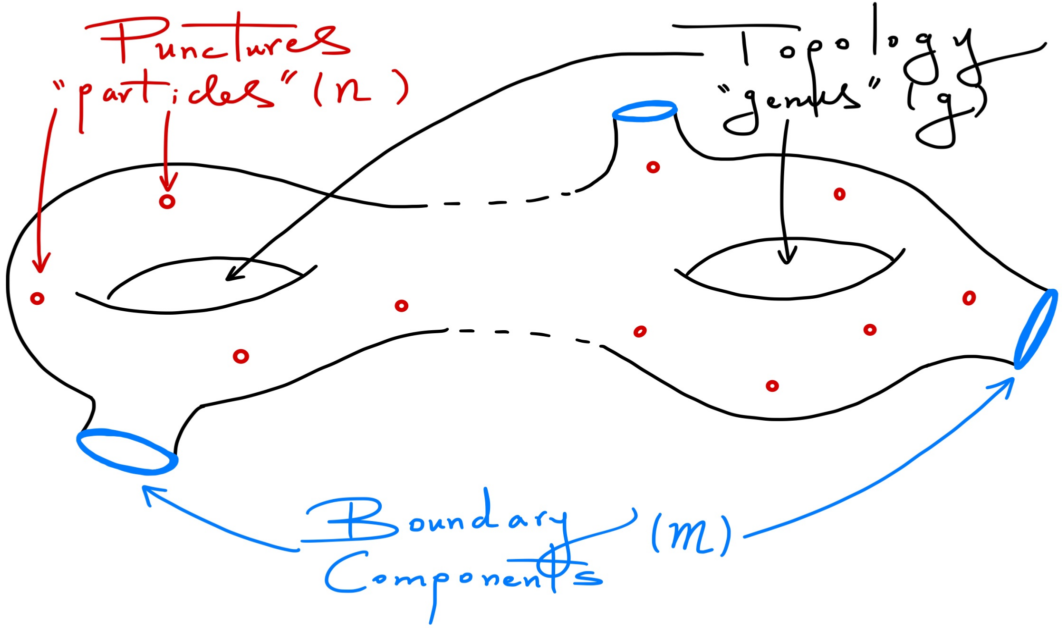
The **moduli space** of the theory is **the moduli space of G-flat connections**

$$\mathcal{M}_f(M^3, G) = \left\{ A \in \Omega^1(M^3, \mathfrak{g}) | F_A \equiv dA + \frac{1}{2} A^2 = 0 \right\} \Big/ \text{Gauge Transformations}$$

[6] G.V.Dunne: Les Houches Lectures (1998)

II. 3D CHERN-SIMONS THEORY

A. Ingredients: What is $\Sigma_{g,n}^{m}$?



B. Moduli space of Chern-Simons theory: Generalities

The **Riemann-Hilbert correspondence** allows us to provide a **topological** description of $\mathcal{M}_f(M^3, G)$

$$\mathcal{M}_f(M^3, G) \cong \mathrm{Hom}(\pi_1(M^3), G)/G = \mathrm{Hom}(\pi_1(\Sigma_{g,n}^m), G)/G$$

More **explicitly** ($\mathrm{L}G \equiv \{g \in C^\infty(\mathbb{R}, G) \mid g(x + 2\pi) = Mg(x), M \in G\}$)

$$\mathcal{M}_f(\Sigma_{g,n}^m, G) \cong \{(C_1, \cdots, C_n, A_1, B_1, \cdots, A_g, B_g, D_1, \cdots, D_m) \in G^{n+2g} \times (\mathrm{L}G)^m \mid \text{constraints}\}/G$$

