Deconfinement in Sp(4) gauge theory

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Deconfinement on the lattice

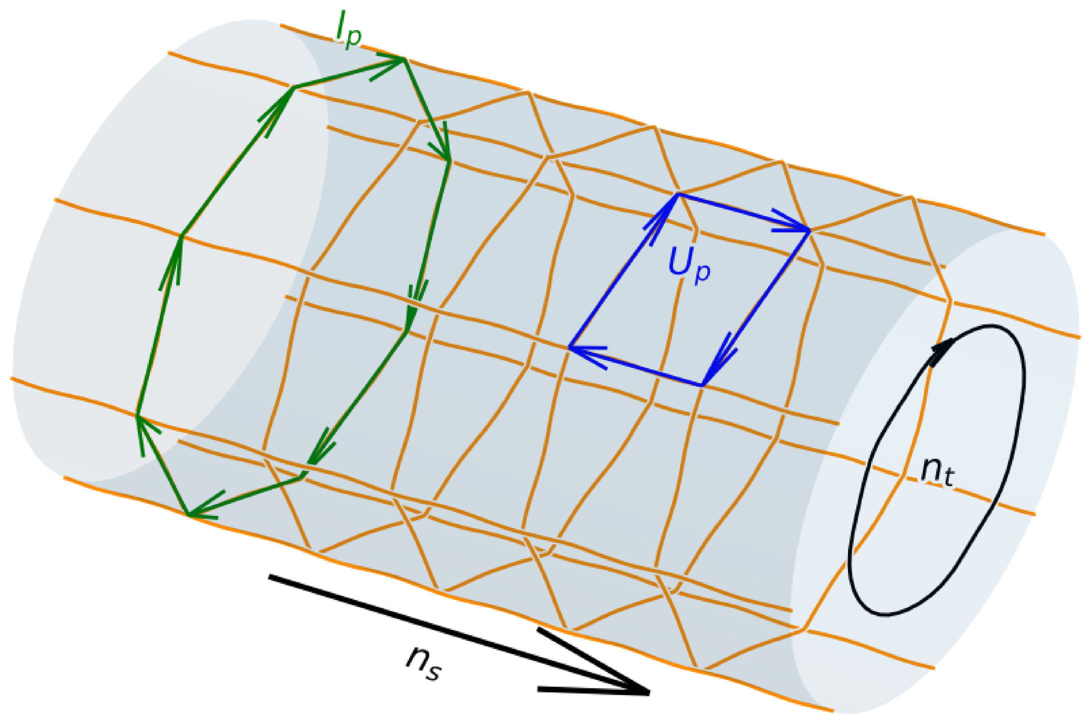

Figure 1. Diagram of a lattice with one spatial, n_s, and one periodic temporal direction, n_τ. Shown in orange are the links, $U_\mu(n_t, \vec{n}_s) \in Sp(4)$. An example of a plaquette, U_p, is shown in blue and a Polyakov loop is shown in green.

Discretise Euclidean spacetime onto a hypercubic lattice of size $\tilde{V} = a^4 N_s^3 \times N_t$, with lattice spacing a and periodic boundary conditions. When N_t is set to be $\ll N_s$, the temperature of the lattice system is given by $1/aN_t$, which is changed by varying the coupling $\beta(a)$. The gauge field consists of elements of $Sp(4)$ living on the links of the lattice, $U_\mu(n_t, \vec{n}_s) \in Sp(4) \subset SU(4)$, obeying $U_\mu \Omega (U_\mu)^T = \Omega$ with $\Omega = i\sigma_2 \mathbb{1}_{2\times 2}$. It is described by the standard Wilson action,

$$S[U] \equiv \frac{6\tilde{V}}{a^4}(1 - u_p[U]), \tag{1}$$

where U is a given configuration and u_p is the average plaquette, $\frac{a^4}{6\tilde{V}} \sum_p U_p$.

Deconfinement on the lattice is associated with the spontaneous breaking of the centre symmetry, for $Sp(4)$ this is $\mathbb{Z}_2$. The order parameter for this phase transition is the average Polyakov loop,

$$\langle l_p \rangle_\beta \equiv \left\langle \frac{1}{4N_s^3} \sum_{\vec{n}_s} \mathrm{Tr} \left(\prod_{n_t=0}^{N_t-1} U_0(n_t, \vec{n}_s) \right) \right\rangle_\beta \begin{cases} = 0 \text{ confined phase} \\ \neq 0 \text{ deconfined phase} \end{cases} . \tag{2}$$

For $Sp(4)$, deconfinement is a first order transition. In the critical region the equilibrium energy distribution of the system will therefore exhibit the characteristic double peak structure, due to the co-existence of phases. The structure of this energy distribution at the critical point, when the two phases are equally likely, can be related to two important quantities when characterising first order transitions: the latent heat, jump in energy at the critical temperature, and the surface tension, which relates to the probability of tunneling between phases.

