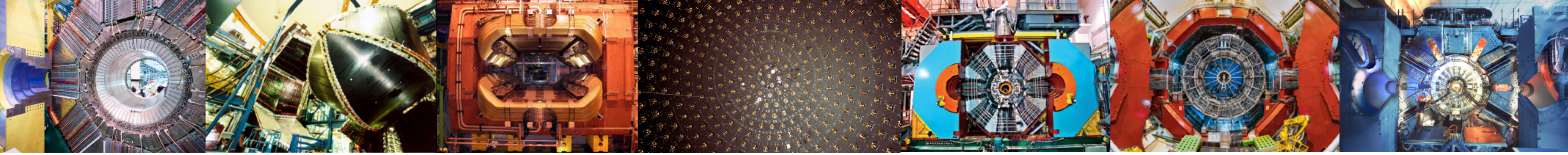
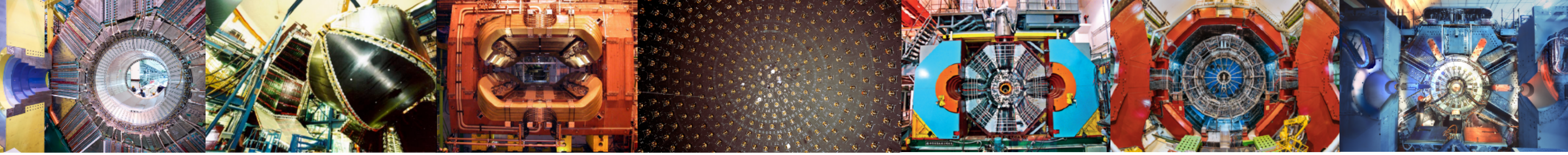


Approaching the Inverse Problem

William I. Jay

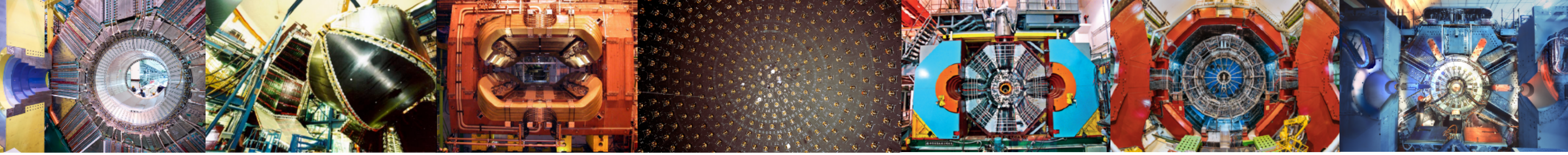
Lattice 2024 — Liverpool, England — 29 July 2024



Approaching Spectral Densities

William I. Jay

Lattice 2024 Liverpool, England 29 July 2024



Outline

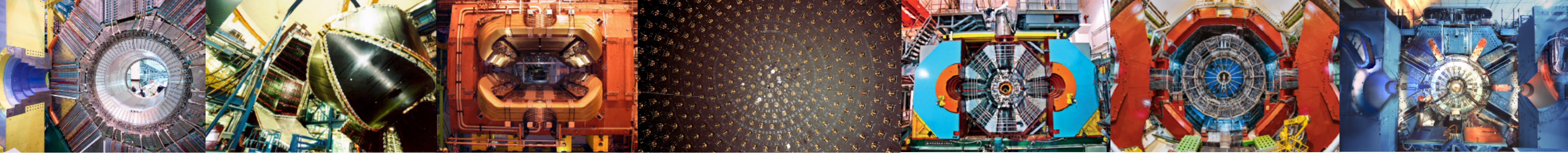
- Connections to the conference
- Motivation and Context
- Spectral Densities and Lattice QCD
    - Role of smearing
    - Analytic continuation

Not a comprehensive review.

Impossible to review all the exciting work in this area in the time allotted.

My topical presentation reflects my interests (and probably biases).

Apologies for all the excellent work not mentioned.



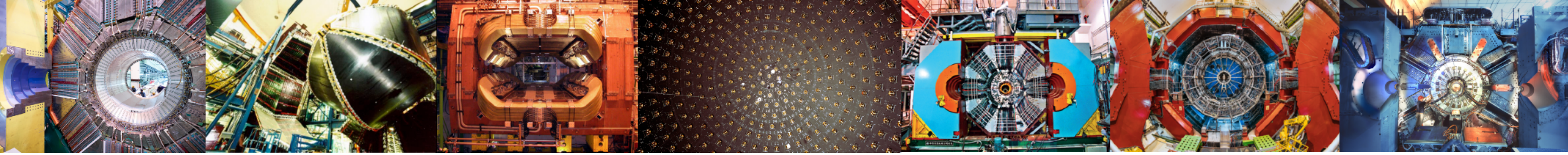
Recent adjacent discussion

Recent plenary talks at Lattice conference series

- J. Bulava's plenary talk at Lattice 2022
  - *Spectral Reconstruction of Inclusive Rates* [link]
  - *PoS* LATTICE2022 (2023) 231 [arXiv:2301.04072]
- J. Liang's plenary talk at Lattice 2019
  - Hadronic Tensor and Neutrino-Nucleon Scattering [link]
  - *PoS* LATTICE2019 (2020) 046 [arXiv:2008.12389]

Lattice@CERN 2024: Inverse Problems 8-12 July 2024 [Indico Link]

- Week of talks/discussion about this topic. Check out the workshop webpage!



Neighboring talks at this conference

More than a dozen presentations related to spectral densities

Monday

- **Shear viscosity from quenched to full lattice QCD** — *Pavan Pavan* — 11:55 - 12:15
- **Thermal photon production rate from lattice QCD** — *Dibyendu Bala* — 12:15 - 12:35
- **Sparse modeling study to extract spectral functions from lattice QCD data** — *Junichi Takahashi* — 15:35 - 15:55
- **Pseudo-scalar meson spectral properties from spatial hadron correlators** — *Tristan Ueding* — 15:55 - 16:15

Tuesday

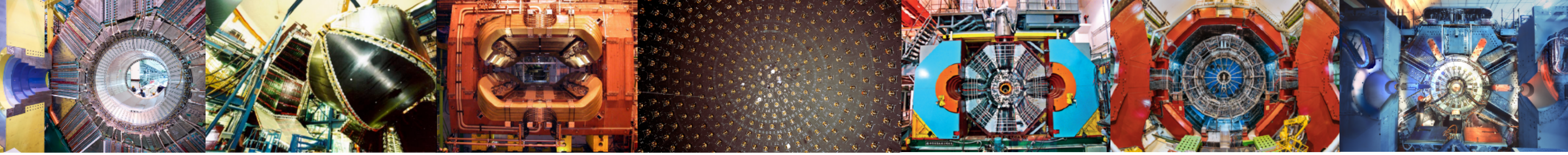
- **Virtual radiative Leptonic decays of charged Kaons** — *Roberto Di Palma* — 11:15 - 11:35
- **Inclusive semileptonic $D_s\mapsto X \ell \nu$ decay from lattice QCD** — *Dr Alessandro De Santis* — 15:05 - 15:25
- **Semileptonic Inclusive Decay of the D_s Meson** — *Christiane Groß* — 15:25 - 15:45
- **The Cabibbo Angle from Inclusive τ Decays** — *Giuseppe Gagliardi* — 16:55 - 17:15

Wednesday

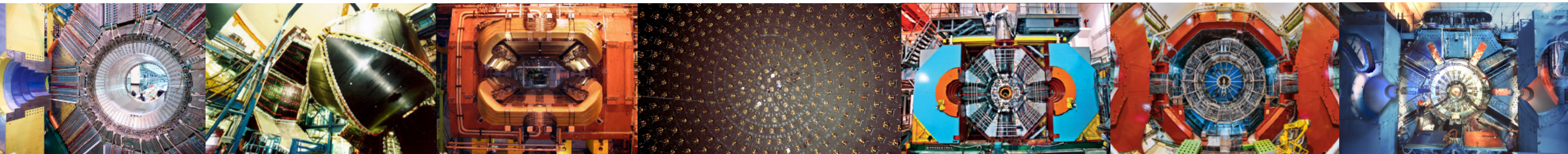
- **Spectroscopy of lattice gauge theories from spectral densities** — *Niccolo Forzano* — 11:35 - 11:55
- **Quarkonia Spectral Functions from (2+1)-flavor QCD using Non-perturbative Thermal Potential** — *Dr Sajid Ali* — 12:35 - 12:55
- **NRQCD Bottomonium spectrum at non-zero temperatures using Backus-Gilbert regularisations** — *Antonio Smecca* — 11:15 - 11:35
- **NRQCD Bottomonium at non-zero temperature using using time-derivative moments** — *Rachel Horohan D'arcy* — 11:35 - 11:55

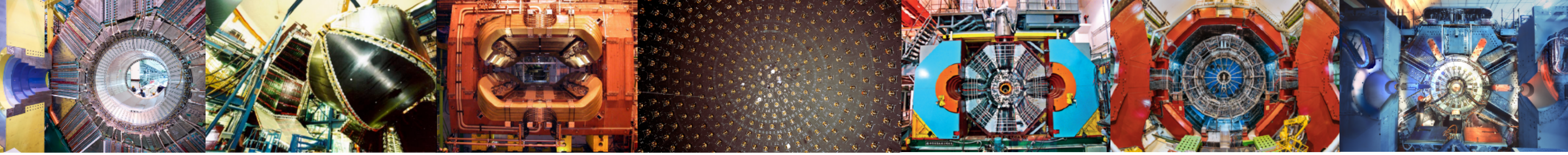
Friday

- **Spectral densities from Euclidean-time lattice correlation functions** — *Matteo Saccardi* — 15:15 - 15:35
- **Progress in Reconstructing the Hadronic Tensor from Euclidean Correlators** — *Douglas Stewart* — 15:15 - 15:35