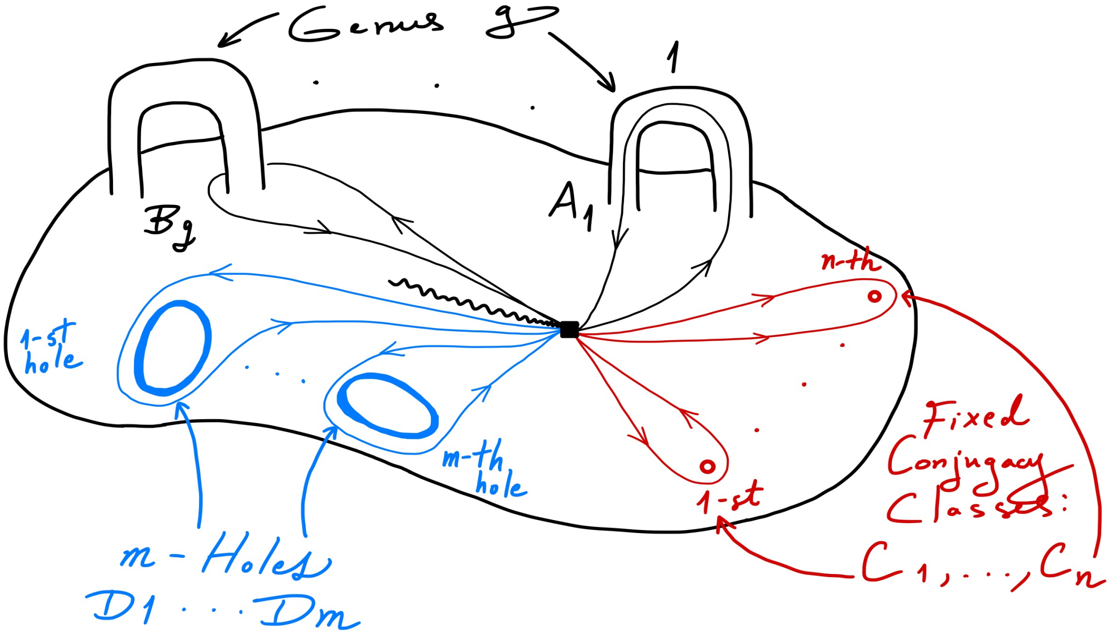
with the **constraints**:

- **Topological:** $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] C_n \cdots C_1 \mathrm{Hol}(D_m) \cdots \mathrm{Hol}(D_1) = 1_G$
- **Wigner:** $C_i \in \mathcal{C}_i$ (fixed conjugacy classes) for $i = 1, \cdots, n$.

Theorem [[7] Atiyah-Bott (1983), [8] Fock-Rosly (1992)]**: The space $\mathcal{M}_f(\Sigma_{g,n}^m, G)$ is Poisson!**

[9] M.Audin: Chapter V of *Torus Actions on Symplectic Manifolds* (2004)

C. Moduli space of Chern-Simons theory: A picture



III. 2D WZNW THEORY

A. Ingredients

The **Input** of a **2d WZNW theory** consists of:

- Your favourite 2-manifold M^2 and Lie group G.
- A non-degenerate ad-invariant bilinear form $\mathrm{Tr} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

The **action** of the theory is

$$\boxed{S_{WZNW}(g) = -\frac{k}{2\pi}\int_{M^2 \cong \partial M^3} d^2x \mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial^i g) + \frac{k}{3\pi}\int_{M^3} d^3x \epsilon^{ijk}\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g)}$$

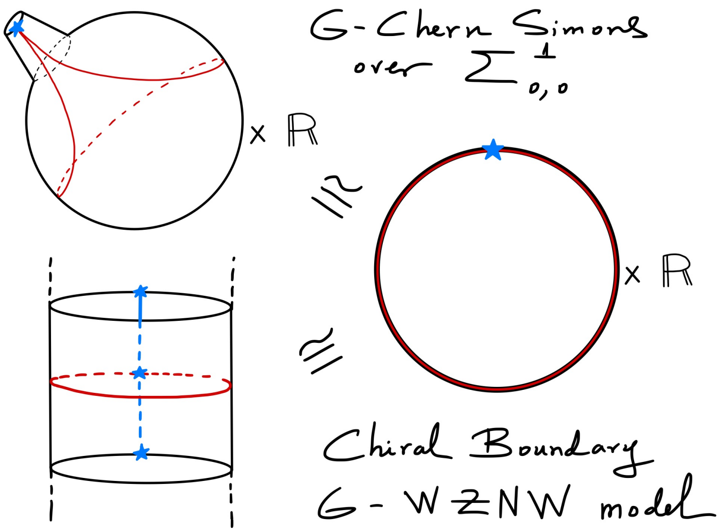
The **moduli space** of the theory is **the space of maps** $g : M^2 \to G$ satisfying

$$\boxed{\partial_+[g^{-1}(x^+,x^-)\partial_- g(x^+,x^-)] = 0 = \partial_-[\partial_+ g(x^+,x^-)g^{-1}(x^+,x^-)]}$$

