Exact results in large-N CFTs

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Supervised by John Wheater

The road to results

Tools invented for quantum gravity allow us to exactly solve new CFTs.

[‘t Hooft ‘74] realised that the large N limit gives us control over aspects of theories; for example, if we’re at strong coupling:

$$g \gg 1 \qquad\qquad \frac{1}{N} \ll 1$$

We will walk through the justification for large N:

- First vectors models (1D) – (too) easy
- then matrix models (2D) - hard
- then tensors models (>3) - possible

- Why they don’t immediately work for QG.
- At RG fixed points of tensor models, we will find nontrivial CFTs.
- Find exact solutions for these CFT data.

Why the large N limit?

- *Classicalisation*: essentially, we are doing mean field.
- If we have:
  1) lots of degrees of freedom
  2) 'large' symmetry groups G

  Then G-invariant quantities self-average, and become more classical.

$$N_{\text{dof}} \to \infty$$

- Simplify, or even become analytically solvable.
- Just as $Z[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]}$ is dominated by $S'[\phi] = 0$ as "$\hbar \to 0$", if we can change variables:

$$Z[J] = \int \mathcal{D}\sigma \, e^{\frac{i}{N} T[\sigma]}$$

Then we can exactly solve this with $T'[\sigma]$ in the $N \to \infty$ limit.

$O(N)$ vector model

$$\phi_a \to O_a^b \phi_b$$

N scalars, transforming in the fundamental of $O(N)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) - \frac{\lambda}{4!}\phi^4$$

There is a unique interaction invariant under $O(N)$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_a \partial^\mu \phi_a - m^2 \phi_a \phi_a\right) - \frac{g}{N4!}(\phi_a \phi_a)^2$$

Choose optimal scaling, 1/N.

Think of the indices ‘a’ as flowing along propagators;

$$\frac{i\delta^{ab}}{p^2 - m^2} \qquad a \text{———} b$$

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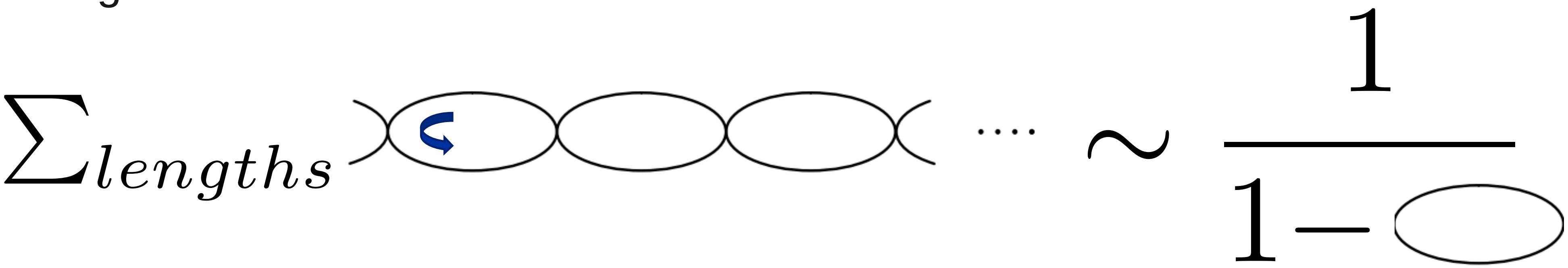
Unfortunately, it’s boring:

Essentially, the only contributions at LO in N are infinite bubble chains.

→ geometric series. Solvable!



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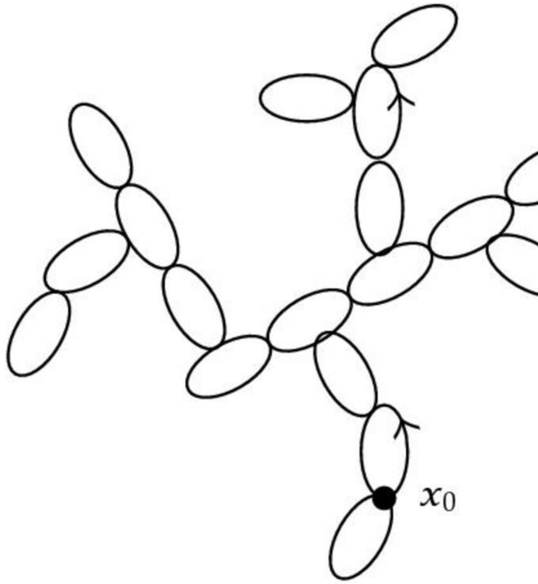

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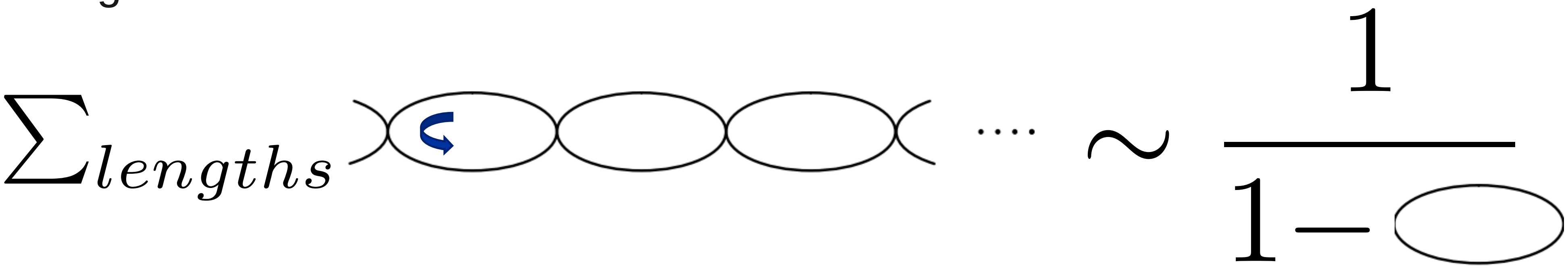

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‘Upgrade’ to the $O(N)xO(N)$ matrix model:

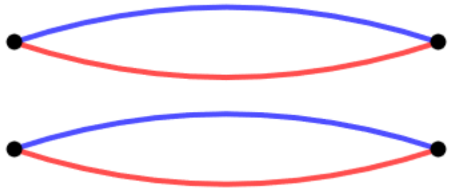
$O(N) \times O(N)$ matrix model

$$\phi_{ab} \to O_a^c P_b^d \phi_{cd}$$

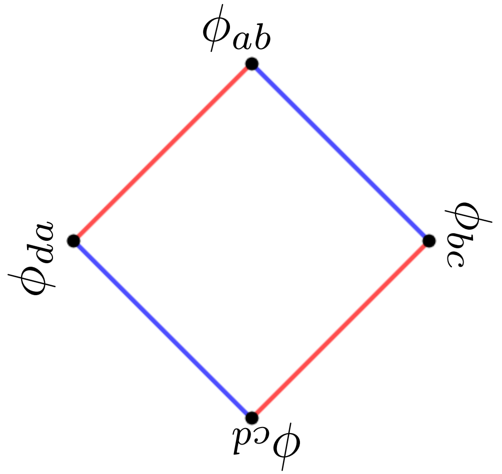
Graphical notation for the indices gives a convenient pictorial representation:

