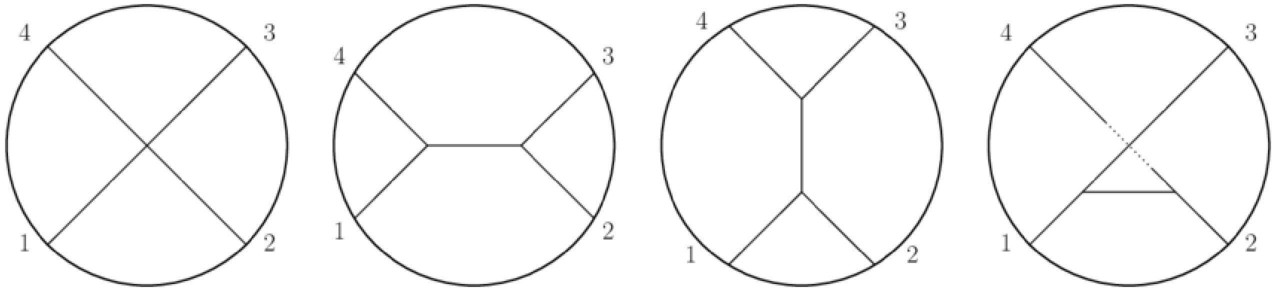

Conformal 4-Point Functions

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OPE and Bootstrapping

The operator product expansion (OPE) says the product of two fields located is equal to a sum over the spectrum of primary operators acted on by a derivative operator:

$$\phi(x_1)\phi(x_2) = \sum_{\Delta,l} c_{\phi\phi O^{(l)}}\, C\left(x_{12}, \partial_{x_2}\right)_{\mu_1 \ldots \mu_l} O^{(l)}_{\mu_1 \ldots \mu_l}(x_2)$$

Then, by using the well-known forms of the 2- and 3-point functions, the 4 point function can be expressed as [Dolan and Osborn, 2001]:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\Delta,l} \frac{1}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}} c^2_{\phi\phi O^{(l)}} G^{(l)}(\Delta, l, u, v)$$

For a free theory, we can just do Wick contractions to compute the LHS and by comparing orders in u and v, we can compute the OPE coefficients. For interacting theories, things become more complicated.

Bootstrapping involves determining the consequences of the symmetries of the CFT and then imposing constraints based on these [Ferrara, Grillo, Gatto 1973, Polyakov 1974]. The crossing symmetry equation is obtained by bootstrapping:

$$\sum_{\Delta,l} c^2_{\Delta,l} \left(\frac{v^{\Delta_\phi} G_{\Delta,l}(u,v) - u^{\Delta_\phi} G_{\Delta,l}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} \right) = 1$$

Mellin Space

By the AdS-CFT correspondence, the scattering amplitude of states in string theory/supergravity on AdS is equal to correlation functions in the dual CFT. Even to $O\left(\frac{1}{N^2}\right)$ and in the supergravity limit, the expressions in real space are very complicated and only a handful of correlators have been computed explicitly.

However, by instead looking at the Mellin transform of the correlator, vastly simpler equations can be found. The Mellin transform of the interacting part of $\langle O_{p_1} O_{p_2} O_{p_3} O_{p_4} \rangle$ is given by:

$$M(s,t,\sigma,\tau) = \int_0^\infty dV\, V^{-\frac{t}{2}+\frac{p_1+p_4}{2}-1} \int_0^\infty dU\, U^{-\frac{s}{2}+\frac{p_3+p_4}{2}-L-1} G(U,V,\sigma,\tau)$$

where $O_p(x,y) = y^{i_1} \dots y^{i_p}\, Tr\left(\phi_{i_1}(x) \dots \phi_{i_p}(x)\right)$. For type IIB supergravity on $AdS_5 \times S^5$ and to $O\left(\frac{1}{N^2}\right)$, the Mellin amplitude [Rastelli and Zhou, 2016] has been found by bootstrapping.

$$\frac{M(s,t,\sigma,\tau)}{\Gamma_{p_1 p_2 p_3 p_4}} = \sum_{\substack{i+j+k=L-2 \\ 0 \le i,j,k \le L-2}} \frac{a_{ijk}\sigma^i \tau^j}{(s - s_M + 2k)(t - t_M + 2j)(\tilde{u} - u_M + 2i)}$$

where $\Gamma_{p_1 p_2 p_3 p_4}$ is a product of Gamma functions, a_{ijk} is a set of constants, σ and τ are simple functions of y_i and L, s_M, t_M and u_M depend on p_1, p_2, p_3 and p_4.

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Dispersive Sum Rules

The Viraosoro-Shapiro amplitude is a much-researched object in string theory. It is the scattering amplitude of four gravitons. The 4 point function of the stress tensor is equal to the AdS version of the amplitude.

To leading order in $1/c$, the form of the $1/\lambda$ expansion of Mellin amplitude can be found, with undetermined coefficients of the $1/\lambda$ terms.

$$M(s,t) = \frac{8}{(s-2)(t-2)(u-2)} + \sum_{a,b}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \left(\tilde{\alpha}_{a,b} + \frac{1}{\lambda^{\frac{1}{2}}} \tilde{\beta}_{a,b} + \frac{1}{\lambda} \tilde{\gamma}_{a,b} + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right)$$

By making use of the partial wave expansion, there has been recent work carried out to find an expression for the AdS amplitude [Alday, Hansen and Shapiro, 2022].

The derivation of the dispersive sum rules begins by considering the Mellin amplitude divided by a finite product with zeroes at integer values:

$$\frac{M(s,t)}{\Pi_{i=1}^{q}(s-2-2i)(t-2-2i)} = \oint_s ds' \frac{1}{s'-s} \frac{M(s', 4-u-s')}{\Pi_{i=1}^{q}(s'-2-2i)(2-u-s'-2i)}$$

The contour is then deformed to infinity, picking up poles from the Mellin amplitude which appear at $s = \tau + 2m$ and $t = 4 - u - \tau - 2m$.

Dispersive Sum Rules

However, at order $O(1/c)$, there is a bound on the Mellin amplitude, referred to as the bound on chaos:

$$\lim_{s\to i\infty} |M(s,t)| \leq \frac{1}{|s|^2}$$

This bound means that the contour integral at infinity will vanish. Thus, the previous expression above is equal to minus the sum of the residues of the integrand at the poles of $M(s,t)$. Thus, an infinite set of sum relations for the Mellin amplitude are found in terms of OPE data.

$$\frac{M(s,t)}{\Pi_{i=1}^{q}(s-2-2i)(t-2-2i)} = \sum_{\tau,l} c_{\tau,l}^2 \omega_{\tau,l}(s,t;q)$$

$$\omega_{\tau,l}(s,t;q) = \sum_{m=0} \frac{Q_{l,m}^{\tau+4,d=4}(u-4)}{\Pi_{i=1}^{q}(2-2i-2m-u-\tau)(-2-2i+2m+\tau)} \left(\frac{1}{s-\tau-2m} + \frac{1}{t-\tau-2m}\right)$$

With these dispersive sum relations, the corrections can be expressed individually in terms of the OPE data of the stringy operators.

In the same vein as this work, similar dispersive sum relations may be derivable, which may allow different correlators to be computed and this is a possible direction that I may go in for my PhD project.

Thank you for listening!