[10] P.Furlan, L.Hadjivanov and I.Todorov: *Canonical approach to the WZNW model* (2014)

IV. (QUASI)POISSON EQUIVALENCE OF 3D CS AND 2D WZNW THEORIES [11]

A. The structures



A. Quasi-Poisson structure

$$\{g(x) \overset{\otimes}{,} g(y)\} = (g(x) \otimes g(y))\Omega \text{sign}(y - x)$$

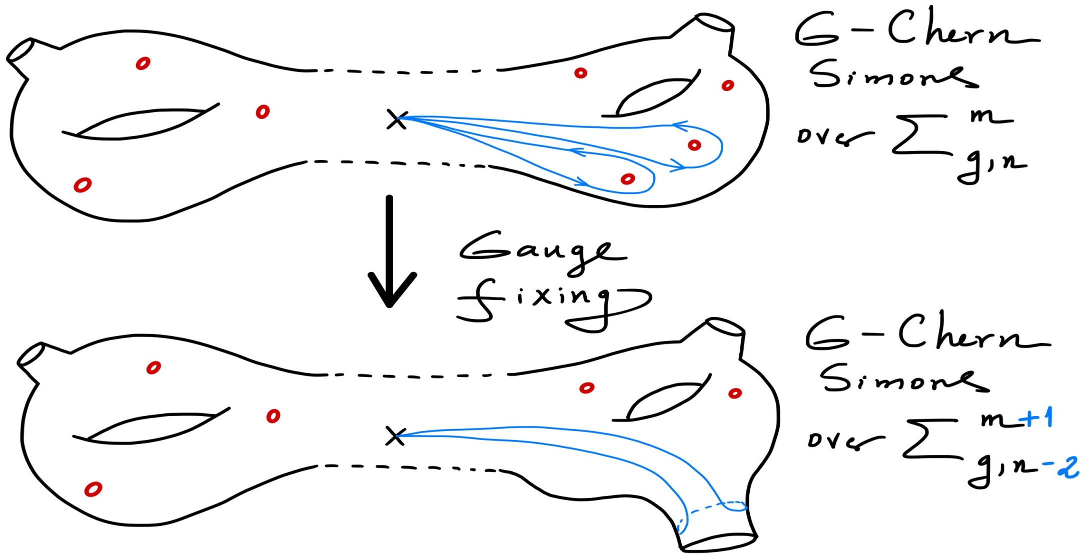
[12] (LeBlanc 2007), [13] (Balog et.al 2000) and [14] (Klimčík 2014)

B. Poisson structure

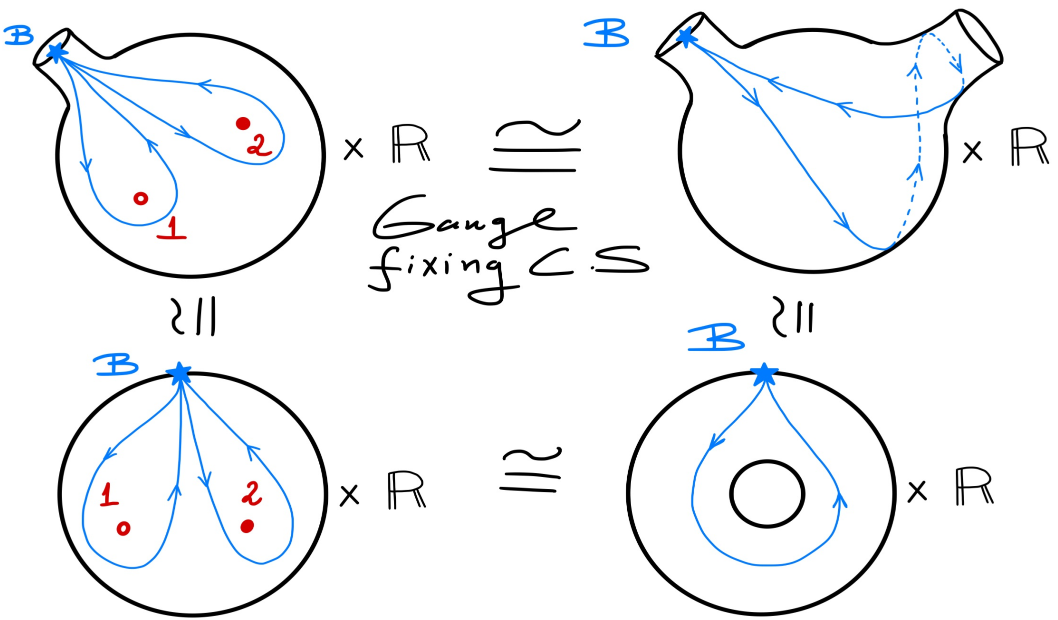
$$\Omega \to \Omega + \hat{r}(\mathbf{Extra})$$