One colour per index position, and lines indicate contraction.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_{ab} \partial^\mu \phi_{ab} - m^2 \phi_{ab}\phi_{ab}\right) - \frac{\lambda_{\mathrm{dt}}}{4!}\left(\phi_{ab}\phi_{ab}\right)^2$$



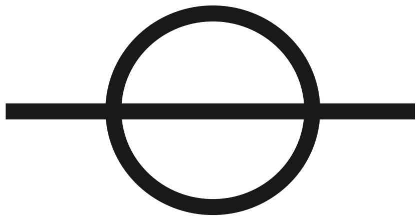
$$\mathcal{L} \supset -\frac{\lambda_{st}}{4!N}\left(\phi_{ab}\phi_{bc}\phi_{cd}\phi_{da}\right)$$

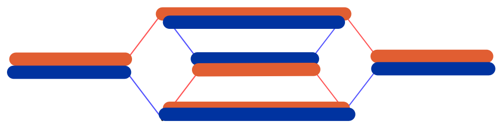


$$\frac{i\delta^{ab}\delta^{cd}}{p^2 - m^2}$$

$O(N) \times O(N)$ matrix model – propagator corrections

$$(\text{Propagator})^{-1} \supset$$





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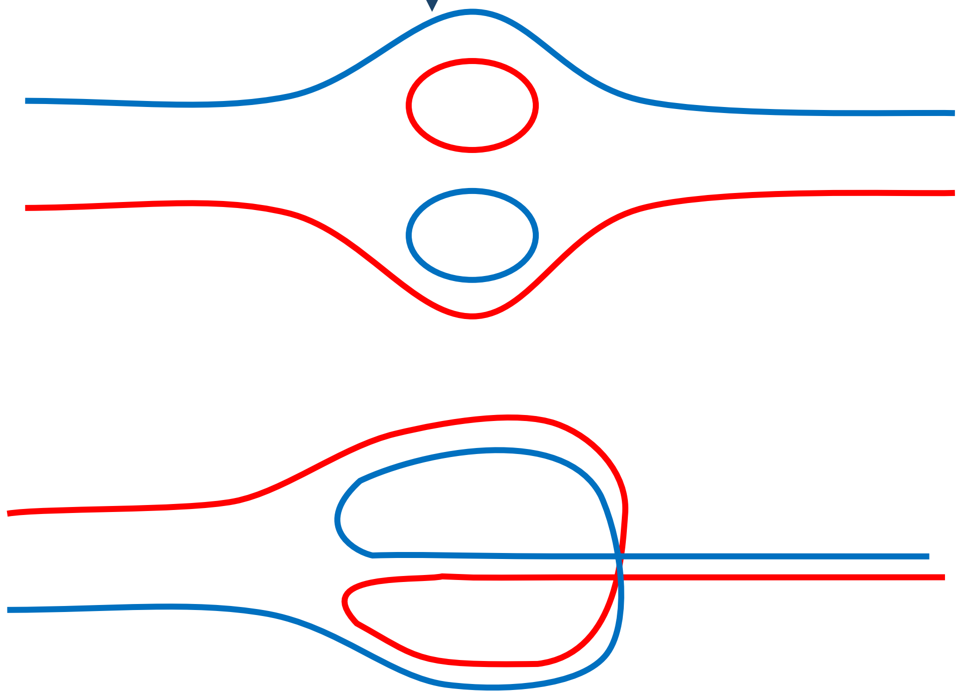

$$\sim O(1)$$

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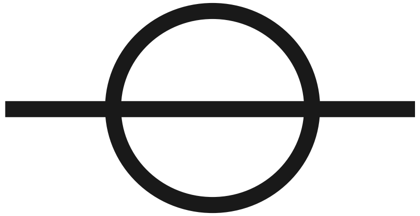
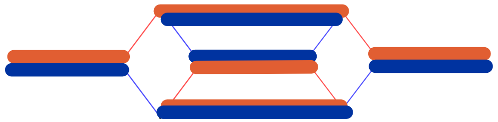
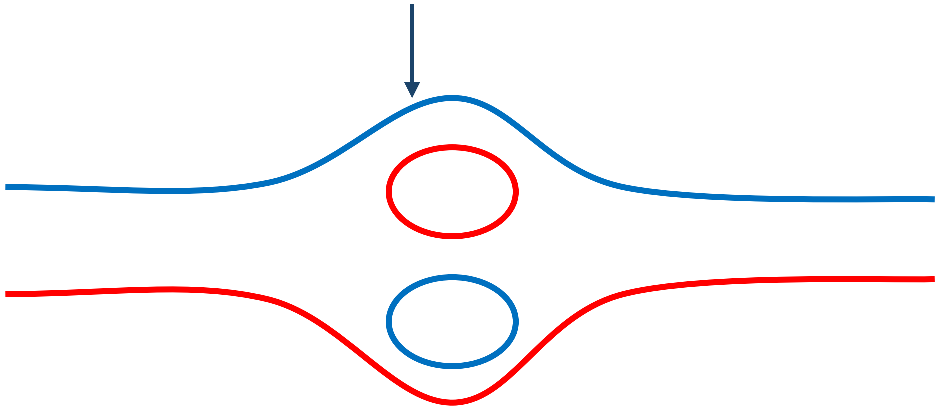
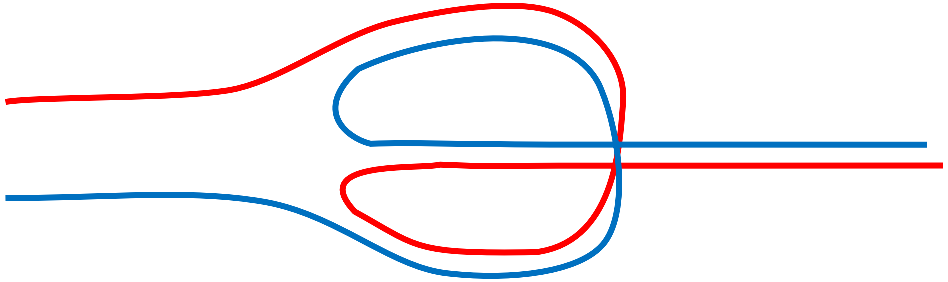
$(\text{Propagator})^{-1} \supset$ 



$(\text{Propagator})^{-1} \supset$  $\sim O(1)$

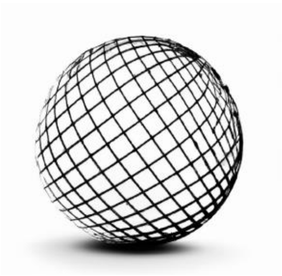
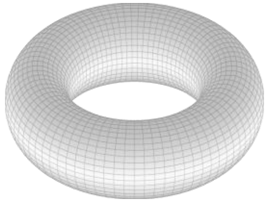
 $\sim O\left(\frac{1}{N}\right)$

Planar graphs

Upshot: the diagrams are organised by a genus expansion.

So, if we're at strong coupling, the leading order would have lots and lots of vertices, and would be planar.

This looks like the topological expansion of the 2D dynamical gravity partition function:

$Z =$  $+$  $+ \ldots$

Have we invented gravity?

$O(N)$ vector model:

- Infinitely long chains → dynamical "1D gravity"

$O(N)$ matrix model:

- Graphical expansion in terms of surface genus - discretised dynamical 2D gravity

$N \to \infty$ gives a 'continuum limit' of 2D QG.

