A brief geometrical introduction to the Penrose Transform

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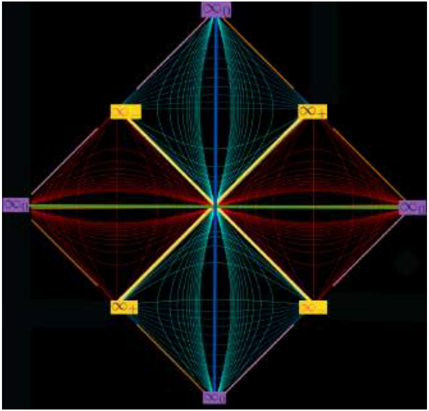
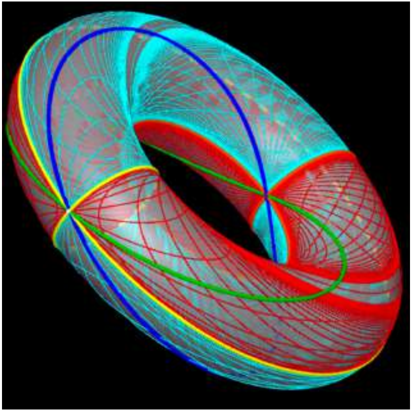
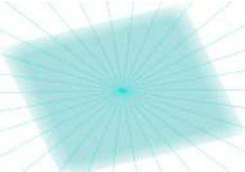




Double Fibration

$$\begin{array}{ccc} & F_{12} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{CP}^3 & & G_{2,4}(\mathbb{C}) \end{array}$$

Considering $\pi_1(S_1, S_2) = S_1$ y $\pi_2(S_1, S_2) = S_2$ we establish a correspondence, by taking $A \subset \mathbb{CP}^3$ to $c(A) := \pi_2(\pi_1^{-1}(A)) \subset G_{2,4}(\mathbb{C})$ and reciprocally, for $B \subset G_{2,4}(\mathbb{C})$ we take $c^{-1}(B) := \pi_1(\pi_2^{-1}(B)) \subset \mathbb{CP}^3$.



The space of null-lines

$$\Sigma(Z^0, Z^1, Z^2, Z^3) := Z^0\bar{Z}^2 + Z^1\bar{Z}^3 + Z^2\bar{Z}^0 + Z^3\bar{Z}^1$$

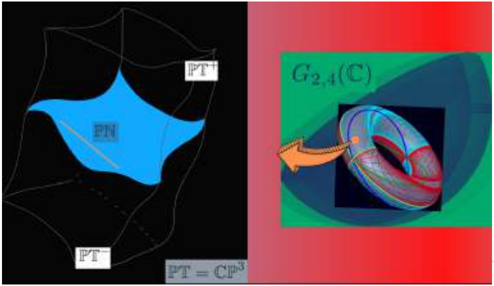
Definition

We define $\mathbb{PN}$, $\mathbb{PT}^+$ *and* $\mathbb{PT}^-$ *as*

$$\mathbb{PN} := \{[Z^0 : Z^1 : Z^2 : Z^3] \in \mathbb{PT} \mid \Sigma(Z^0, Z^1, Z^2, Z^3) = 0\}$$

$$\mathbb{PT}^+ := \{[Z^0 : Z^1 : Z^2 : Z^3] \in \mathbb{PT} \mid \Sigma(Z^0, Z^1, Z^2, Z^3) > 0\}$$

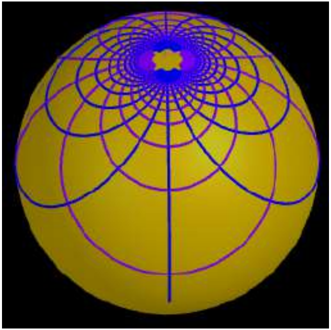
$$\mathbb{PT}^- := \{[Z^0 : Z^1 : Z^2 : Z^3] \in \mathbb{PT} \mid \Sigma(Z^0, Z^1, Z^2, Z^3) < 0\}$$



Light rays

Proposition

There is a 1-1 correspondence between light-rays in real space-time and proyective twistors in $\mathbb{PN}$. Hence $\mathbb{PN}$ is the space of all space-time light-rays. Moreover, $L_{\vec{x}} := c^{-1}(\mathcal{E}(\vec{x}))$ is intrinsically the celestial sphere of the event $\vec{x}$.



