Quark Mass Anomalous Dimension in the mMOM Scheme to the Five Loop Level

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Abstract

We calculate the five-loop contribution to the quark bi-linear, ghost, gluon and quark field anomalous dimension for the minimal momentum subtraction (mMOM) scheme when fixed in the linear covariant gauge. The coupling constant gauge parameter plane is then analysed for fixed points with the corresponding critical exponents evaluated. which is used to consider bounds on the conformal window. Based on: J.A. Gracey R.H. Mason, J. Phys. A56 (2023), 085401 [arXiv:2210.14604]; and J.A. Gracey, R.H. Mason, T.A. Ryttov R.M. Simms, Phys.Rev.D 108 (2023) 4, 045006 [arXiv:2306.09056].

The mMOM Scheme

When loop corrections are included in the calculation of physical variables in quantum field theory, unconstrained loop momenta lead to divergent integrals and infinite values for measurable quantities. We can regularise these divergences with dimensional regularisation by setting the number of spacetime dimensions d to $4-2\epsilon$. Calculations are then renormalized by rescaling variables such that the dependence on ϵ^{-n}, where $n>0$, is removed in all calculations so the limit $\epsilon \to 0$ can be taken to regain our original system. During this process the ϵ^0 part can also be modified, and the way this is done is dictated by the renormalization scheme. Due to renormalization group invariance a measurable quantity should not be dependent on the scheme to order in truncation.

A typical class of schemes are the MOM schemes in which the finite parts of the field renormalization constants are defined such that there are no loop corrections to the two-point functions when evaluated at a characteristic renormalization scale. However, this does not provide a definition for the coupling anomalous dimension. In the mMOM scheme [1] we use

$$Z_g^{\overline{\text{MS}}}\sqrt{Z_A^{\overline{\text{MS}}}}Z_c^{\overline{\text{MS}}} \;=\; Z_g^{\text{mMOM}}\sqrt{Z_A^{\text{mMOM}}}Z_c^{\text{mMOM}} \tag{1}$$

which is defined such that the observed property that the ghost gluon vertex is finite to all orders in the Landau gauge is extended to include a general gauge parameter α [2], where $\overline{\text{MS}}$ is a scheme where no finite contributions are included in the renormalization constant.

Anomalous Dimensions

Renormalization introduces an unphysical renormalization scale, μ, which the fields and couplings varying according to the equation

$$\gamma_\phi(a,\alpha) = \frac{d\ln Z_\phi}{dl} = \beta(a,\alpha)\frac{\partial \ln Z_\phi}{\partial a} \;+\; \alpha\gamma_\alpha(a,\alpha)\frac{\partial \ln Z_\phi}{\partial \alpha} \tag{2}$$

where $l=\ln\frac{\mu^2}{\Lambda^2}$ and $\phi \in \{A,c,\psi\}$. Under the structure of a scheme change, if we have the anomalous dimension of e.g. the $\overline{\text{MS}}$ scheme to the $N+1$ loop level, then we only have to directly calculate any new one to the N loop level and use the scheme change equation

$$\begin{aligned}\gamma_\phi^{\text{mMOM}}\left(a_{\text{mMOM}},\alpha_{\text{mMOM}}\right) = \Bigg[&\gamma_\phi^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}\right)+\beta^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}}\right)\frac{\partial}{\partial a_{\overline{\text{MS}}}}\ln C_\phi\left(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}\right)\\ &+\alpha_{\overline{\text{MS}}}\,\gamma_\alpha^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}\right)\frac{\partial}{\partial \alpha_{\overline{\text{MS}}}}\ln C_\phi\left(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}\right)\Bigg]_{\substack{\overline{\text{MS}}\to\\ \text{mMOM}}}\end{aligned} \tag{3}$$

to find it to the $N+1$ loop level, where $C_\phi(a,\alpha) \;=\; Z_\phi^{\text{mMOM}}/Z_\phi^{\overline{\text{MS}}}$. We have calculated the renormalization constants to the four loop level by following the definitions of the renormalization constants for the mMOM scheme using Green's functions from [3] and used the five-loop calculations made for the $\overline{\text{MS}}$ scheme [4-6] to find the field and coupling anomalous dimensions in mMOM to five loop, where the β-function had already been presented to this level in [3]. For example the gluon field anomalous dimension is given by