So, to obtain higher dimensions – just add another index?

- Originally, it was thought that this could work to give discretisations of extended gravitational objects (D-branes, etc.). See [Rivasseau '16, Gurau '16]

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Generalise again.

$O(N)^3$ tensor model

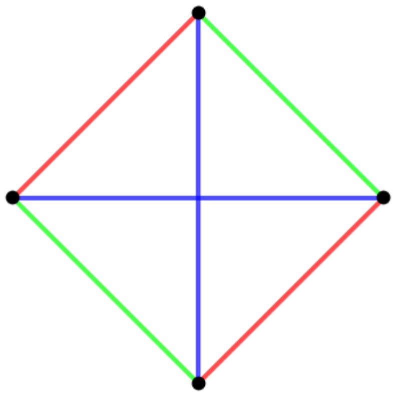
$$\phi_{abc} \to O_a^d P_b^e Q_c^f \phi_{def}$$

Now: the main object of study: tensor models.

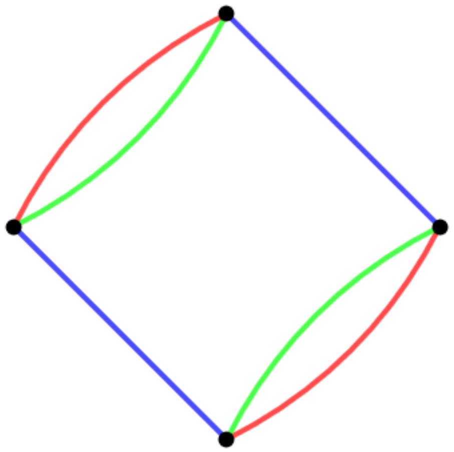
One boson in the tri-fundamental – $O(N)^3$. Generalising from before:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_{abc} \partial^\mu \phi_{abc} - m^2 \phi_{abc}\phi_{abc}\right) + \text{couplings}$$

$\frac{\lambda_t}{N^{3/2}}$

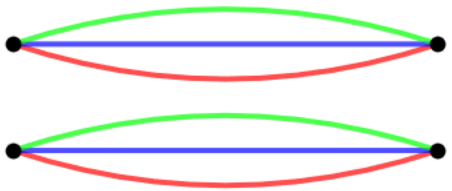


$\frac{\lambda_p}{N^2}$



$\times 3$

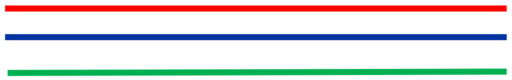
$\frac{\lambda_{dt}}{N^3}$



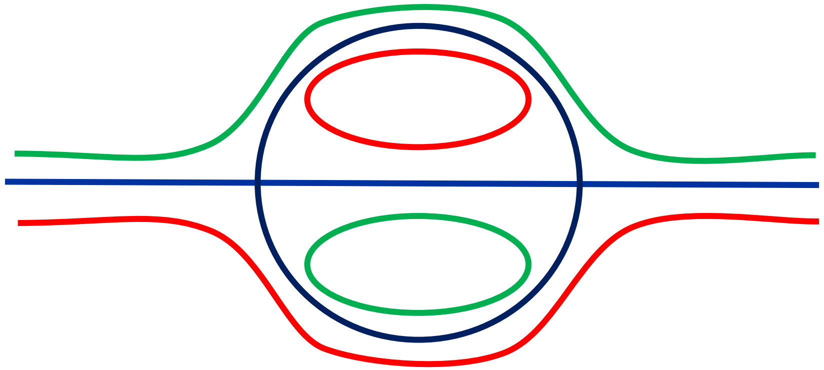
Melonic graphs

Once again: index loops give a factor of N

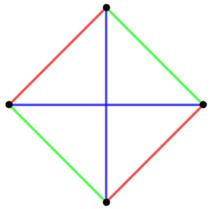
Propagator $\frac{i\delta^{ab}\delta^{cd}\delta^{ef}}{p^2 - m^2}$

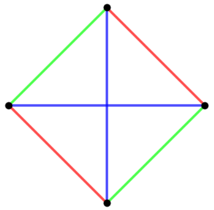


Example LO correction:



from





Stalled for a while, until the *melonic limit* was found: a particular scaling of coupling constants [Bonzom et al. '11; Witten '16].

Melonic graphs

Large-N expansions are now indexed by Gurau degree:

$$\mathcal{A} = \sum_{\omega \in \mathbb{N}_0/2} N^{-\omega} F(G_\omega)$$

$$\omega(\mathcal{G}) = \frac{1}{2} \sum_{\mathcal{J}} g(\mathcal{J}) = \frac{(D-1)!}{2} \left(D + \frac{D(D-1)}{4} V(\mathcal{G}) - F(\mathcal{G}) \right)$$

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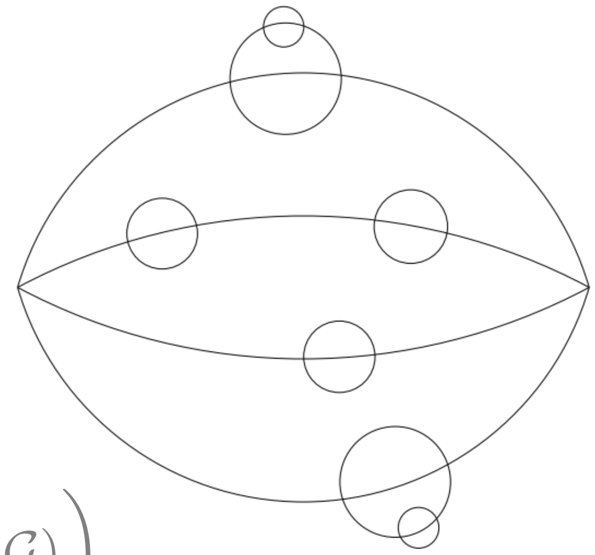

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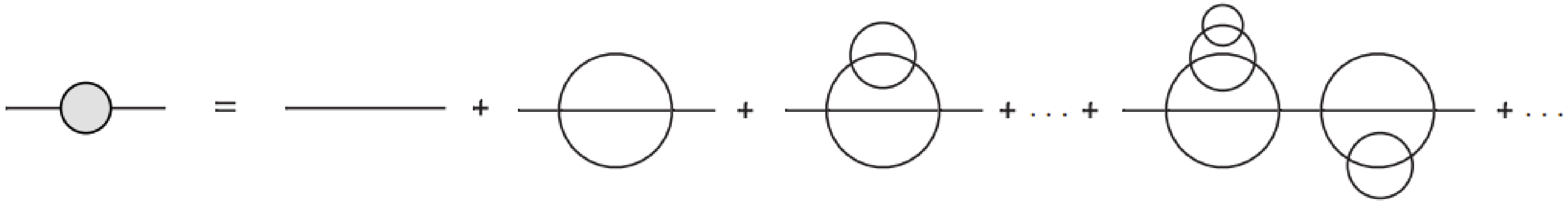
Luckily, the leading order, $\omega = 0$, has a nice graphical interpretation: the melons. Thus $F(G_0)$:



Clearly $G_0 \subset$ Planar diagrams. Observe summable 'multidimensional geometric sequence': this is the **melonic truncation [Gurau '19].**