[15] (Meusburger et.al 2012), [16] (Boalch 2014), [17] (MP et.al 2023) [18] (Feher 2001).

B. Gauge fixing in 3d CS theory



[19] C.Meusburger and T.Schonfeld: *Gauge fixing in (2+1)-gravity (Dirac brackets)* (2011)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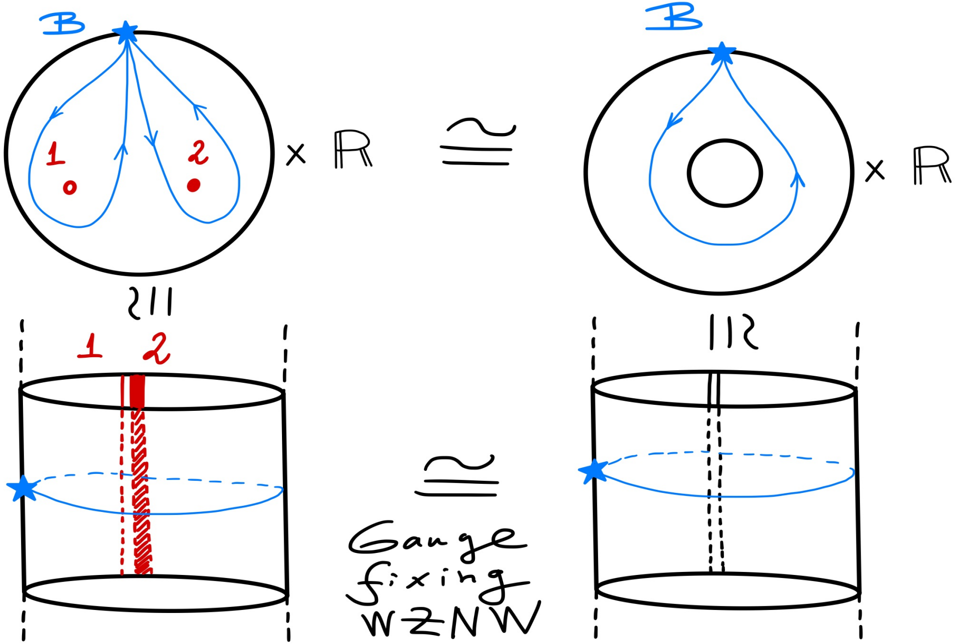
C. Gauge fixing correspondence



V. OUTLOOK AND CONCLUSIONS

- Using simple **topological arguments** the **equivalence** between the **(Quasi)Poisson structures** of **3d CS** (a TFT) and **2d WZNW** (a CFT) is straighforward.
- The same **topopological arguments** provide understanding about how the **(Dirac) gauge fixing** translates from one theory to the other.
- **Affinization:** Now you consider the **Loop group** of your favourite **Lie group** (**Application:** (1+1)d and 2d Integrable systems).
- **Enhancement to Super groups:** Now you consider your favourite **Super Lie group** (**Application:** 3d Super-gravity Holography).

THANK YOU!

VI. REFERENCES

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[3] J. Fjelstad and S. Hwang, Physics Letters B **466**, 227 (1999).

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