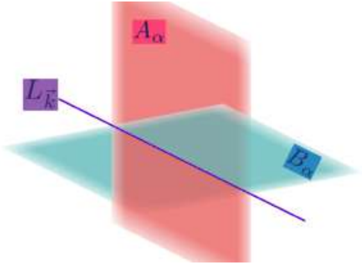
Twistorial solution to the wave equation

I) We begin by choosing $[A_\alpha], [B_\alpha] \in \mathbb{PT}^*$ to define

$$f(Z^\alpha) := \frac{1}{A_\alpha(Z^\alpha)B_\alpha(Z^\alpha)}$$

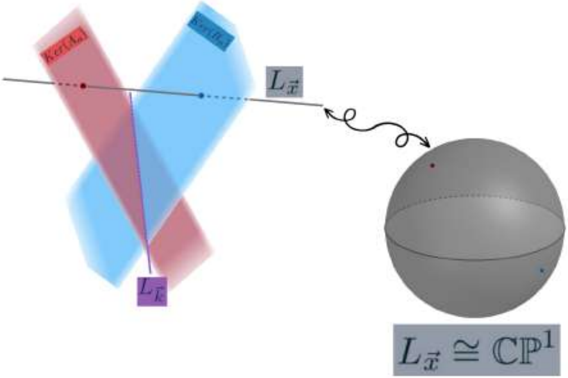
writing $A_\alpha(Z^\alpha) = A_0 Z^0 + A_1 Z^1 + A_2 Z^2 + A_3 Z^3$.

We denote $L_{\vec{k}} := Ker(A_\alpha) \cap Ker(B_\alpha)$



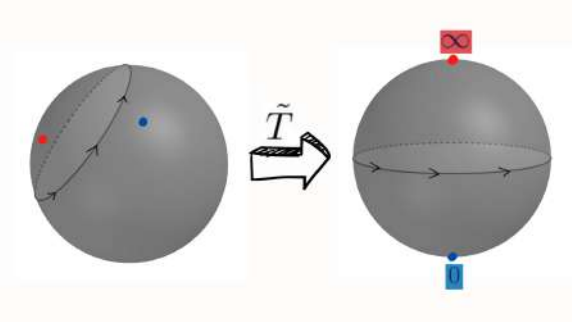
Twistorial solution to the wave equation

II) Then, fixing the previous planes, we take $x \in G_{2,4}$ and *restrict* f to L_x , so $f|_{L_x}$ is a meromorphic function over the Riemann sphere with two poles.



Twistorial solution to the wave equation

II) Then we consider a contour between the two poles and compute *the residue* of $f|_{L_x}(Z^2 dZ^3 - Z^3 dZ^2)$ on one of them, the result will be a complex number.



Twistorial solution to the wave equation

III) Varying x (but keeping k fixed) we obtain a function from $G_{2,4}$ to the complex numbers. It turns out that this function Is a solution to the wave equation $\Box\phi = 0$!

$$\phi(t, x, y, z) := \frac{1}{2\pi i} \oint_{\mathcal{C}} f|_{L_{\vec{x}}}(Z^2, Z^3)(Z^2 dZ^3 - Z^3 dZ^2)$$

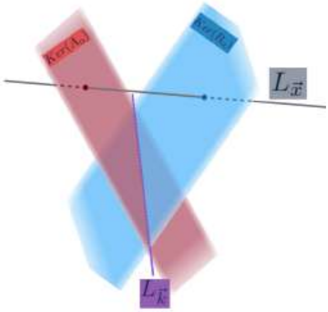
This way of assigning to every homogeneous polynomial of degree -2 a solution to the complexified wave equation is known as the **Penrose Transform**.