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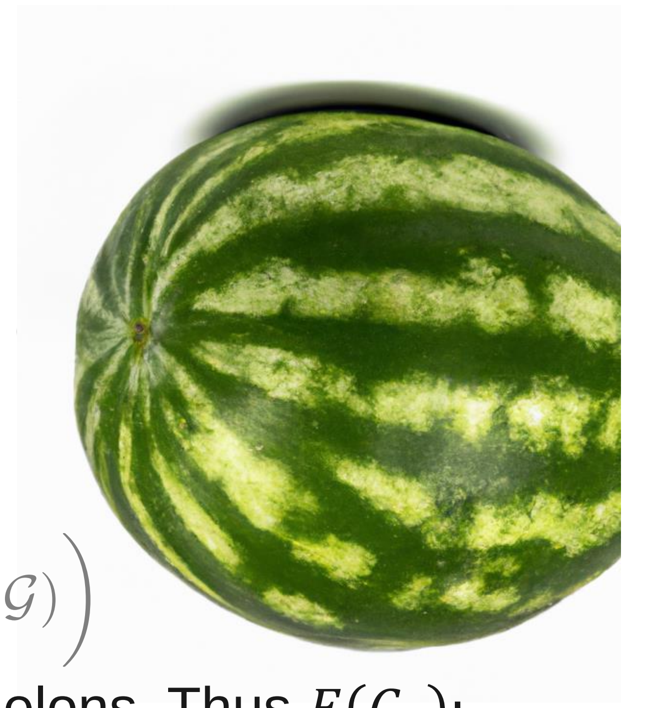

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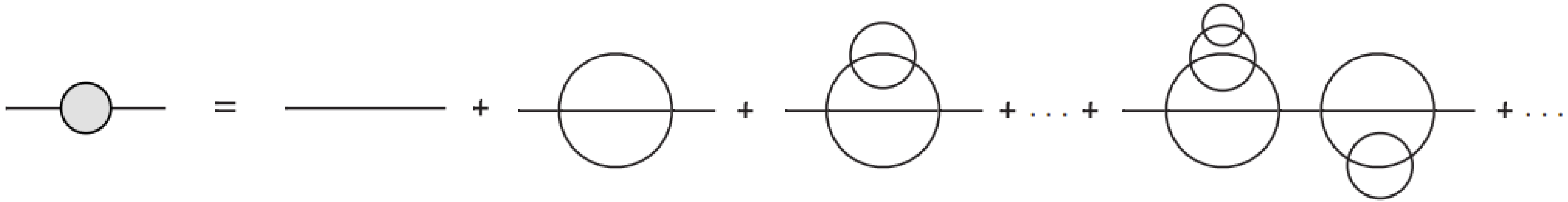

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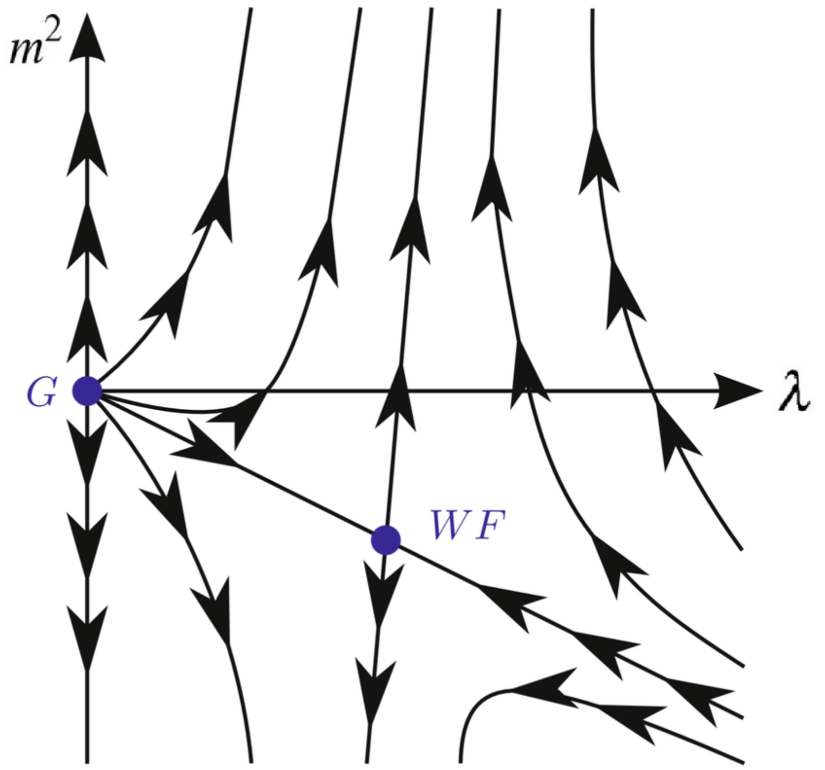
Unfortunate news: degenerate spaces

- Sadly, the spaces generated have
  - Hausdorff (ball) dimension 2
  - Spectral (drunk) dimension 4/3
- Secretly a 2D gravity all along.
  - Unsurprising, from planarity.
- But: we have gained access to these non-Lagrangian theories, defined solely by a graphical expansion of Feynman diagrams.
- More tractable than the matrix models – we can make progress.
- Find their RG fixed points: are they interesting as CFTs?
- Let’s take an example. First, backtrack:

A reminder: the Wilson-Fisher fixed point

- Solving the 3D Ising model is hard: strong coupling. No small parameter to expand in.
- Generally applicable lesson: we can invent one. See: the hydrogen atom, the N-body problem…
- Idea: get an idea for the structure of the RG flow by taking $D = 4 - \epsilon$ [Wilson '71].
- Then throw caution to the wind and take $\epsilon = 1$.
- Solve for the quantum dimension of the scalar field at that fixed point:



$$\Delta_\phi = \frac{D-2}{2} + \gamma_\phi, \quad \gamma_\phi = \#\epsilon^2 + \#\epsilon^3 + \dots$$

CFT exploration: the 3D Yukawa-ish model.

We make the arbitrary choice to work with the following Lagrangian in D dimensions:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi\phi\right) + \bar{\psi}(i\not{\partial} - m)\psi$$
$$- \frac{\lambda}{4!}\phi^4 - \frac{g}{2}\phi^2\bar{\psi}\psi - \frac{h}{6!}\phi^6$$

Consistently renormalizable theory.

Promote all the fields to tensors $\phi \rightarrow \phi_{abc}$ with N^3 degrees of freedom.

Now find fixed points, and understand them.

