Worldline Monte Carlo: A numerical method for a first quantised approach to QFT.

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Introducing the Worldline formalism

Path integrals in QFT usually refer to functional integrals over field configurations as in the second quantised approach. In the **worldline formalism**, we study QFT as path integrals over relativistic point particles[1, 2].

- Propagator (scalar QED) - $D^{x'x}[A] = \langle x'|(-(\partial_\mu + ieA_\mu)^2 + m^2)^{-1}|x\rangle$:

$$D^{x'x}[A] = \int_0^\infty dT\, e^{-im^2T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, e^{i\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + e\dot{x}\cdot A(x(\tau))\right]}$$

- The (one-loop) effective action (scalar QED) $i\Gamma^1[A] = \ln \mathrm{Det}[-(\partial_\mu + ieA_\mu)^2 + m^2]$:

$$i\Gamma^1[A] = \int_0^\infty \frac{dT}{T}\, e^{-im^2T} \oint_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{i\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + e\dot{x}\cdot A(x(\tau))\right]}.$$

There exist different methods for the calculation of worldline path integrals, two of them are:

(i) Analytic Gaussian evaluation or "string-inspired" approach, based on the use of worldline Green functions.

(ii) A direct numerical calculation of the path integral (worldline Monte Carlo).

Worldline Monte Carlo

Worldline Monte Carlo expresses the path integrals (in Euclidean space) as an **expectation value** of the Wilson line → suitable for Monte Carlo!

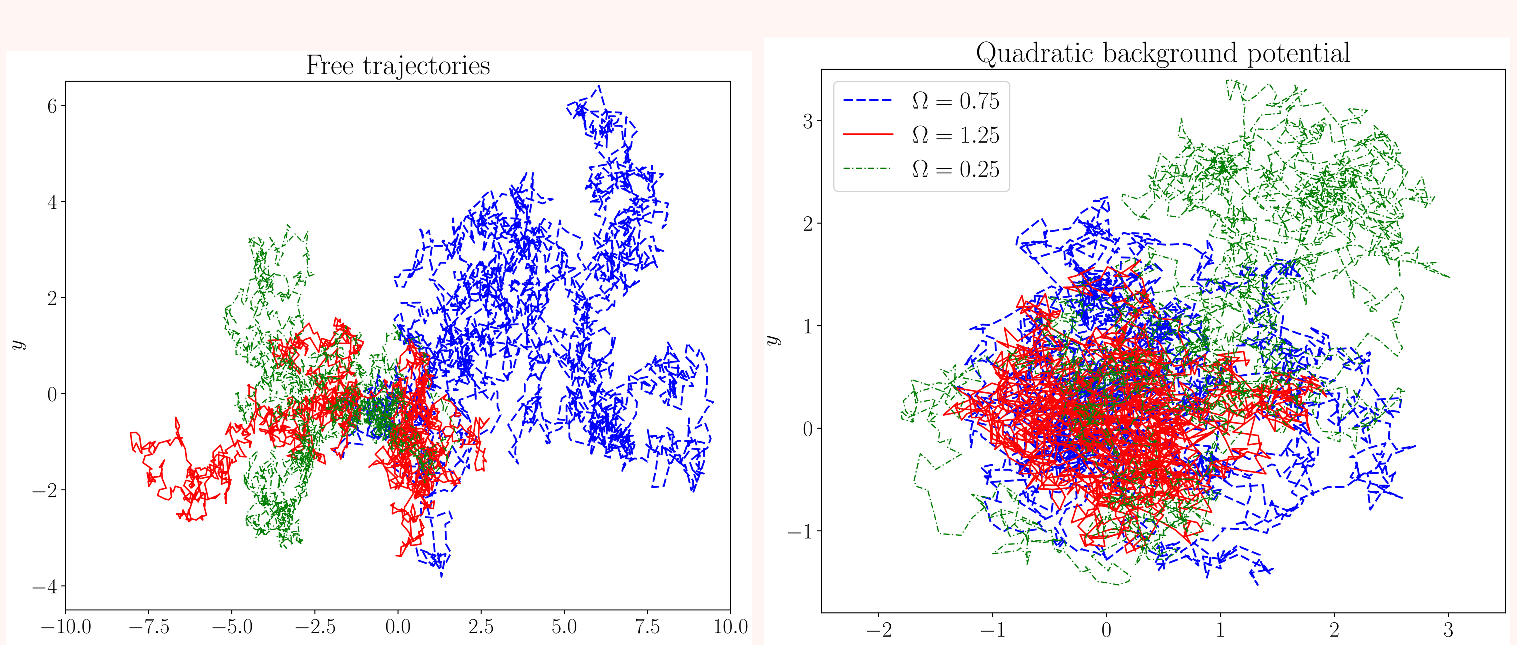
$$\int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + V(x(\tau))\right]} = \left(\frac{1}{4\pi T}\right)^{\frac{D}{2}} e^{-\frac{1}{4T}(x-x')^2} \left\langle e^{-T\int_0^1 du\, V(x(u))} \right\rangle,$$