Motivation and Context



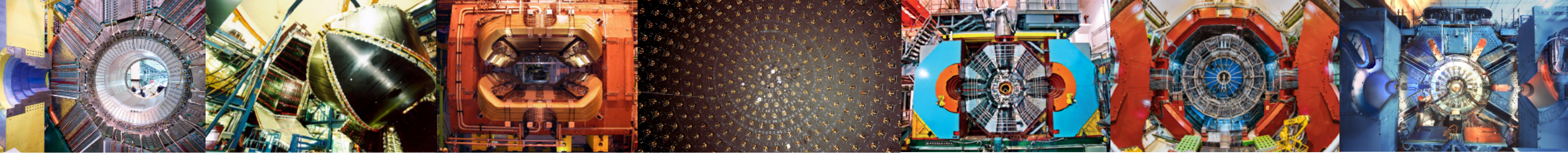


Spectral Densities

$$G(\tau) = \int \frac{d\omega}{2\pi} \rho(\omega) e^{-\omega\tau}$$

Euclidean correlation function
Evaluate with Lattice QCD

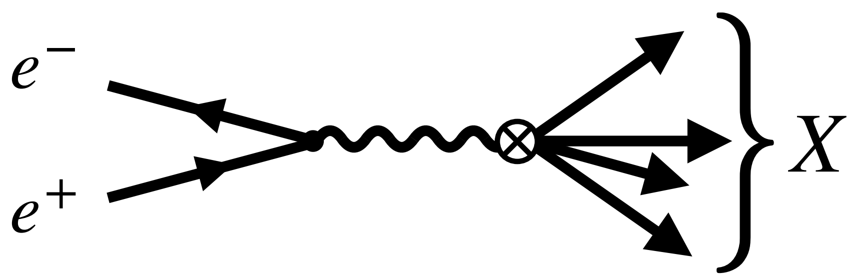
Spectral density
Compute from $G(\tau)$?



Spectral functions for inclusive observables

The R-ratio: $e^+e^- \to \text{hadrons}$

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



QCD correlation function

$$\int d^4x\, e^{iq\cdot x} \langle \varnothing \,|\, [j_\mu^{\text{EM}}(x), j_\nu^{\text{EM}}(0)] \,|\, \varnothing \rangle$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\rho(q^2)$$

Connection via the optical theorem

$$\rho(s) = \frac{R(s)}{12\pi^2}$$

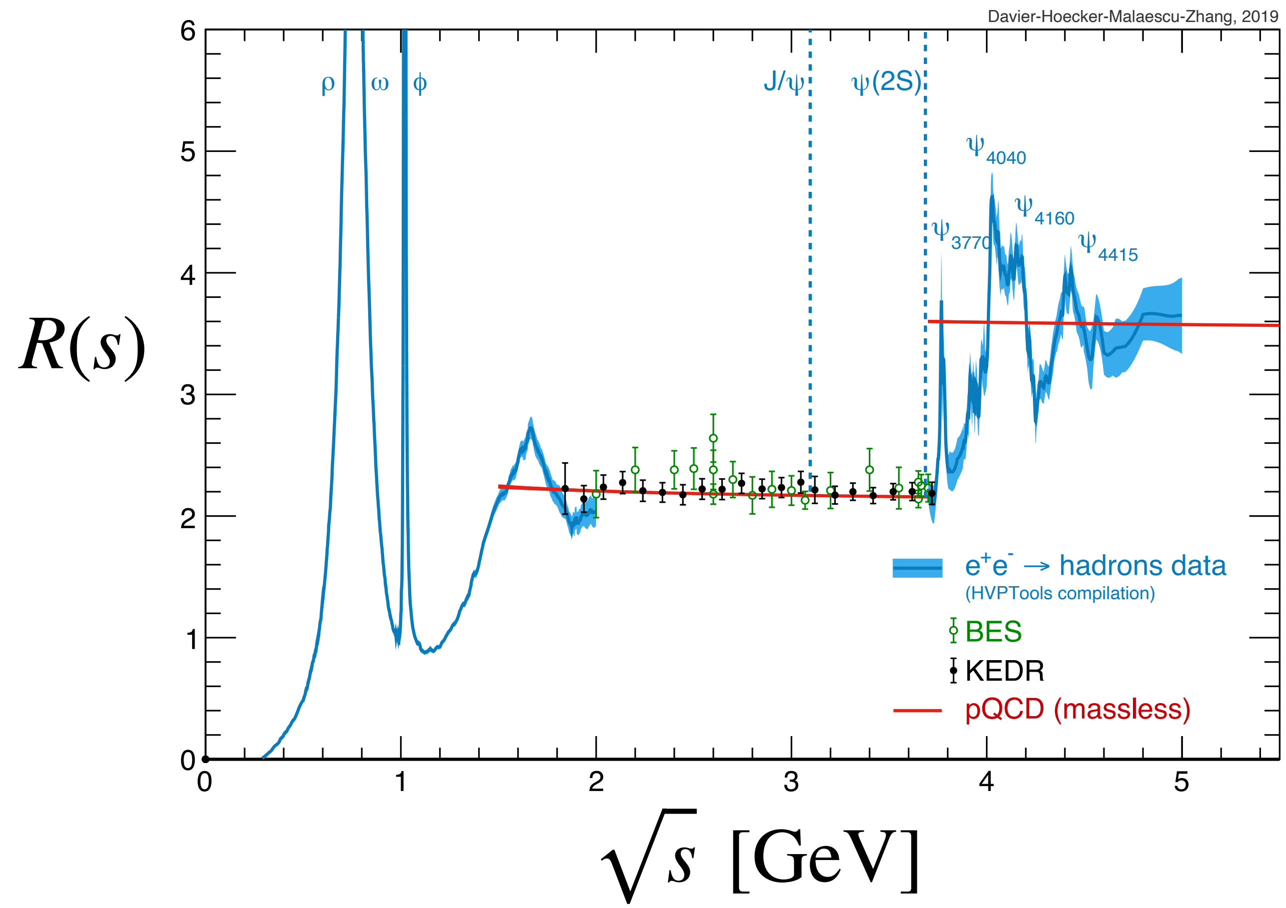
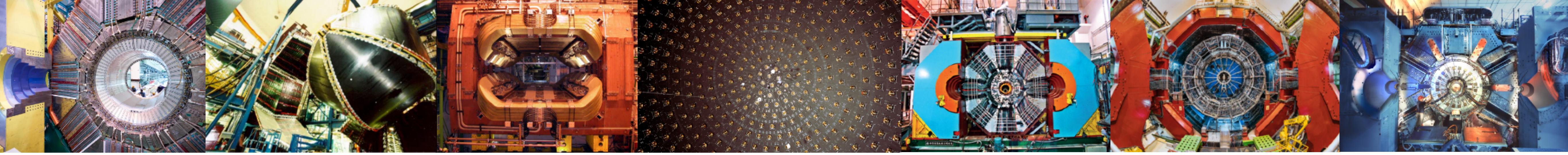


Figure: arXiv:1908.00921
Davier, Hoecker, Malaescu, Zhang



Spectral functions for inclusive observables

Other high-impact observables

- Hadronic width of the tau $\tau \to X$
    - $\langle 0 | [J^{V-A}, J^{V-A}] | 0 \rangle$

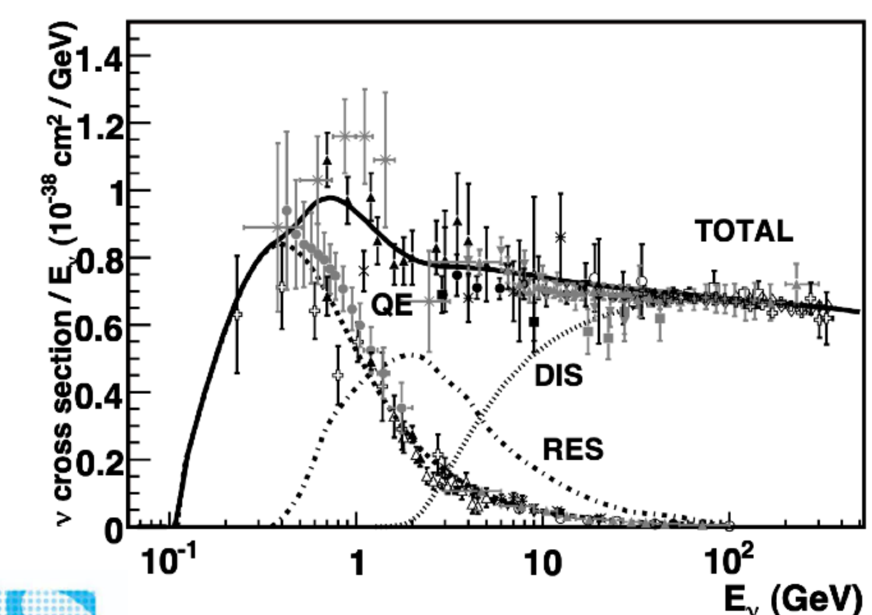
  } $|V_{ud}|, |V_{us}|$ "Cabibbo anomaly"

- Inclusive semileptonic decays $B \to X\ell\nu$
    - $\langle B | [J^{V-A}, J^{V-A}] | B \rangle$

  } $|V_{cb}|, |V_{ub}|$ "Inclusive-exclusive tensions"

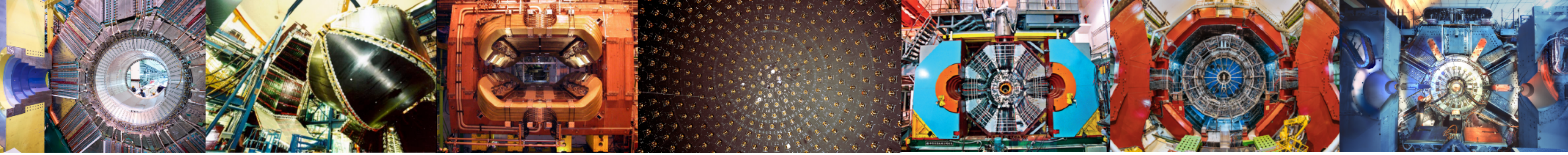
- Inclusive neutrino-nucleon scattering $\nu_\ell N \to \ell X$
    - $\langle N | [J^{V-A}, J^{V-A}] | N \rangle$