Melonic model

$$\phi_{abc} \to O_a^d P_b^e Q_c^f \phi_{def}$$

One fermion, one boson, in the tri-fundamental – $O(N)^3$:

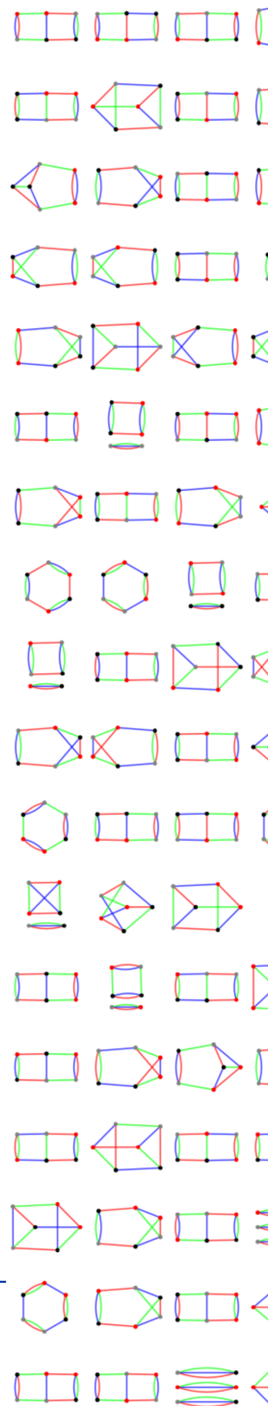
$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_{abc} \partial^\mu \phi_{abc} - m^2 \phi_{abc}\phi_{abc}\right) + \bar{\psi}_{abc}(i\not{\partial} - m)\psi_{abc} - V(\phi_{abc}, \psi_{abc})$$

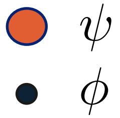
The potential contains 14 $\phi^2\bar{\psi}\psi$ invariants, 5 ϕ^4 invariants, and 8 ϕ^6 invariants, with 17 associated coupling constants (assuming index symmetry).

This situation only gets worse for higher rank fields $\phi_{abcde\ldots}$, or under symmetry breaking.

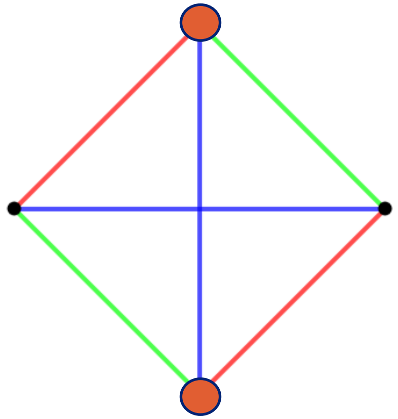
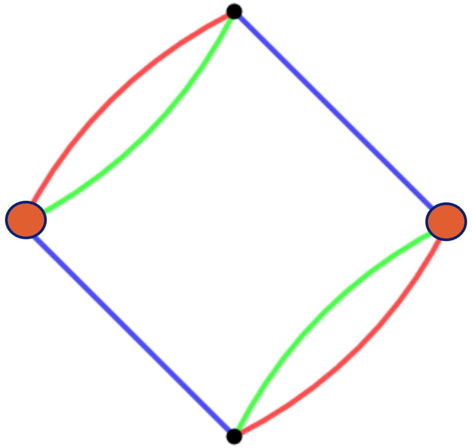
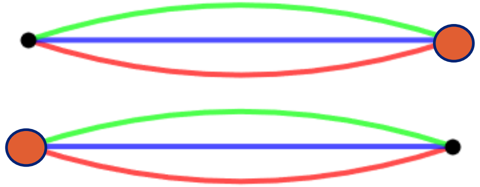
Can just do standard RG analysis in $D = 3 - \epsilon$, as well as to leading order in 1/N.

So, let's assume the following form for the potential, using the graphical notation:





Melonic CFT

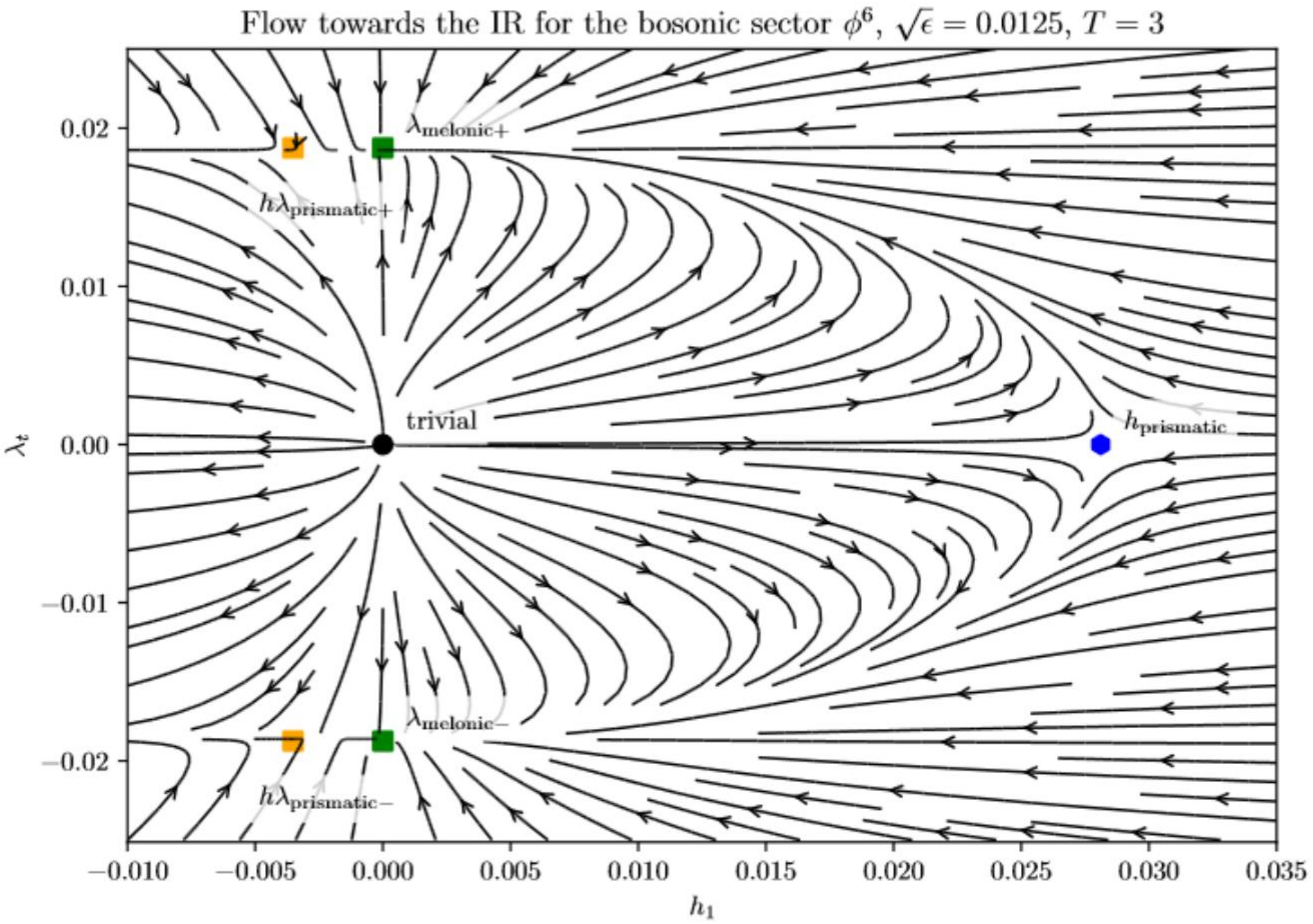
$\frac{\lambda_t}{N^{3/2}}$  $\frac{\lambda_p}{N^2}$  $\times 3$ $\frac{\lambda_{dt}}{N^3}$ 

Assuming no SSB (a significant assumption – see [Rychkov, Stergiou '18]);