$$\begin{aligned}\gamma_A^{\text{mMOM}}(a,0)\Big|^{SU(3)} = &\left[\frac{2}{3}N_f-\frac{13}{2}\right]a+[...]\,a^2+[...]\,a^3+[...]\,a^4\\
&+\Bigg[\frac{1760}{27}\zeta_5N_f^4-\frac{71363464263}{16384}\zeta_7-\frac{16520894997}{2048}-\frac{16359945025}{6912}\zeta_5N_f\\
&-\frac{8539017539}{20736}\zeta_3N_f+\frac{3150668061}{2048}\zeta_3+\frac{23633674547}{18432}\zeta_7N_f\\
&+\frac{29623505625}{4096}\zeta_5+\frac{327291152977}{124416}N_f-\frac{613783375}{2592}N_f^2\\
&-\frac{25237429}{432}\zeta_7N_f^2-\frac{979887}{64}\zeta_3^2N_f-\frac{292855}{54}\zeta_5N_f^3-\frac{51647}{36}\zeta_3^2N_f^2\\
&-\frac{1861}{27}N_f^4-\frac{304}{27}\zeta_3N_f^4+\frac{2240}{27}\zeta_3^2N_f^3+\frac{129545}{162}\zeta_3N_f^3\\
&+\frac{2420705}{324}N_f^3+\frac{13148441}{2592}\zeta_3N_f^2+\frac{246191965}{1296}\zeta_5N_f^2\\
&+\frac{671260095}{2048}\zeta_3^2\Bigg]a^5 \;+\; O(a^6)\end{aligned} \tag{4}$$

where N_f is the number of active fermions and we have truncated the lower loop orders and presented in SU(3) and the Landau gauge ($\alpha=0$) for brevity, but have been calculated for a general gauge parameter and group. Note the gauge parameter α runs according to the equation $\frac{d\alpha}{dl}=-\alpha\gamma_A(a,\alpha)=\alpha\gamma_\alpha(a,\alpha)$ for the linear covariant gauge.

Quark Mass Anomalous Dimension

In [7] the Green's function for the quark-bilinear operator was calculated to the four-loop level including the finite contribution. This allowed us to evaluate the five-loop anomalous dimension for this operator. The quark-bilinear operator would describe the massive term in the Lagrangian and is given by $G=\langle\psi(p)[\bar\psi\psi](0)\bar\psi(-p)\rangle$ with the renormalization constant given by

$$\bar\psi_0\psi_0 \to Z_{\bar\psi\psi}\bar\psi\psi = \frac{Z_{\bar\psi\psi}}{Z_\psi}\bar\psi_0\psi_0 = Z_m\bar\psi_0\psi_0. \tag{5}$$

Calculating the running of this operator

$$\begin{aligned}\gamma_m^{\text{mMOM}}(a,0)\Big|^{SU(3)} = &-4a+[...]\,a^2+[...]\,a^3+[...]\,a^4\\
&+\Bigg[\frac{3576071485}{27648}\zeta_7-\frac{75504232175}{7776}+\frac{9610932889}{5832}N_f+\frac{17917034005}{31104}\zeta_5\\
&+\frac{187324052147}{31104}\zeta_3-\frac{310328447}{432}\zeta_3^2-\frac{257106335}{324}\zeta_3N_f\\
&-\frac{180251015}{1944}\zeta_5N_f-\frac{22459484}{243}N_f^2-\frac{4778536}{81}\zeta_7N_f-\frac{60928}{81}\zeta_3^2N_f^2\\
&-\frac{28096}{81}\zeta_3N_f^3-\frac{1600}{9}\zeta_5N_f^3-\frac{352}{27}N_f^4+\frac{1372}{3}\zeta_7N_f^2+\frac{464038}{243}N_f^3\\
&+\frac{948548}{27}\zeta_3N_f^2+\frac{1850845}{243}\zeta_5N_f^2+\frac{6570181}{162}\zeta_3^2N_f\Bigg]a^5+O(a^6).\end{aligned} \tag{6}$$

Critical Exponents

The conformal window is defined as the region in which we have a number of active fermions such that we find perturbatively accessible fixed points, where the running of the coupling constants goes to zero

$$\beta(a_{\text{fp}},\alpha_{\text{fp}})=0 \qquad \text{and} \qquad \alpha_{\text{fp}}\gamma_\alpha(a_{\text{fp}},\alpha_{\text{fp}})=0. \tag{7}$$

The closest non-trivial fixed point to the axis in the Landau gauge is the well-known Banks-Zaks fixed point (BZ) at a_{BZ} [8,9] which is stable in the coupling constant direction but unstable along the gauge-parameter direction. There is a consistent infra-red stable fixed point (IRS) at $a \approx a_{BZ}$ and $\alpha \approx -3$, found for RI$'$ scheme in [10]. If we evaluate the anomalous dimensions of an operator at a fixed point we find the quantum correction to the renormalization dimension of that operator. Below we present a collection of anomalous dimensions evaluated at these fixed points.