  } →



Formaggio and Zeller
Rev.Mod.Phys. 84 (2012) 1307-1341
arXiv:1305.7513

- Transport coefficients in hot QCD
    - $\langle 0 | [T_{\mu\nu}, T^{\mu\nu}] | 0 \rangle_{1/\beta \neq 0}$



Spectral functions for inclusive observables

Other high-impact observables

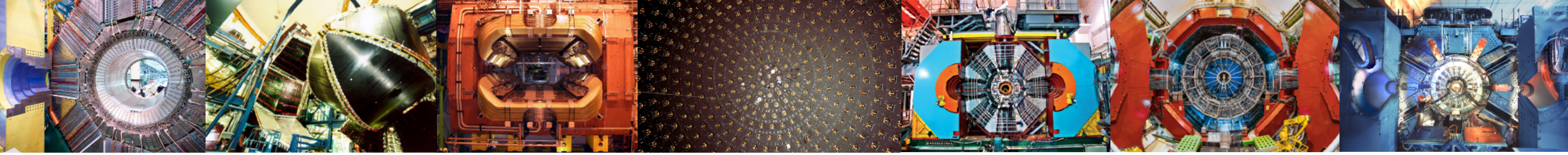
- Hadronic width of the tau $\tau \to X$
  - $\langle 0 | [J^{V-A}, J^{V-A}] | 0 \rangle$

$|V_{ud}|, |V_{us}|$
"Cabibbo anomaly"

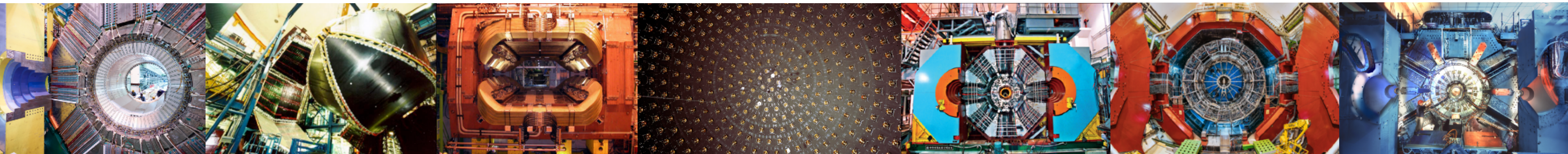
- Inclusive semile
  - $\langle B | [J^{V-A}, J$
- Inclusive neutri
  - $\langle N | [J^{V-A}, J$
- Transport coeffi
  - $\langle 0 | [T_{\mu\nu}, T^{\mu$

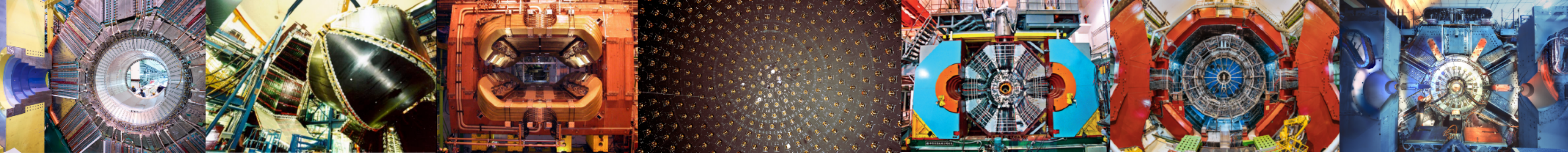
Connections to these topics are all being discussed in the parallel talks highlighted earlier — check them out!

Formaggio and Zeller
Rev.Mod.Phys. 84 (2012) 1307-1341
arXiv:1305.7513



The Inverse Problem





The Inverse Problem

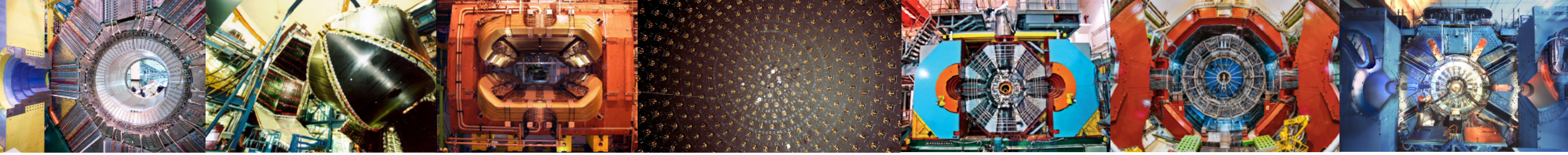
$$G(\tau) = \int \frac{d\omega}{2\pi} \rho(\omega) e^{-\omega\tau}$$

Euclidean correlation function

Evaluate with Lattice QCD

Spectral density

Compute from $G(\tau)$?

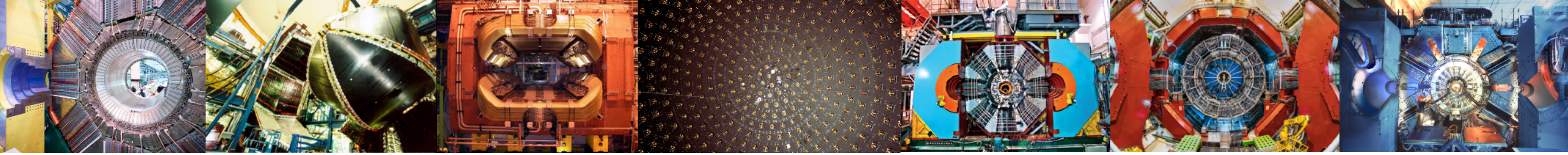


The Inverse Problem

$$G(\tau) = \int \frac{d\omega}{2\pi} \rho(\omega) e^{-\omega\tau}$$

Intertwined conceptual/technical challenges:

1. Calculation in finite volume deforms the spectrum.
2. Euclidean data is available at a finite set of points.
3. Statistical uncertainty is present.



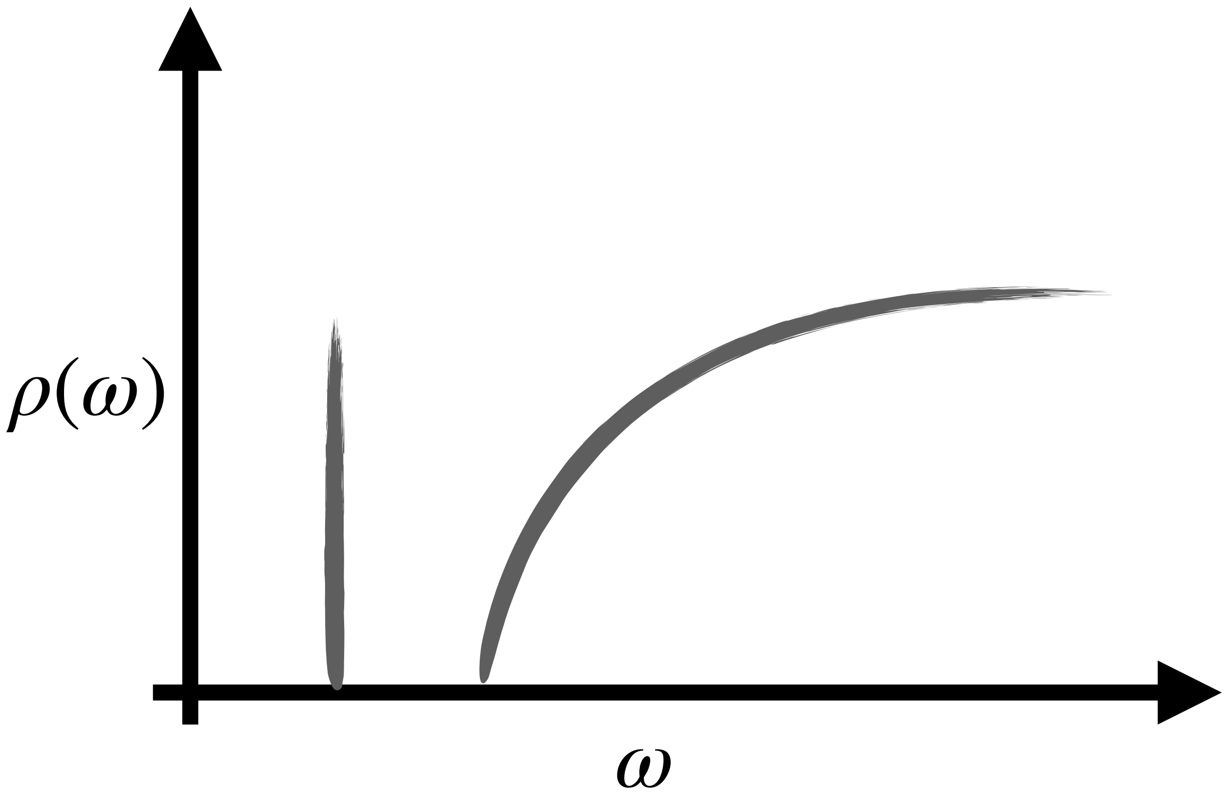
Spectral Densities

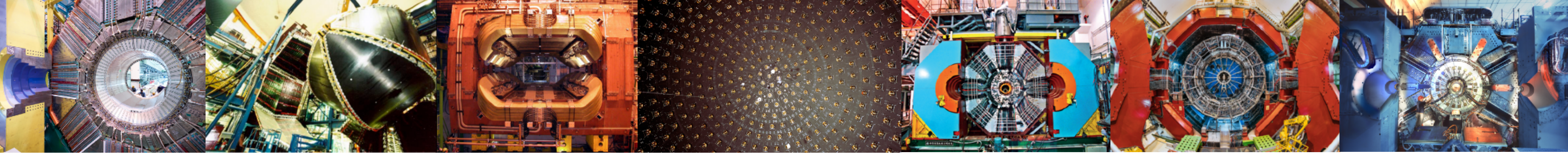
The deformation of finite volume

Consider inclusive electron-proton scattering

Elastic scattering: $\rho(\omega) \sim \delta(\omega - E_p) \times (\text{form factor})^2$