In $D = 3 - \epsilon$, at leading order in 1/N and fourth loop order, we find a Wilson-Fisher-like fixed point for non-zero tetrahedra λ_t coupling; all other couplings must be zero.

$$\lambda_t = \pm\left(\frac{3\sqrt{\epsilon}}{\sqrt{T+1}} + \frac{(11T^2+26T+2)\epsilon^{3/2}}{4(T+1)^{5/2}} + O(\epsilon^{5/2})\right), \text{ for } T = \mathrm{Tr}(\mathbb{I}_s)$$

Unusually, it has $\sqrt{\epsilon}$ scaling. But there many others:

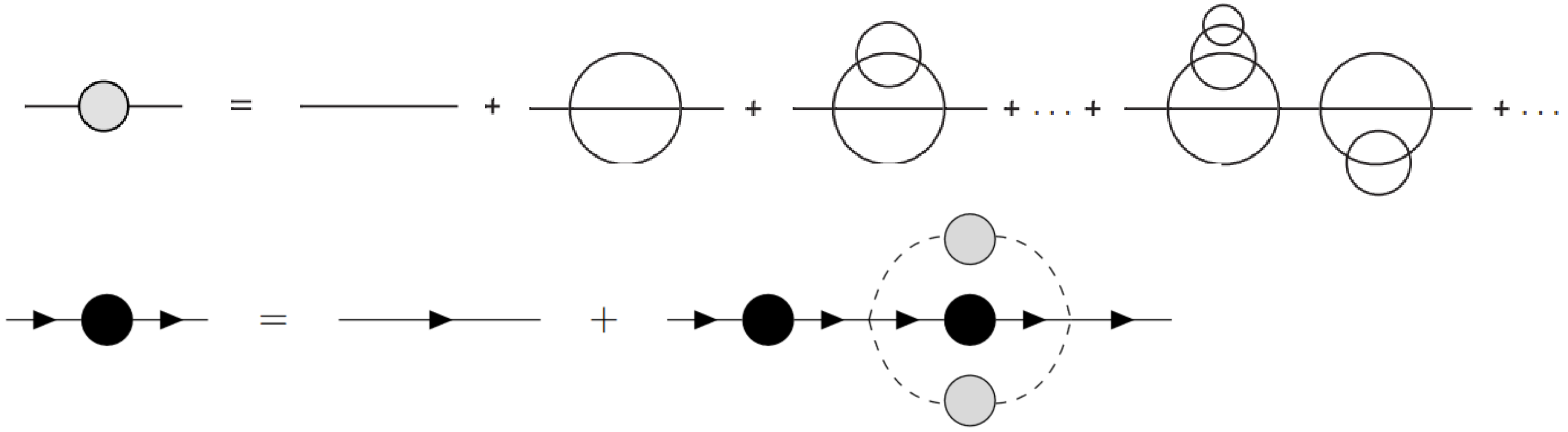


Flow towards the IR for the bosonic sector ϕ^6, $\sqrt{\epsilon} = 0.0125$, $T = 3$
$\lambda_{\text{melonic}+}$
$h\lambda_{\text{prismatic}+}$
trivial
$h_{\text{prismatic}}$
$\lambda_{\text{melonic}-}$
$h\lambda_{\text{prismatic}-}$
0.02
0.01
0.00
−0.01
−0.02
λ_t
−0.010
−0.005
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
h_1

Now some exact results:

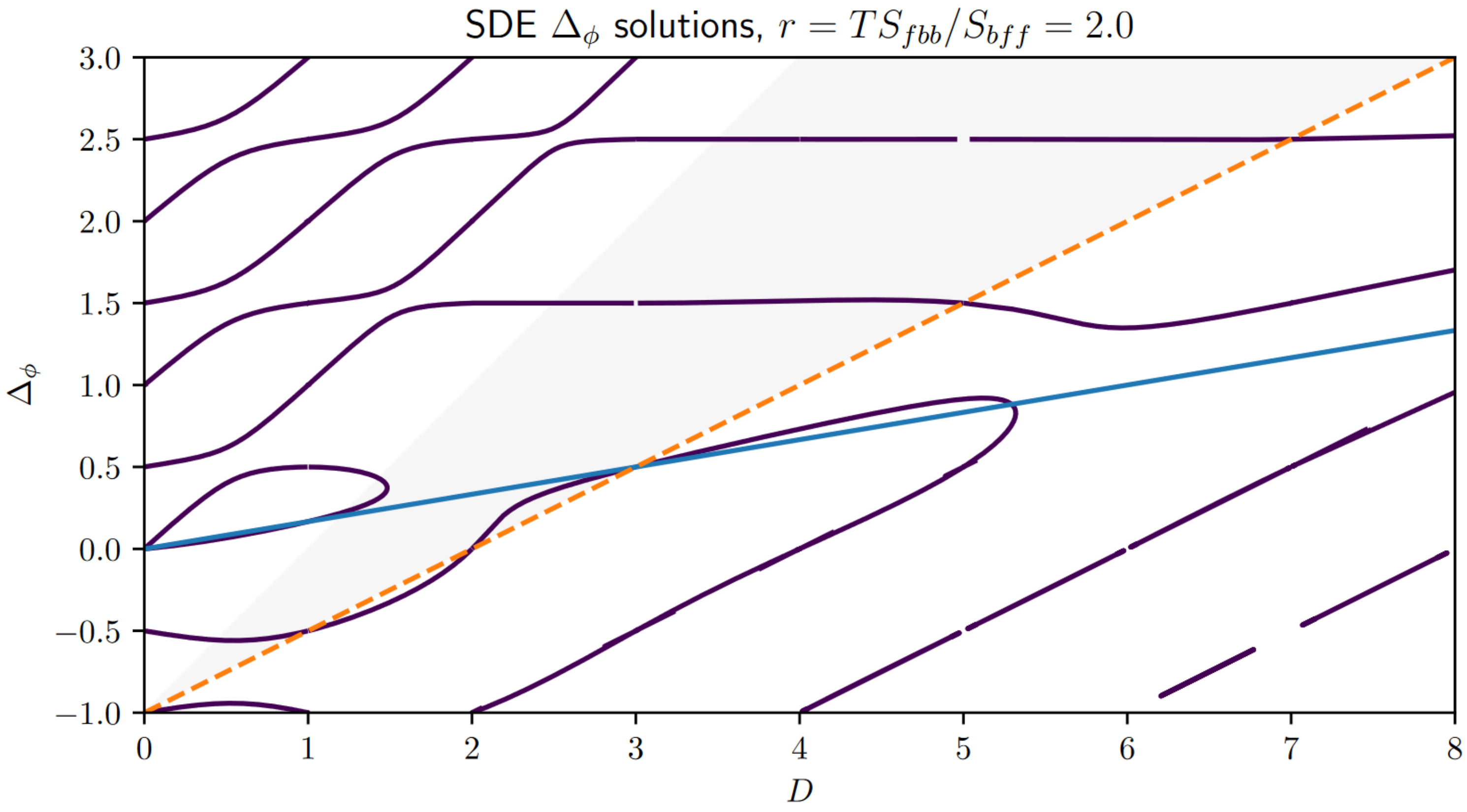
CFT data completely summarised by $\{\Delta_i, J_i, C_{ijk}\}$.