The density of states

We define the partition function at a coupling, β, as,

$$Z_\beta \equiv \int [DU_\mu] e^{-\beta S[U]} = \int dE e^{-\beta E} \int [DU_\mu] \delta(S[U] - E) = \int dE \rho(E) e^{-\beta E}, \tag{3}$$

in the last equation we have introduced the density of states, $\rho(E)$, which counts the number of configurations with energy (action) E. Expectation values of observables with explicit dependence on the energy, $O(E)$, such as the average plaquette, become one-dimensional integrals over the energy,

$$\langle O(E) \rangle_\beta = \int dE O(E) \rho(E) e^{-\beta E}. \tag{4}$$

We can also directly determine the energy distribution, and therefore the plaquette distribution at a coupling β,

$$P_\beta(u_p) = \frac{1}{Z_\beta} \rho(E) e^{-\beta E}|_{E = 6\tilde{V}(1-u_p)/a^4}. \tag{5}$$

LLR method

Following Ref. [3] and Ref. [4], we approximate the density of states as a piecewise log linear function in the energy,

$$\ln \rho(E) \approx a_n(E - E_n) + c_n. \quad E_n - \frac{\Delta_E}{4} \le E \le E_n + \frac{\Delta_E}{4} \tag{6}$$

for $n = 1, .., 2N - 1$. We choose the range of energy values, $E_1 \to E_{2N-1}$, to cover all relevant physics.The c_n coefficients are set by continuity at the boundary between intervals,

$$c_n = \frac{\Delta_E}{4} a_1 + \frac{\Delta_E}{2} \sum_{k=2}^{n-1} a_k + \frac{\Delta_E}{4} a_n, \quad c_1 = 0. \tag{7}$$

To find the Taylor coefficients a_n, we will solve the equation,

$$\langle\langle E - E_n \rangle\rangle_n(a_n) = \langle\langle u_p - (u_p)_n \rangle\rangle_n(a_n) = 0 \tag{8}$$

where E_n is the centre of the energy interval and the $(u_p)_n$ is the corresponding plaquette value, $1 - E_n a^4/6\tilde{V}$. The double angle bracket $\langle\langle ... \rangle\rangle_n(a_n)$ indicates the expectation value of configurations at a coupling a_n, for configurations restricted to the nth energy interval. We solve this equation iteratively using the Robbins-Monro method,

$$a_n^{(m+1)} = a_n^{(m)} - \frac{12}{\Delta_E^2} \langle\langle E - E_n \rangle\rangle_n(a_n^{(m)}), \tag{9}$$

starting with some initial guess of $a_n^{(1)}$. If we take $m \to \infty$, we will gain the true a_n value, however in practise this is not possible. The iterations are therefore truncated, introducing a systematic error, which is estimated by repeating the calculation and treating the resulting error statistically.

Once the coefficients, a_n, have been calculated we can now compute the density of states, $\rho(E)$, with Eq. 6. Using this we can reconstruct the plaquette distribution with Eq. 5. In the top plot of Fig. 2 we show the a_n coefficients against $(u_p)_n$, and the lower plot shows the plaquette distribution at the critical coupling. The dashed horizontal line in the top figure shows the line when $a_n = \beta$, the vertical lines show corresponding $(u_p)_n$ values. Through continuum extrapolations, the difference between the two peaks of the plaquette distribution, $\Delta\langle u_p \rangle$, can be related to the latent heat of the continuum theory[5]. The non-invertible structure of the function $a_n((u_p)_n)$ leads to the double peak structure of the plaquette distribution.