Inelastic scattering: $\rho(\omega) \sim \Theta\left(\omega - M_N - M_\pi\right) \times \left(\text{phase space}\right) \times |\mathscr{M}|^2$





Spectral Densities

The deformation of finite volume

Consider inclusive electron-proton scattering

Elastic scattering: $\rho(\omega) \sim \delta(\omega - E_p) \times (\text{form factor})^2$

Inelastic scattering: $\rho(\omega) \sim \Theta\left(\omega - M_N - M_\pi\right) \times \left(\text{phase space}\right) \times |\mathscr{M}|^2$

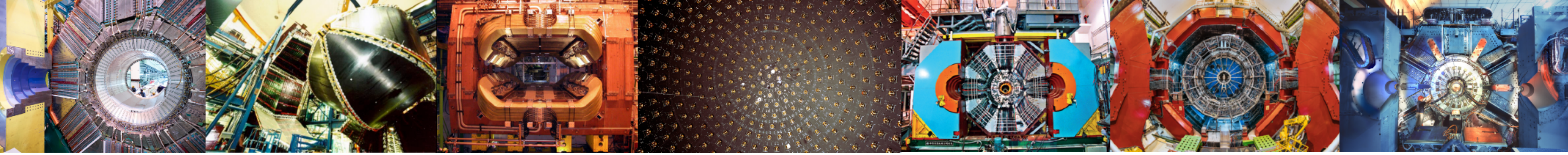
But… a QM system in a box has a discrete spectrum.

See plenary talks by
Felix Erben — Tues 9:00
Nilmani Mathur — Sat 9:00

Finite-volume formalism in elastic region

M. Lüscher (1986)
L. Lellouch and M. Lüscher (2001)

…and many, many other contributors!



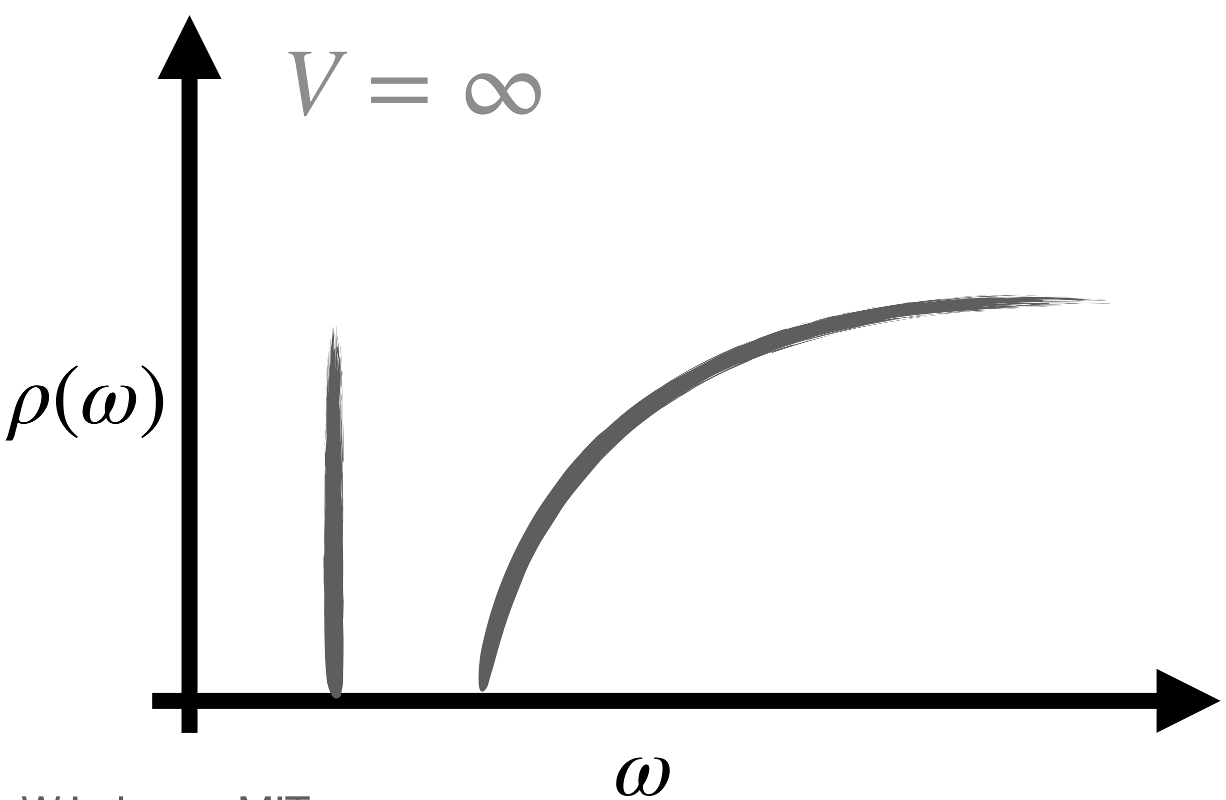
Spectral Densities

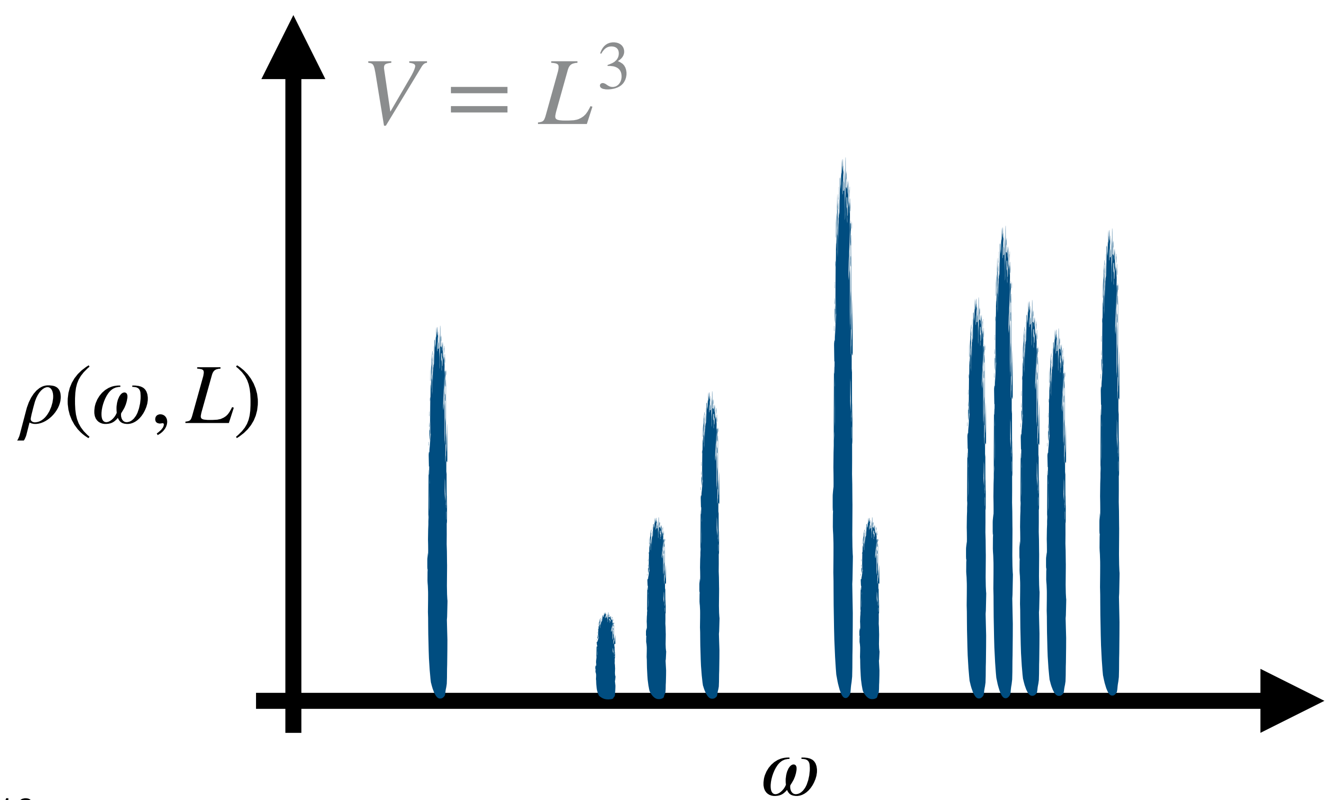
The deformation of finite volume

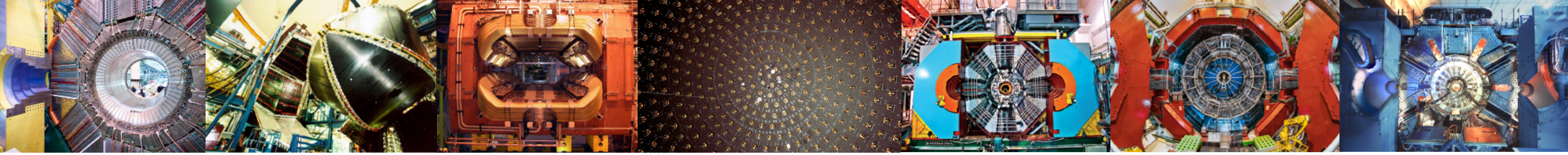
Consider inclusive electron-proton scattering

Elastic scattering: $\rho(\omega) \sim \delta(\omega - E_p) \times (\text{form factor})^2$

Inelastic scattering: $\rho(\omega) \sim \Theta\left(\omega - M_N - M_\pi\right) \times \left(\text{phase space}\right) \times |\mathscr{M}|^2$