Melonic truncation permits solution of Schwinger-Dyson equations [Klebanov et al., 2017]:



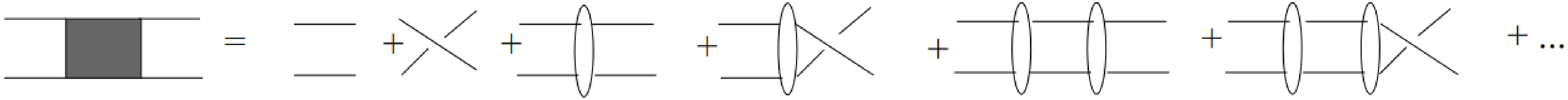
So; ansatz a conformal solution $= \frac{A}{|x|^{2\Delta}}$, and solve in the IR limit.

Conformal dimensions



$$\gamma_\phi = \frac{\epsilon}{3} + \frac{\epsilon^2}{81} + \frac{(428 - 27\pi^2)\,\epsilon^3}{8748} + \left(-\frac{7\zeta(3)}{108} - \frac{4\pi^2}{2187} + \frac{6299}{59049}\right)\epsilon^4 + O\left(\epsilon^5\right)$$

Truncation: solving the rest.



$$= \frac{1}{1 - \mathbb{K}_{4\mathrm{pt}}}$$

Roughly: if we have an operator of the form $:\phi\partial^{2n}\phi:$, then we can connect each leg. If we know a finite expression for the kernel, we can just diagonalise that, and solve for the unit eigenvalues of this integral operator , i.e. $k(\Delta) = 1$:

~~“Acting with the kernel once shouldn’t affect this CFT operator.”~~

Finally: unpleasant expressions, exact answers

For spin zero, the equations for the $\{\Delta_i, c_{ijk}\}$s are relatively compact:

$$g_{s=0}(BB, F\bar{F}_{\not{x}}) \equiv -\frac{\Gamma\left(\Delta\phi+\frac{1}{2}\right)\Gamma\left(\frac{D+1}{2}+\Delta\phi\right)\Gamma\left(\frac{1}{2}(-2\Delta\phi+\tau+1)\right)\Gamma\left(\frac{1}{2}(D-2\Delta\phi-\tau+1)\right)}{\Gamma\left(\frac{1}{2}-\Delta\phi\right)\Gamma\left(\frac{1}{2}(D-2\Delta\phi+1)\right)\Gamma\left(\Delta\phi+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(D+2\Delta\phi-\tau+1)\right)}$$

$$-\frac{6T\Gamma\left(\frac{D+1}{2}+\Delta\phi\right)^2\Gamma\left(\frac{D}{2}-\Delta\phi\right)^2\Gamma\left(\Delta\phi-\frac{\tau}{2}\right)\Gamma\left(\frac{1}{2}(-2\Delta\phi+\tau+1)\right)\Gamma\left(\frac{1}{2}(D-2\Delta\phi-\tau+1)\right)\Gamma\left(-\frac{D}{2}+\Delta\phi+\frac{\tau}{2}\right)}{\Gamma\left(\frac{1}{2}-\Delta\phi\right)^2\Gamma(\Delta\phi)^2\Gamma\left(\Delta\phi+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(D+2\Delta\phi-\tau+1)\right)\Gamma\left(D-\Delta\phi-\frac{\tau}{2}\right)\Gamma\left(\frac{1}{2}(D-2\Delta\phi+\tau)\right)}$$

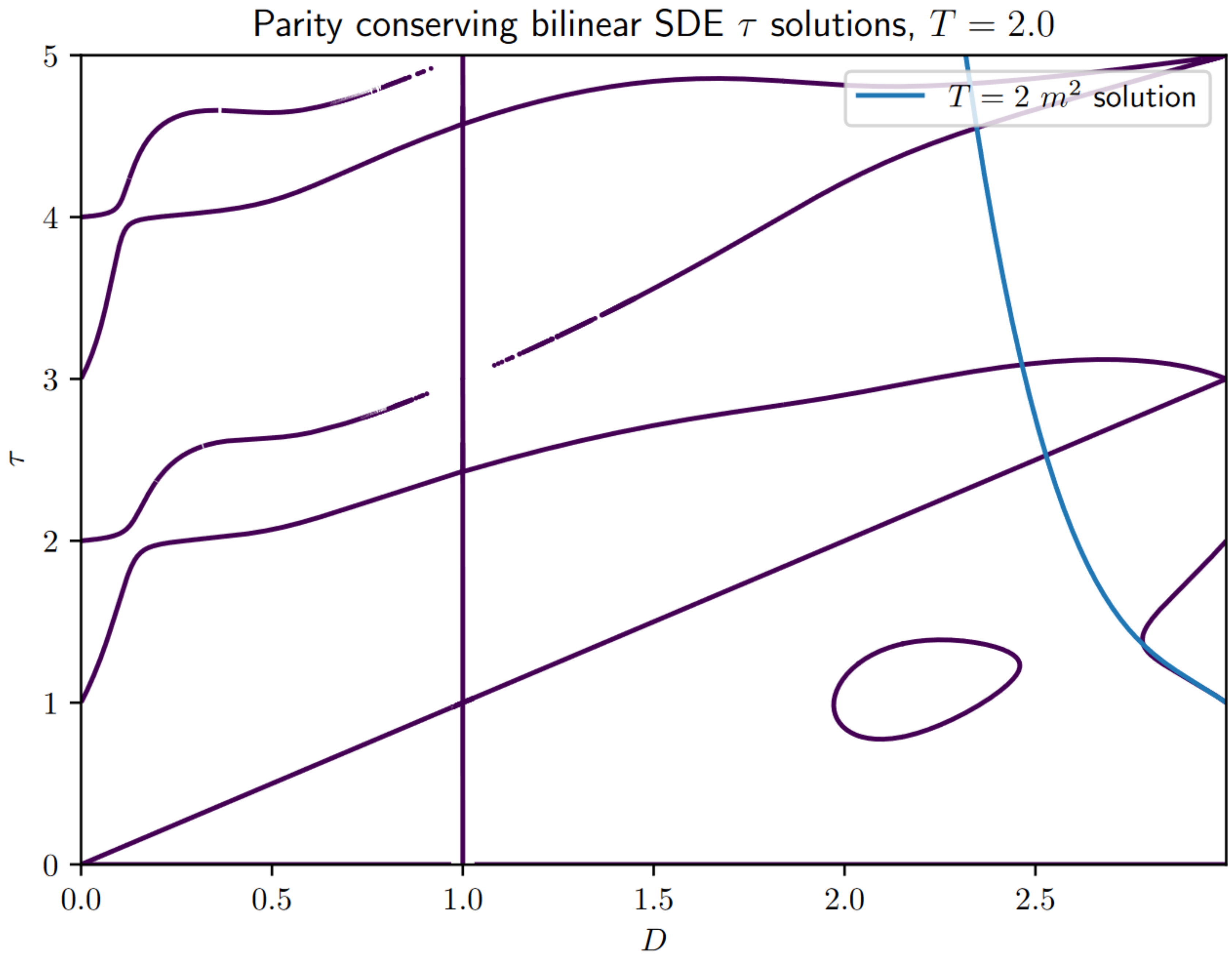
$$+\frac{2T((\pi\Delta\phi)\cos)\Gamma\left(\frac{1}{2}(D-2\Delta\phi+1)\right)\Gamma\left(\frac{D+1}{2}+\Delta\phi\right)\Gamma\left(\frac{D}{2}-\Delta\phi\right)^2\Gamma\left(\Delta\phi-\frac{\tau}{2}\right)\Gamma\left(-\frac{D}{2}+\Delta\phi+\frac{\tau}{2}\right)}{\pi\Gamma(\Delta\phi)^2\Gamma\left(D-\Delta\phi-\frac{\tau}{2}\right)\Gamma\left(\frac{1}{2}(D-2\Delta\phi+\tau)\right)}$$

$$c_{\phi\phi(J,m)} = \mu^d_{\Delta_\phi}\left(\Delta_{m,J}, J\right)\operatorname{Res}\left[\frac{1}{1-g(\Delta,J)}\right]_{\Delta=\Delta_{m,J}} = \frac{\mu^d_{\Delta_\phi}\left(\Delta_{m,J}, J\right)}{g'\left(h_{m,J}, J\right)}$$

Function of *continuous* spin [Kravchuk, Simons-Duffin ‘19].