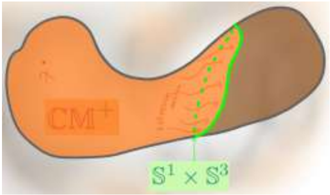
Twistorial solution to the wave equation

To assure that ϕ is well-defined, we restrict the domain to the **future cone**

$$\mathbb{CM}^+ := \{x \in G_{2,4}(\mathbb{C}) : L_x = c^{-1}(\{x\}) \subset \mathbb{PT}^+\}$$

so we have that for every $x \in \mathbb{CM}^+ \quad L_x \cap L_k = \varnothing$.





Twistorial solution to the wave equation

The correspondence between fields↔homogeneous polynomials of degree -2 is not 1 a 1 so there is some redundancy

$$\oint_{\mathcal{C}\subset\mathbb{CP}^1} (f+g^{+})|_{L_{\vec{x}}}(Z^2,Z^3)(Z^2dZ^3 - Z^3dZ^2)$$

$$= \oint_{\mathcal{C}\subset\mathbb{CP}^1} f|_{L_{\vec{x}}}(Z^2,Z^3)(Z^2dZ^3 - Z^3dZ^2)$$

$$\text{moreover } f_{01} \sim f_{01} + (h_1 - h_0)$$

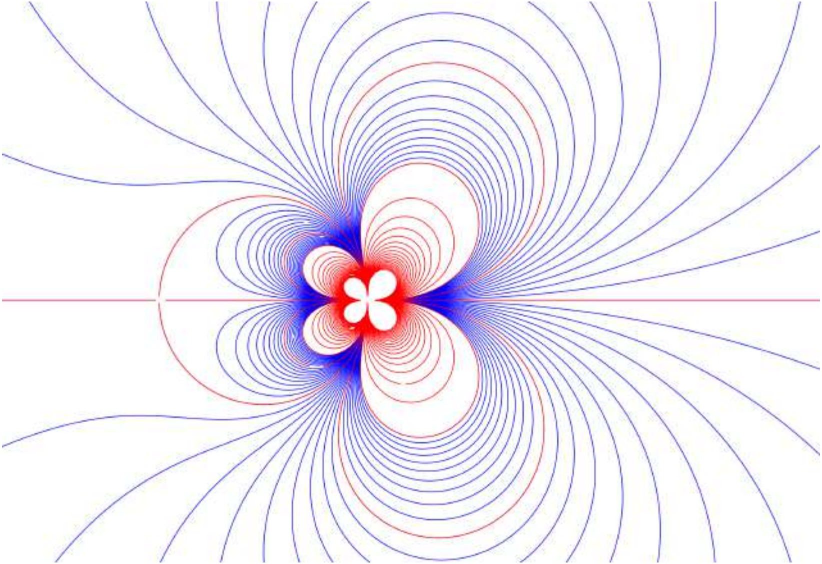
so that

Theorem

The geometric procedure defined by the integral formula leads to an isomorphism

$$\check{\mathrm{H}}^1(\mathbb{PT}^+, \mathcal{O}_{\mathbb{PT}^+}(-2)) \cong \{\textit{Holomorphic positive frequency solutions}$$

$$\textit{to complexified wave equation } \bar{\mathcal{M}}^{1,3}_{\mathbb{R}}\}$$



¡Thanks for your attention!

References

Atiyah, M. (1979). *Geometry of yang-mills fields* (1°). Academia Nazionale dei Lincei Scuola Normale Superiore.

Eastwood, M. (1951). Twistor theory (the penrose transform). *Mathematical Institute, Oxford*, 1–11.

Huggett, S. (2022). *Twistor geometry*. http://stephenhuggett.com/Twistorgeometry.pdf

Huggett, S., & Tod, K. (1985). *An introduction to twistor theory* (2°). Cambridge University Press.

Penrose, R. (1979). On the twistor descriptions of massless fields. *Twistor Newsletter*.

Penrose, R. (2004). *El camino a la realidad* (5°). Debate.

Schottenloher, M. (2008). *A mathematical introduction to conformal field theory* (2°). Springer.