where we have defined $x(u) = x + (y-x)u + \sqrt{T}q(u)$.
We treat the expectation value as:

$$\left\langle e^{-T\int_0^1 du\, V(x(u))} \right\rangle = \frac{\int \mathcal{D}q\, e^{-T\int_0^1 du\, V(x(u))}\, \mathcal{P}[\{q(u)\}]}{\int \mathcal{D}q\, \mathcal{P}[\{q(u)\}]};\ \mathcal{P}[\{q(u)\}] \propto \exp\left(-\frac{1}{4}\int_0^1 du\, \dot{q}^2\right).$$

Worldline Monte Carlo: discrete sum over $\mathbf{N_L}$ loops, and discretise each one at $\mathbf{N_p}$ proper time points:

$$\int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + V(x(\tau))\right]} \overset{\text{M.C.}}{\to} \frac{e^{-\frac{(x-x')^2}{4T}}}{(4\pi T)^{\frac{D}{2}}} N_L^{-1} \sum_{\substack{n=1 \\ v_i \sim \wp(v)}}^{N_L} e^{-\frac{T}{N_p}\sum_{i=1}^{N_p} V\left(x_n(\frac{i}{N_p})\right)}$$



(a) (b)

Effective action in scalar QED

To illustrate the method, consider the unrenormalized Euclidean one-loop effective action in the worldline representation, where the Wilson loop is introduced[3]:

$$\Gamma^1[A] = \frac{1}{(4\pi)^{D/2}} \int d^Dx_0 \int_{1/\Lambda^2}^\infty \frac{dT}{T^{(D/2)+1}} e^{-m^2T} \langle W \rangle + \text{c.t.};\ W = e^{-ie\int_0^T d\tau\, \dot{x}\cdot A(x(\tau))}.$$

For a constant external magnetic field $\mathbf{B} = B\hat{e}_z$, $A_\mu = (0,0,Bx,0) = (0, Bx\hat{e}_y)$ and in $D=4$ we have:

$$\Gamma^1 = \frac{1}{(4\pi)^2} \int_{1/\Lambda^2}^\infty \frac{dT}{T^3} e^{-m^2T} \int d^4x_0 \left(\frac{eBT}{\sinh(eBT)} - 1\right) + \text{c.t.};\ \langle W \rangle = \frac{eBT}{\sinh(eBT)}.$$