Find the dimensions of the spin-zero mixing family of operators, all of which mix nontrivially:

$$\{:\bar{\psi}(\partial^2)^n\ \not{\partial}\psi:, :\phi(\partial^2)^n\phi:\}$$


Parity conserving bilinear SDE τ solutions, $T = 2.0$
$T = 2\ m^2$ solution
τ
D
0
1
2
3
4
5
0.0
0.5
1.0
1.5
2.0
2.5

Takeaways

- The large N expansion is an invaluable tool to understand theories beyond the weak coupling perturbative picture.

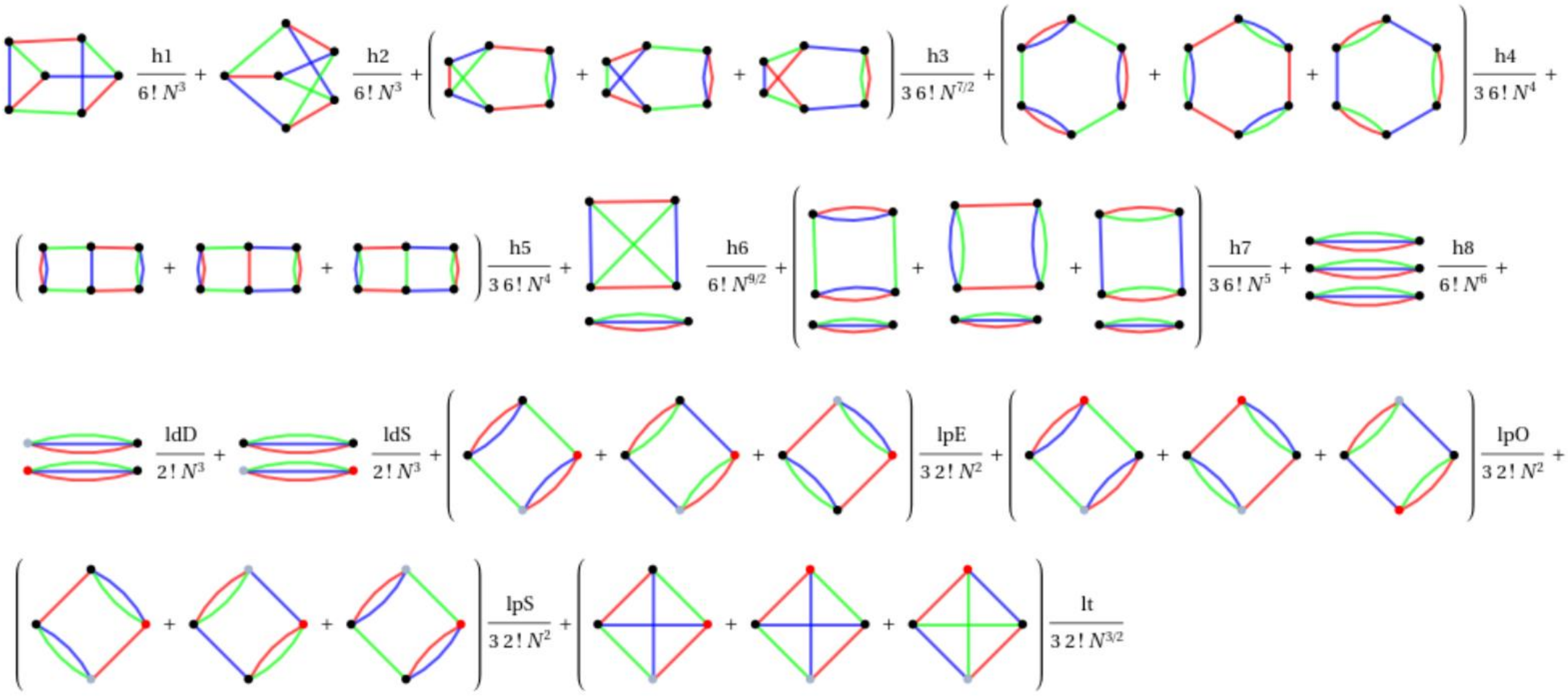
- Recent identification of a novel large-N expansion, applicable to QFTs of various dimensions: the melonic limit.

- This melonic expansion gives strong control over the theory in certain limits. This permits exploration of the space of CFTs.

- And, in particular – access to melonic CFTS and their full set of data.

Thank you

Backup: the full potential



Backup: other known melonic CFTs

Model	Fermions	Symmetry	Key features
GW [1]	Real	$O(n)^{\frac{D(D+1)}{2}}$	SYK-like without disorder
KT [2]	Real	$O(N)^3$	Scaling dimensions, spectrum
KT [2]	Complex	$O(N)^3$	Scaling dimensions, SYK-like
$Sp(N)$ [3]	Complex	Irreducible	Non-zero tetrahedron
		rank-3 $Sp(N)$	SYK-like
Higher rank [2,4]	Real	$O(N)^{q-1}$	Spectrum, chaos, bulk
		$q \geq 6$	Growing number of invariants

[1] [Gurau, Witten,...] [2] [Klebanov, Tarnopolsky, Giombi, Kim, Milekhin, Pallegar, Popov, Zhao,...]
[3] [Carrozza, Pozsgay] [4] [Choudhury, Minwalla,...]

[Harribey '22]

Backup: CFT properties

Model	Sym.	d	FP	Stable	Unitary	NLO
CTKT [1]	$O(N)^3$	$4-\epsilon$	Complex	×	×	stable FP
BGHS [2]	$O(N)^3$	$d<4$	Real	√	√	Non-unitary
Prismatic [3]	$O(N)^3$	$3-\epsilon$	Real	√	√	√
Sextic [4]	$U(N)^3$	$3-\epsilon$	Real	√	×	?
Sextic [4]	$U(N)^3$	$d<3$	Real	√	?	?
Rank 5 [4]	$O(N)^5$	$3-\epsilon$	Trivial	-	-	-

[1][Carrozza,Tanasa,Klebanov,Tarnopolsky,Giombi,...] [2][Benedetti, Gurau, SH, Suzuki]
[3][Giombi,Klebanov,Popov,Prakash,Tarnopolsky] [4][Benedetti,Delporte, SH, Sinha]

[Harribey ‘22]