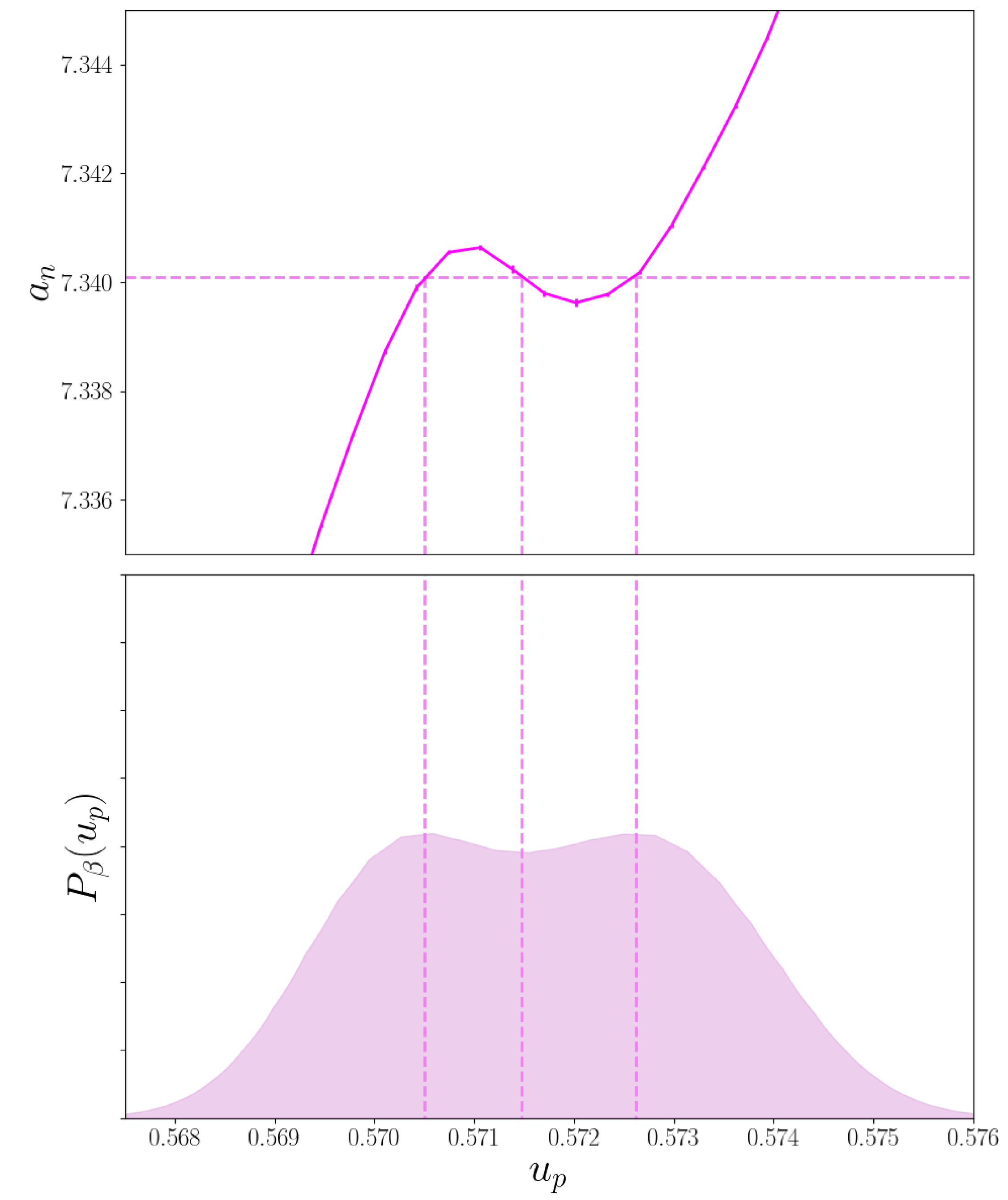

Figure 2. Results for $Sp(4)$ gauge theory on a 4×20^3 lattice with 20 repeats.

Free energy

In analogy with statistical physics we can define a free energy, F, temperature, t and entropy, s,

$$F(t) \equiv E - ts, \quad s \equiv \ln \rho \quad t \equiv \frac{\partial E}{\partial s} \equiv \frac{1}{a_n}, \quad f(t) \equiv \frac{a^4}{\tilde{V}}(F(t) + \Sigma t). \tag{10}$$

We have also introduced a reduced free energy $f(t)$, defined to remove dependence on the arbitrary c_1 term in our ansatz of $\rho(E)$. The free energy can be related to the peaks of the probability distribution through the equation,

$$e^{-\frac{F(t)}{t}} = Z_\beta P_\beta(E)|_{\beta=1/t, E=F(t)+ts}. \tag{11}$$

In Fig. 3, we plot the reduced free kenergy against temperature. It demonstrates the swallow tail structure, characteristic of a first order transition. The plot has three distinct regions, the black lines shows the regime in which only one solution is present. The blue and red lines shows the (meta-)stable and unstable regions respectively.