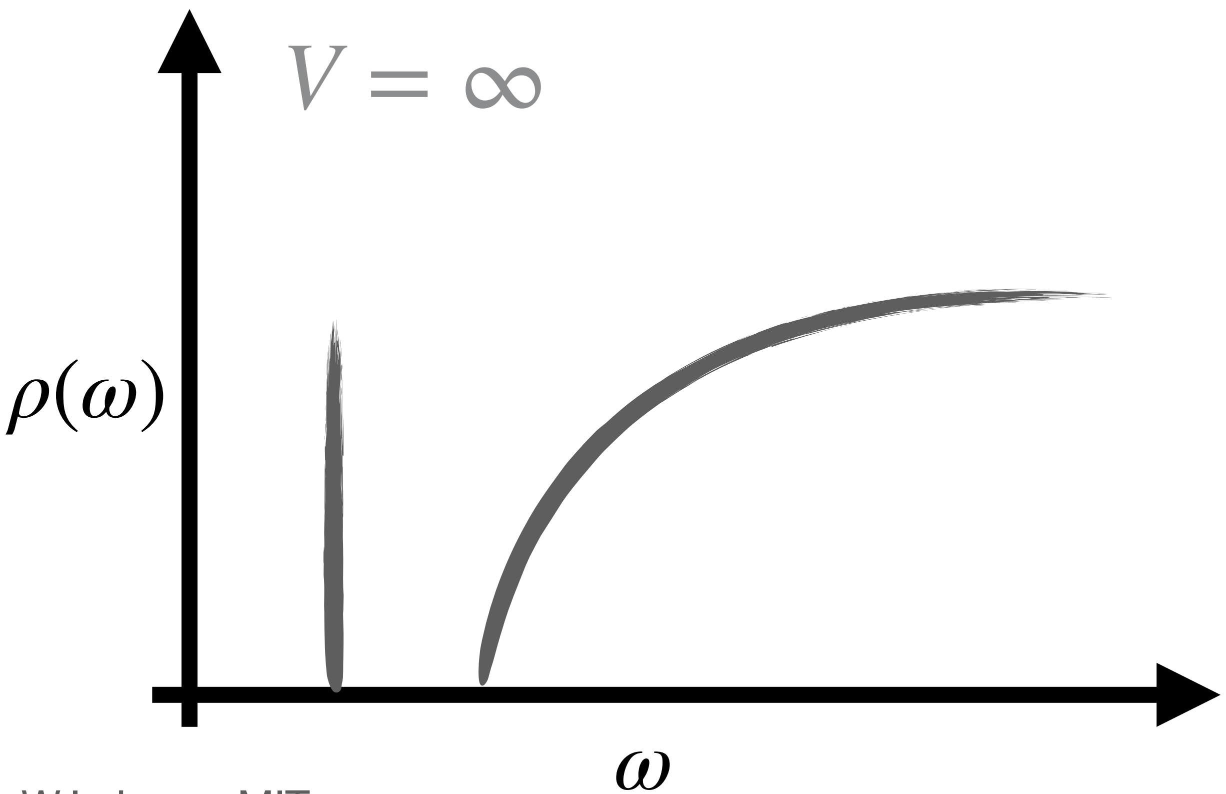


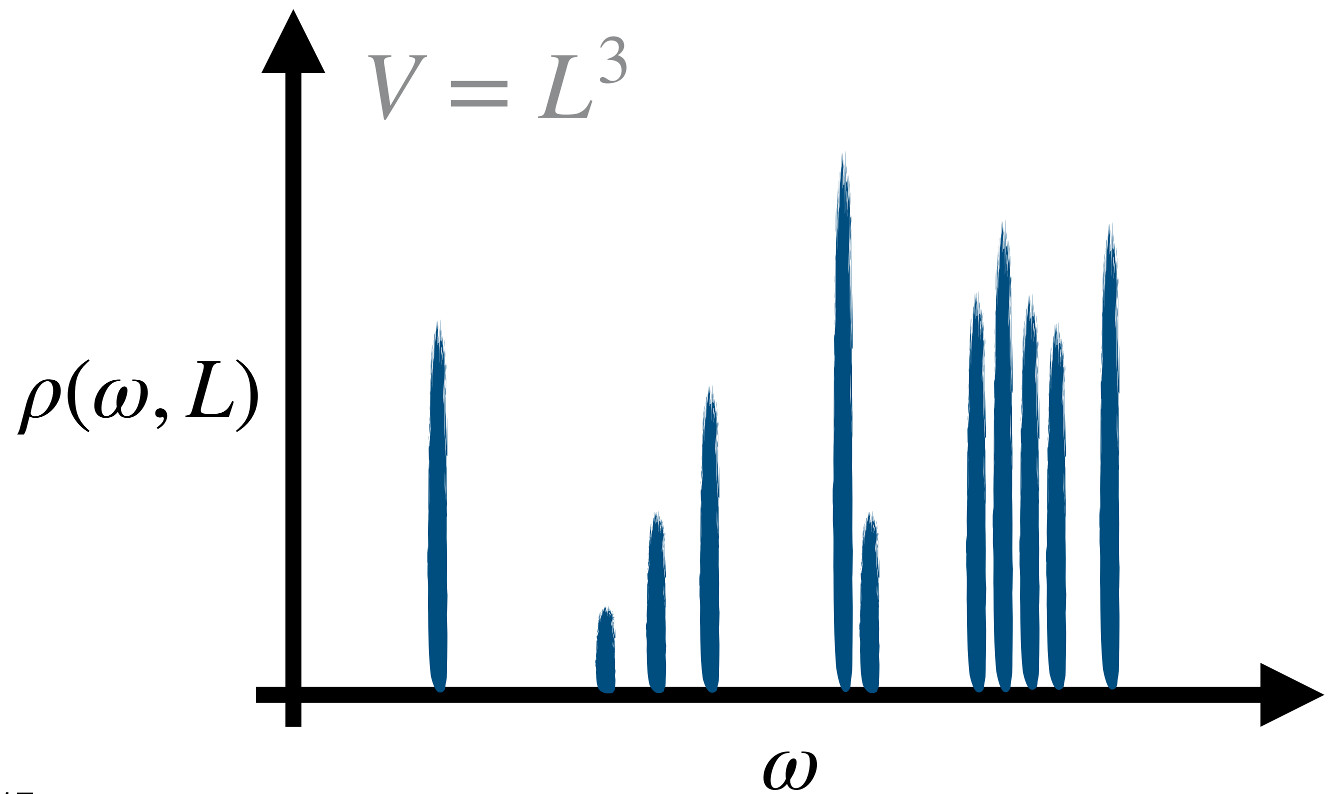


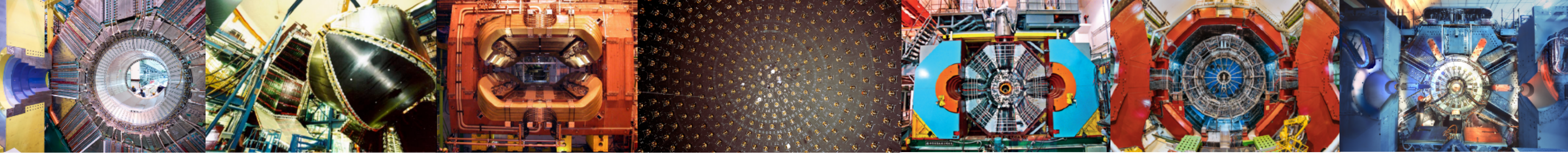
Spectral Densities

The deformation of finite volume

How to reconcile these two pictures?







Spectral Densities

The deformation of finite volume

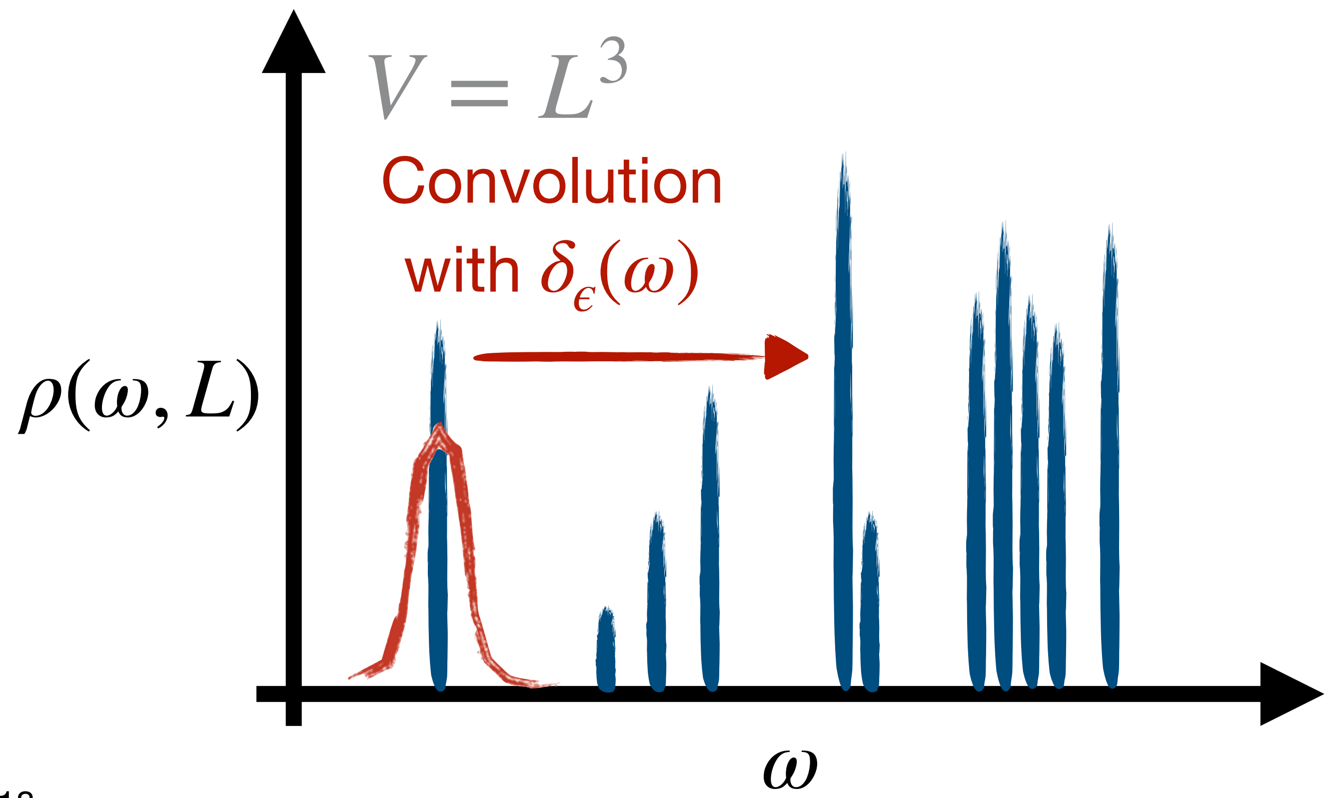
Hansen, Meyer, and Robaina
PRD 96 (2017) 9, 094513
[arXiv:1704.08993]