Ward, R., & Wells, R. (1990). *Twistor geometry and field theory* (1°). Cambridge University Press.

Penrose transform

Writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{i}{\sqrt{2}}\right)\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix}\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} + \begin{pmatrix} A_2 & A_3 \\ B_2 & B_3 \end{pmatrix}$$

our integrand is

$$\omega := f|_{L_{\vec{x}}}(Z^2, Z^3)(Z^2 dZ^3 - Z^3 dZ^2) = \frac{(Z^2 dZ^3 - Z^3 dZ^2)}{(aZ^2 + bZ^3)(cZ^2 + dZ^3)}$$

it turns out that

$$\frac{1}{2\pi i}\oint_{\mathcal{C}\subset\mathbb{CP}^1} \omega = \frac{1}{2\pi i}\oint_{\tilde{\mathcal{C}}\subset\mathbb{C}} \frac{1}{ad-bc}\frac{d\xi}{\xi} = \frac{1}{ad-bc}$$

hence

$$\boxed{\phi(t,x,y,z) = \frac{2}{(A_1B_0 - A_0B_1)||\vec{x}-\vec{k}||^2_{1,3}}}$$

Twistorial solution to the zero-rest-mass field equations

Theorem

(Penrose Transform) *There is a 1-1 correspondence*

$$\check{\mathrm{H}}^1(\mathbb{PT}^+, \mathcal{O}(-2s-2)) \leftrightarrow$$

$$\{\textit{Holomorphic solutions to the zero-rest-mass field equations with helicity } s \textit{ in } \mathbb{CM}^+\}$$

Invariant interpretation

1. We fix two planes $A, B \subset \mathbb{PT}$ so that $L_k := A \cap B \subset \mathbb{PT}^-$. And fix $[f] \in \check{\mathrm{H}}^1(\mathbb{PT}^+, \mathcal{O}_{\mathbb{CP}^3}(-A - B))$.
2. For every $x \in \mathbb{CM}^+ \subset G_{2,4}(\mathbb{C})$ we take the pull-back

$$\iota_x^* \mathcal{O}_{\mathbb{CP}^3}(-A - B) = \mathcal{O}_{L_x \cong \mathbb{CP}^1}(-\tilde{a} - \tilde{b})$$

   where $\tilde{a} = A \cap L_x$ y $\tilde{b} = B \cap L_x$. Also $[f]$ induces an element $\iota_x^*[f] \in \check{\mathrm{H}}^1(L_x, \mathcal{O}_{L_x}(-\tilde{a} - \tilde{b}))$. As ι_x is an inclusion this just *restricts* $[f]$ to L_x.

Invariant interpretation

- The **Serre Duality** tells us that

$$\boxed{\check{\mathrm{H}}^1(L_x, \mathcal{O}_{L_x}(-\tilde{a}-\tilde{b})) \cong \check{\mathrm{H}}^0(L_x, K_{L_x}(\tilde{a}+\tilde{b}))^*}$$

Also it turns out that $K_{L_x}(\tilde{a}+\tilde{b}) := T^*(\mathbb{CP}^1) \otimes [\tilde{a}+\tilde{b}]$ ¡is trivial! hence

$$\boxed{\check{\mathrm{H}}^1(L_x, \mathcal{O}_{L_x}(-\tilde{a}-\tilde{b})) \cong \check{\mathrm{H}}^0(L_x, K_{L_x}(\tilde{a}+\tilde{b}))^* \cong \check{\mathrm{H}}^0(L_x, \mathcal{O}_{L_x}) \cong \mathbb{C}}$$

3. With this **The Penrose transform is the homomorphism** $P : \check{\mathrm{H}}^1(\mathbb{PT}^+, \mathcal{O}_{\mathbb{CP}^3}(-A-B)) \to \check{\mathrm{H}}^0(\mathbb{CM}^+, \mathcal{O}_{\mathbb{CM}^+})$ given by

$$P([f])(x) := S(\iota_x^*[f])$$

whose image consists precisely of the sections that are a solution to the wave equation.