The point at which the two (meta-)stable branches cross corresponds to the critical temperature $t_c = 1/\beta_c$. The difference in free energy between the (meta-)stable and unstable regimes at the critical temperature, $\Delta f(t_c)$, can be related through infinite volume and continuum extrapolations to the surface tension of the transition[5]. This can also be directly related to the plaquette distribution at the critical point through $-\ln(P_{min}/P_{max}) = (\tilde{V}/a^4)\Delta f(t_c)/t_c$, where P_{max} and P_{min} are the height of the maxima and the central minima respectively.

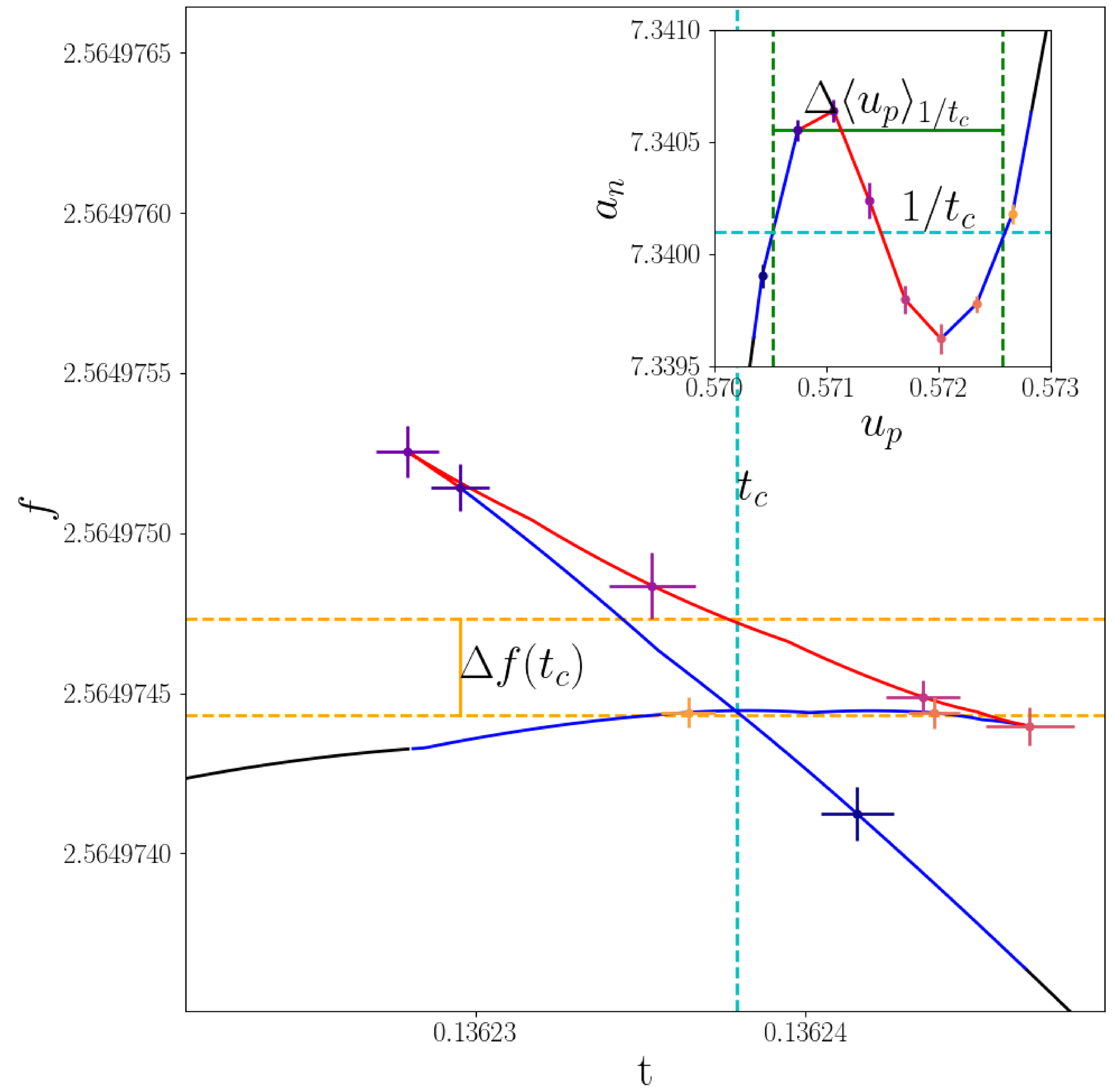

Figure 3. $Sp(4)$ gauge theory on a 4×20^3 lattice with 20 repeats, from Ref. [2].

References

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