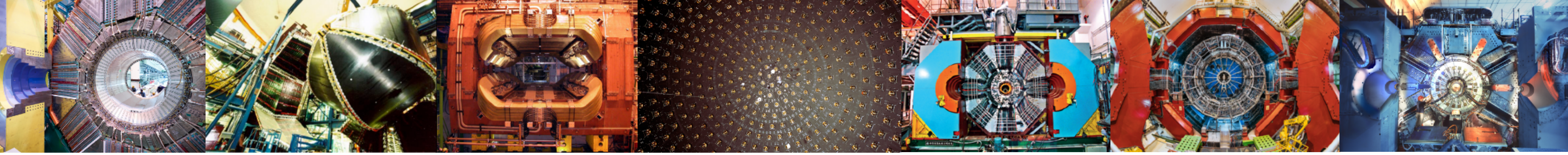
Poggio, Quinn, and Weinberg
PRD 13 (1976) 1958

How to reconcile these two pictures? Smearing.

Choose a smearing kernel $\delta_\epsilon(\omega) =$ 

Define a smeared spectral function $\rho_\epsilon(\omega, L)$





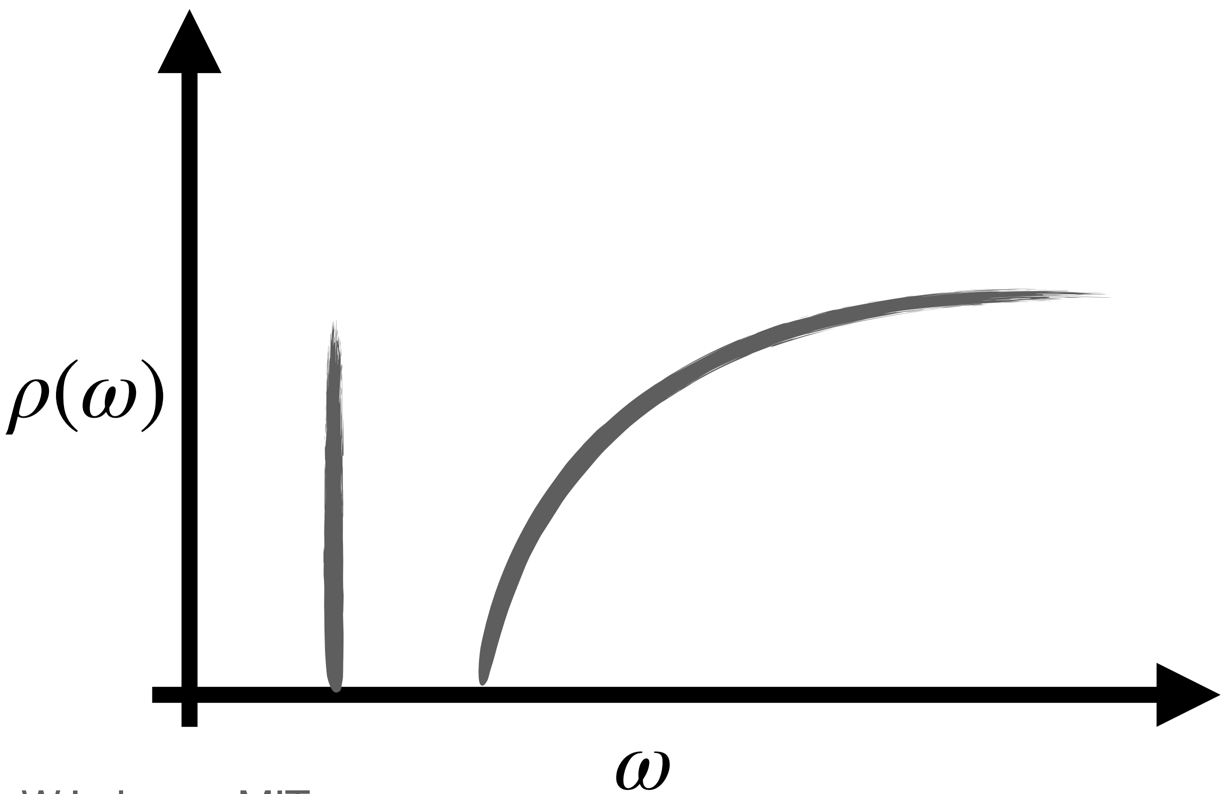
Spectral Densities

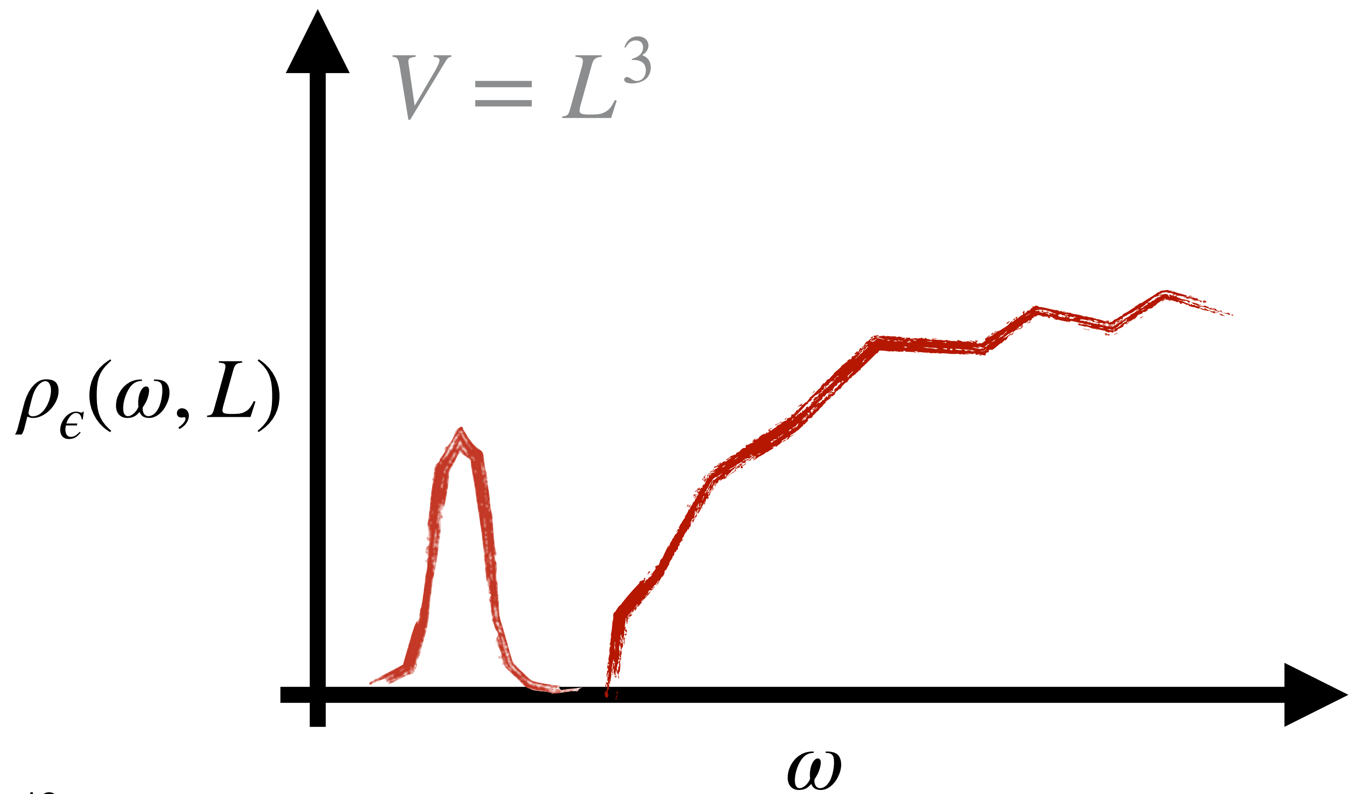
The deformation of finite volume

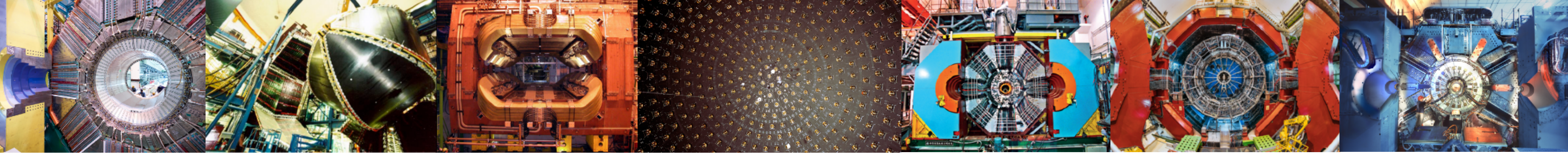
Hansen, Meyer, and Robaina
PRD 96 (2017) 9, 094513
[arXiv:1704.08993]

How to reconcile these two pictures? Smearing.

$$\rho(\omega) = \lim_{\epsilon \to 0} \lim_{L \to \infty} \rho_\epsilon(\omega, L)$$