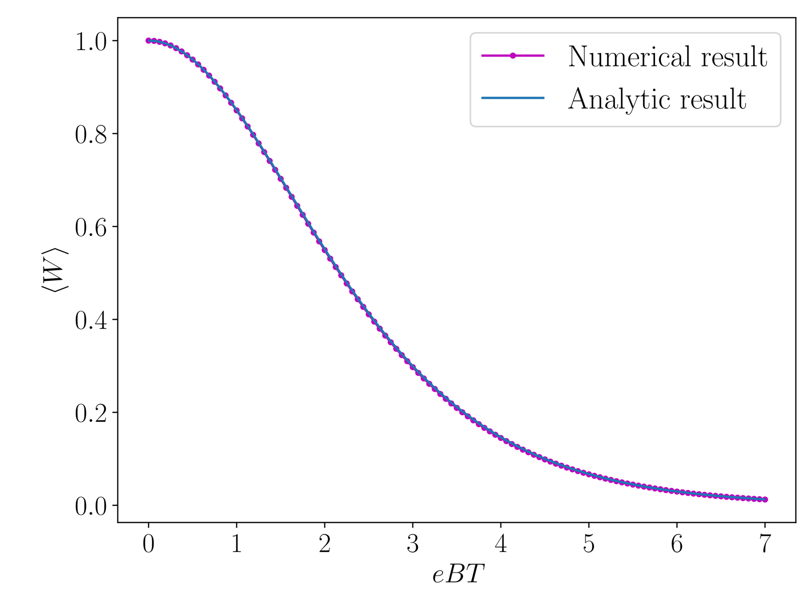


Figure 1. Expectation value of the Wilson loop for a constant magnetic field B.

Quantum mechanics

Worldline MC can provide good estimations of propagators for various systems (for T small) and allow us to give **physical** 'predictions'. For systems with energies bounded from below, the spectral decomposition of the propagator is

$$K(x', x; T) \overset{T\to\infty}{\sim} \psi_0(x')\psi_0^\star(x) \mathrm{e}^{-TE_0}$$

where E_0 is the ground state energy and $\psi_n(x) := \langle x|\Psi_n\rangle$ are energy eigenfunctions. This spoils estimation of E_0 as the asymptotic gradient

$$E_0 = -\lim_{T\to\infty} \frac{\partial}{\partial T} \ln(K(x', x; T)).$$

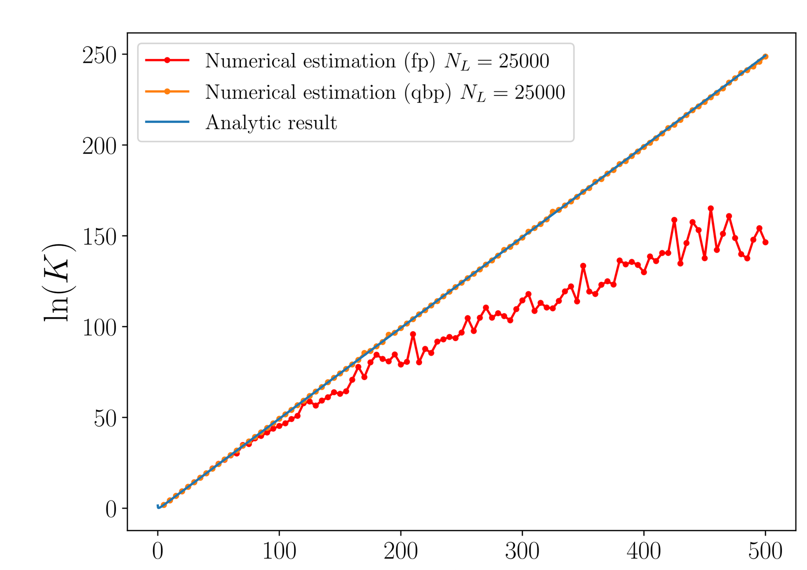


Figure 2. $\ln(K(x', x; T))$ for the Pöschl-Teller potential $V_\lambda(x) = -\frac{\alpha^2}{2m}\frac{\lambda(\lambda+1)}{\cosh^2(\alpha x)}$, ($E_0 = -0.50004391$).

Undersampling problem

Worldline MC suffers from a large time **undersampling** problem[4]:

- Brownian motion trajectories spread as $\sqrt{T}q(u)$, if $T >> 1$ the trajectories no longer sample the potential well.

Late time deviation from linearity ⇒ hard to estimate energies/wavefunctions.