N_f	Type	γ_A	γ_c	γ_ψ	ρ_m
12	BZ	0.135918	-0.067959	-0.001749	0.309177
	IRS	0	-0.123246	-0.117907	0.302656
13	BZ	0.107263	-0.053632	-0.000633	0.228539
	IRS	0	-0.100243	-0.094248	0.227759
14	BZ	0.075242	-0.037621	-0.000158	0.152168
	IRS	0	-0.071998	-0.066518	0.152228
15	BZ	0.043759	-0.021879	-0.000019	0.084206
	IRS	0	-0.042706	-0.038808	0.084220
16	BZ	0.014178	-0.007089	0.000001	0.025902
	IRS	0	-0.014073	-0.012597	0.025902

Table 1. Five loop mMOM scheme $SU(3)$ linear covariant gauge critical exponents.

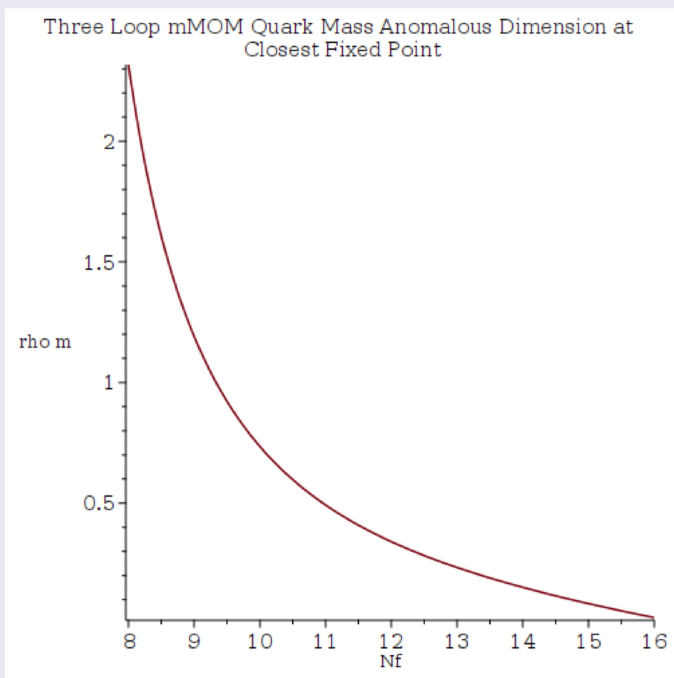
The quantity $\rho_m=-2\gamma_m(a_{\text{fp}},\alpha_{\text{fp}})$ has been evaluated on the lattice for an infrared conformal SU(3) system with $N_f=12$ [11] with values of $\rho_m=0.235(15)$.

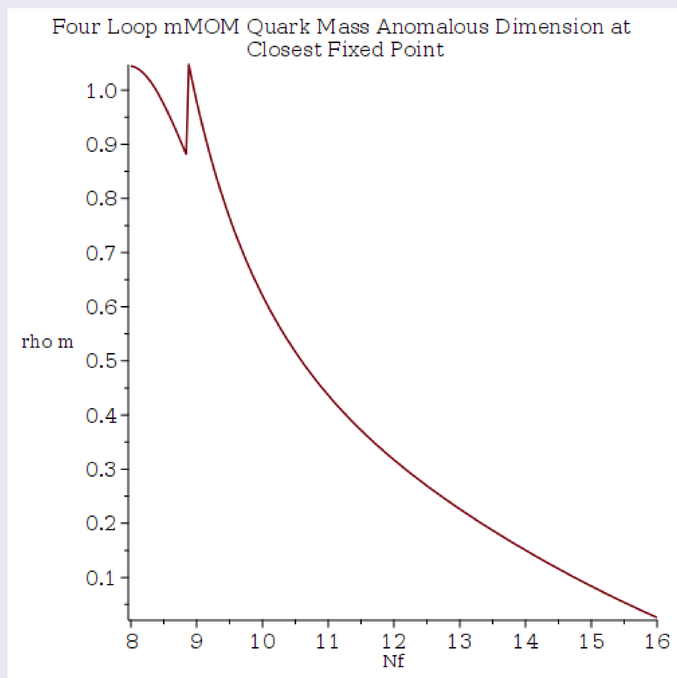
Conformal Window

A leading order approximation of the conformal window is given by $8 \le N_f \le 16$ such that $\beta_0(\alpha)>0>\beta_1(\alpha)$, where $\beta(a,\alpha)=-a^2\sum_{i=0}\beta_i(\alpha)a^i$. However, other suggestions for the lower bound have been suggested, for example several are discussed in [12], including

$$\rho_m = 1 \tag{8}$$

where ρ_m is evaluated at the fixed point with the smallest coupling constant value. Thus if we consider ρ_m for a range of N_f we can try to the point ρ_m crosses these thresholds. These can be plotted as





where we have only plotted the three- and four-loop as the two- and five-loop graphs do not cross $\rho_m=1$ in the minimal range. For these two values we find a boundary on the conformal window of $N_f=9.32$ from the two-loop and $N_f=8.96$ from the four-loop. Repeating this for a variety of different schemes will allow us to put a scheme error on the lower bound of the conformal window.

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