The HLT Algorithm

Hansen, Lupo, and Tantalo
PRD 99 (2019) 9, 094508
arXiv:1903.06476

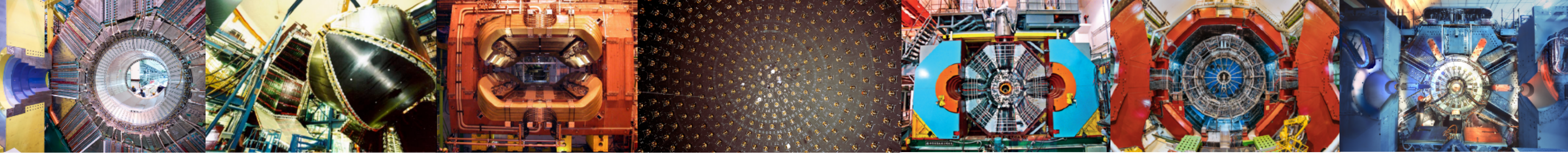
State of the art for practical reconstructions

- Write linear Ansatz for solution:
  - $\rho_\epsilon(\omega) = \sum_t q_t(\omega) C(t) = \int d\omega' \, \rho(\omega') \hat{\delta}_\epsilon(\omega', \omega)$
- Determine coefficients $g_t(\omega)$ by minimizing distance to smearing kernel—which can be chosen freely.
  - $A[q] = \int d\omega' \left\{ \delta_\epsilon(\omega' - \omega) - \hat{\delta}_\epsilon(\omega', \omega) \right\}^2$
  - $B[q] = \text{Var}\left\{ \hat{\rho}_\epsilon(\omega) \right\}$

  Tune λ for tradeoff between bias/variance

  - Minimize the convex sum: $\mathscr{F}_\lambda[q] = (1 - \lambda) A[q] + \lambda B[q]$
- Elegant connection to Bayesian methods / Gaussian processes
  - Del Debbio *et al.* arXiv:2311.18125

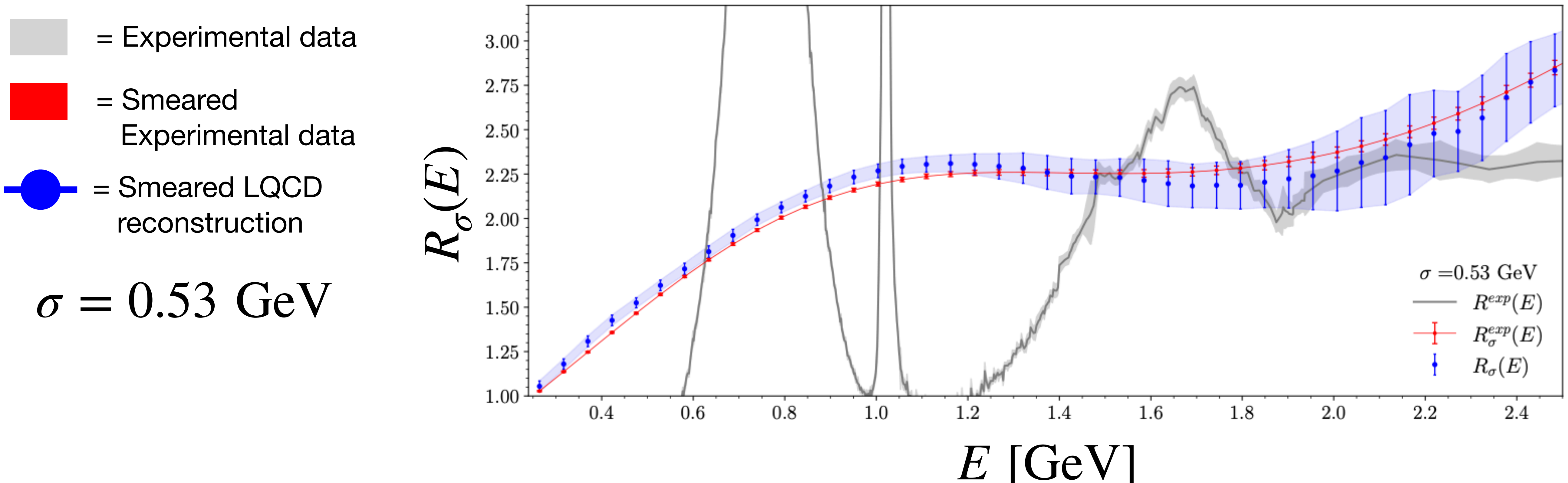
Open-source implementation
github.com/LupoA/lsdensities



Probing the energy-smeared R-ratio

ETMC
PRL 130 (2023) 24, 241901
arXiv: 2212.08467

Spectral reconstruction of $\langle VV \rangle$ correlators with HLT

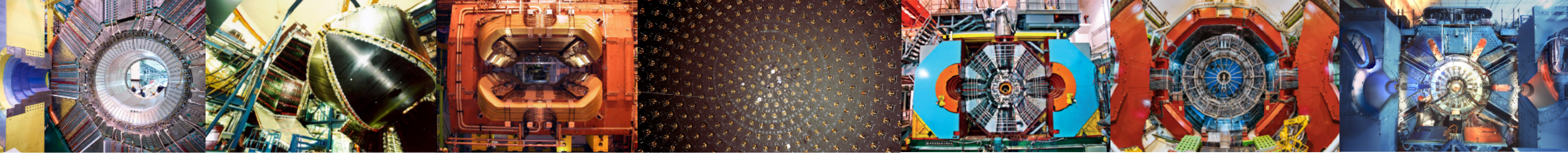


- Gaussian smearing kernel used with the reconstruction using HLT algorithm
- Results presented in continuum limit
- Explicit check of systematic effect of finite volume (B64/B96)
- Same $\langle VV \rangle$ correlators as used in recent ETMC work on μ (g-2) [arXiv:2206.15084]. No QED/SIB corrections.

ID	$L^3 \times T$	a fm	aL fm	m_π GeV
B64	$64^3 \cdot 128$	0.07957(13)	5.09	0.1352(2)
B96	$96^3 \cdot 192$	0.07957(13)	7.64	0.1352(2)
C80	$80^3 \cdot 160$	0.06821(13)	5.46	0.1349(3)
D96	$96^3 \cdot 192$	0.05692(12)	5.46	0.1351(3)

As a matter of principle, how differential could one go?

How much analytic information is contained in, say, O(100) points?

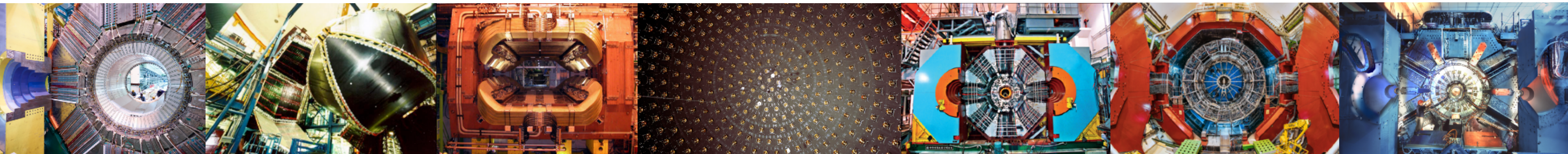


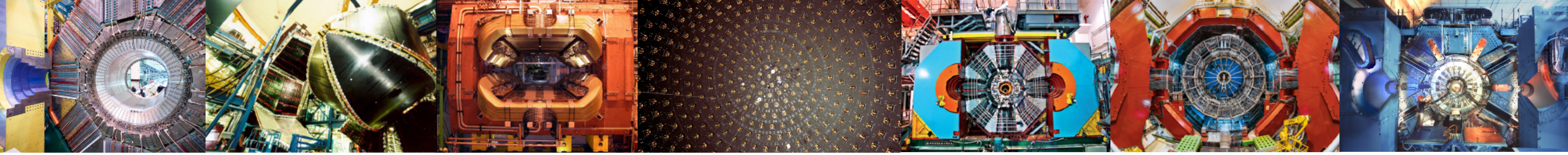
As a matter of principle, how differential could one go?

How much analytic information is contained in, say, O(100) points?