A solution

A background potential can be subtracted directly in the path integral interaction to favor trajectories that better sample the system[5].

$$K = K_\Omega \left\langle e^{-\int_0^T d\tau \left[V(x(\tau)) - \frac{1}{2}m\Omega^2x^2(\tau)\right]} \right\rangle_\Omega \overset{\text{M.C.}}{\to} K_\Omega \sum_{\substack{i=1 \\ \nu_i^\Omega \sim \widetilde{\wp}(v)}}^{N_L} \mathrm{e}^{-\nu_i^\Omega}$$

Grassmann path integrals

In QFT fermions are usually treated via integrals over anti-commuting Grassmann variables. In 1998, M. Creutz presented a method for a direct numerical evaluation of Grassmann path integrals of the form[6]:

$$Z = \int d\psi_N \psi_{N-1} \ldots d\psi_1\, e^{S(\psi)}$$

We implemented the Creutz algorithm for the partition function of a fermionic harmonic oscillator as a benchmark test:

$$Z = \int_A \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^\beta d\tau\, (\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi)} \to e^{\frac{\beta}{2}} \lim_{N\to\infty} \int \left(\prod_{k=1}^N d\bar{\psi}_k d\psi_k\right) e^{-\sum_{k=1}^N \left[\bar{\psi}_k(\psi_k - \psi_{k-1}) + \frac{\beta}{N}\bar{\psi}_k\psi_{k-1}\right]}$$

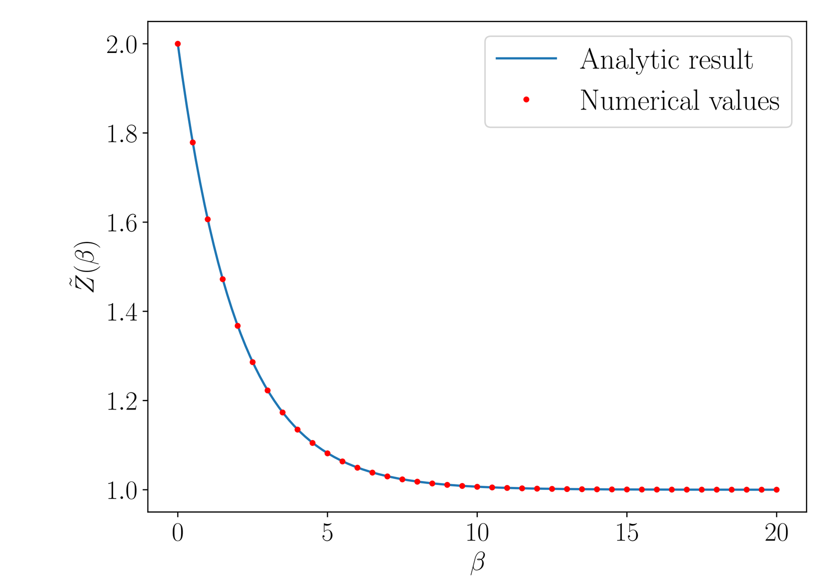


Figure 3. Estimation of $\tilde{Z}(\beta) = e^{-\beta/2}Z(\beta)$ for a fermionic harmonc oscillator ($\omega = 1$).

Further work

We would like to directly evaluate expressions like the effective action in spinor QED:

$$\Gamma_{spin}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int_P \mathcal{D}x\, e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x(\tau))\right)} S[x, A]$$

$$S[x, A] = \mathrm{tr}_\Gamma\, \mathcal{P} \exp\left[-i\frac{e}{4}[\gamma^\mu, \gamma^\nu] \int_0^T d\tau\, F_{\mu\nu}(x(\tau))\right] = \int_A \mathcal{D}\psi\, e^{-\int_0^T d\tau \left(\frac{1}{2}\psi_\mu\dot{\psi}^\mu - ie\psi^\mu F_{\mu\nu}(x(\tau))\psi^\nu\right)}$$

Conclusions

The WMC method has proven to be fast and efficient, generating an ensemble of trajectories that is independent from the potential, and thus universally applicable. However, like any other numerical method, it has limitations, such as the lost of precision at large times. A solution for this problem has been presented only recently. We intend to extend the scope of the method to be able to evaluate path integrals with Grassmann variables in spinor QED.

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