Patrick Oare
MIT →BNL

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190



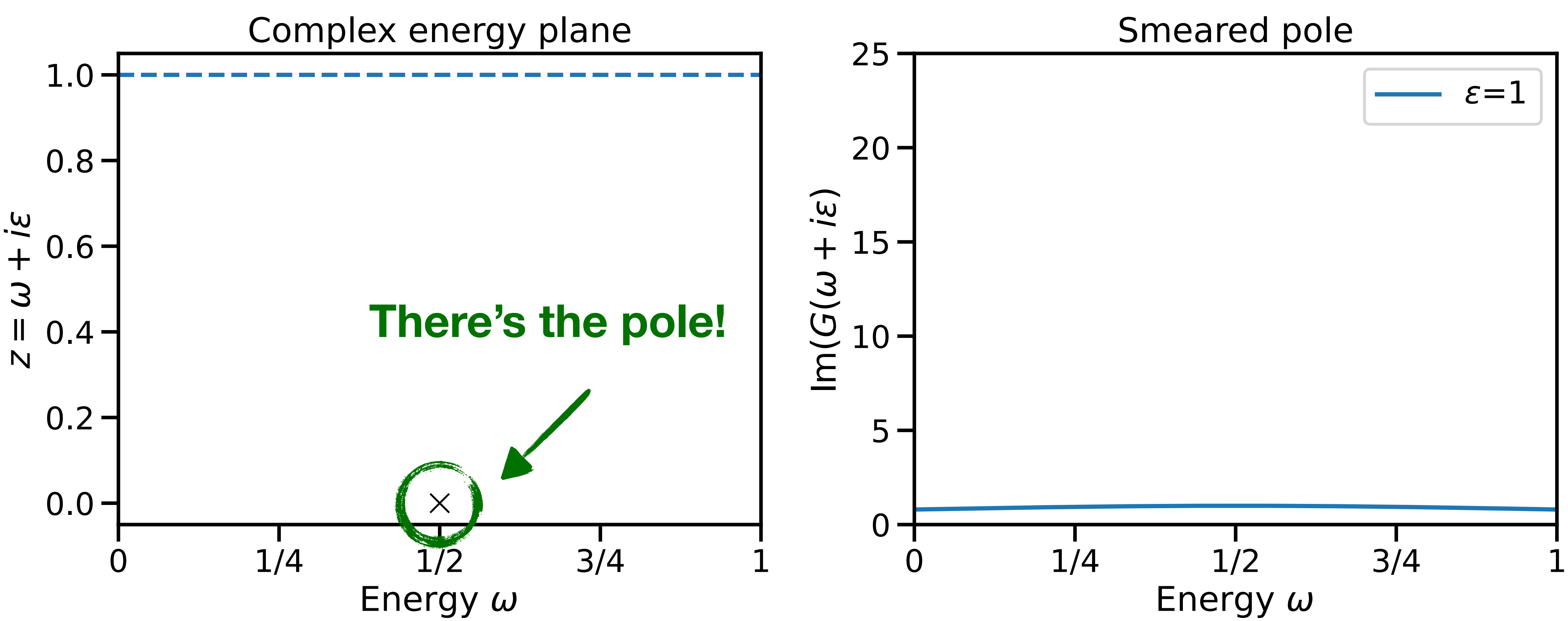


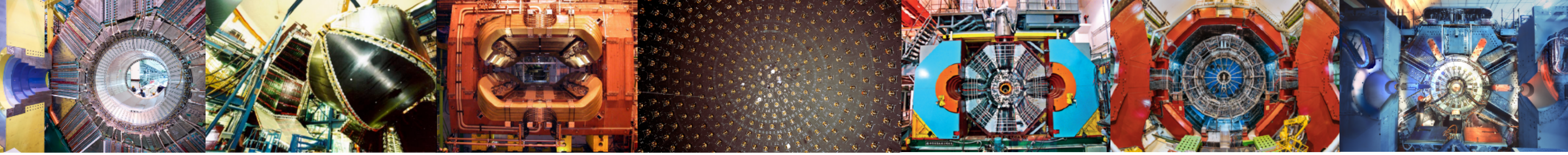
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.



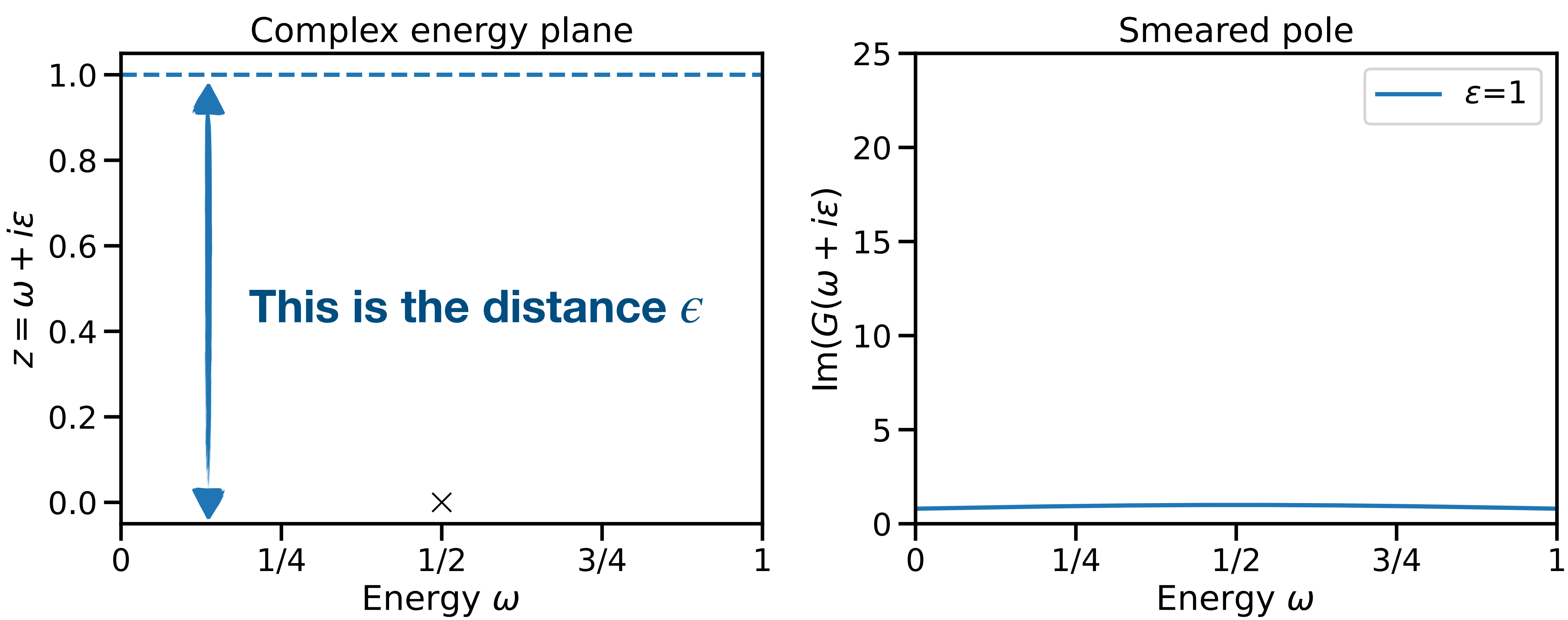


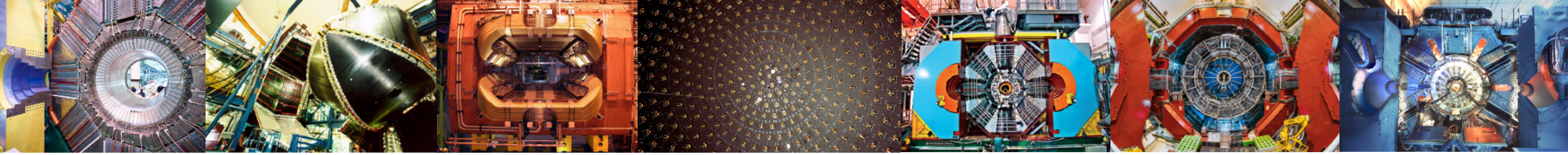
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.



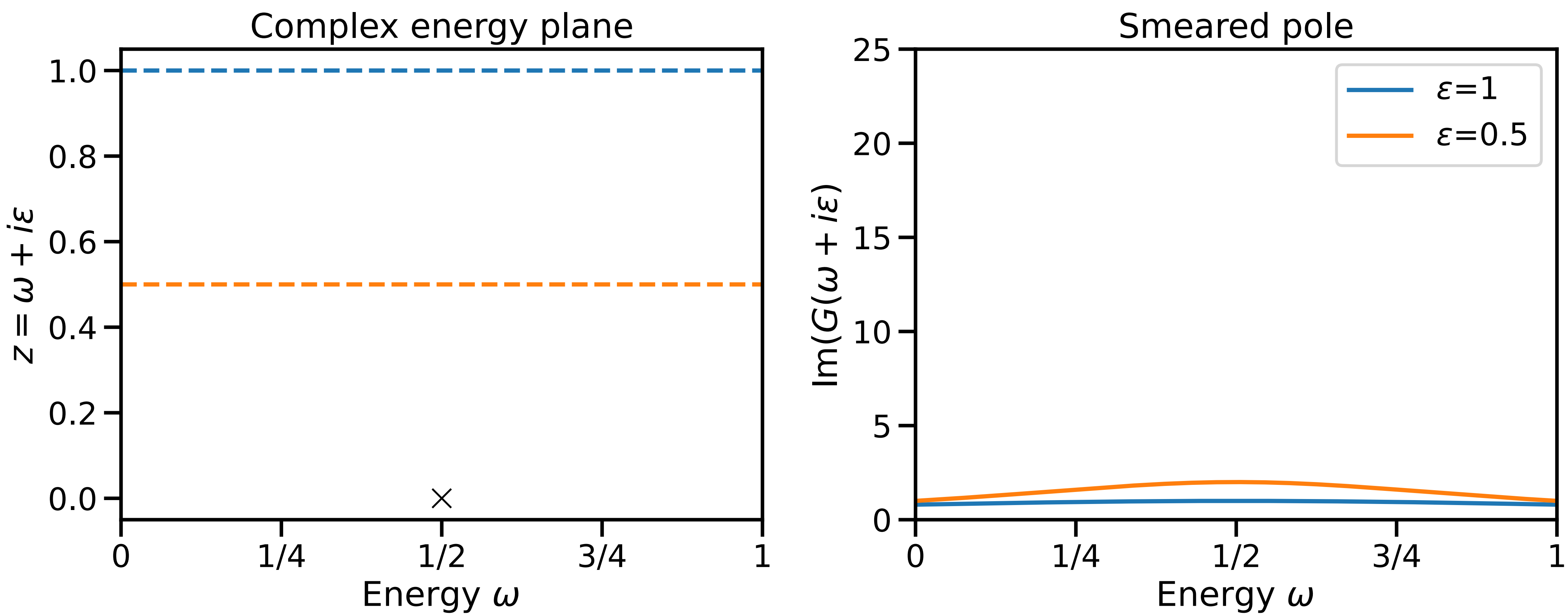


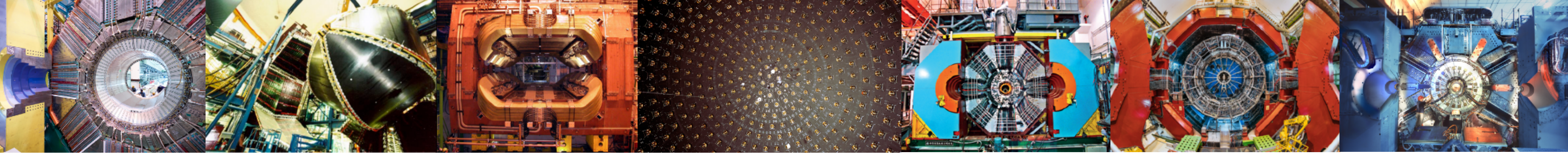
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.



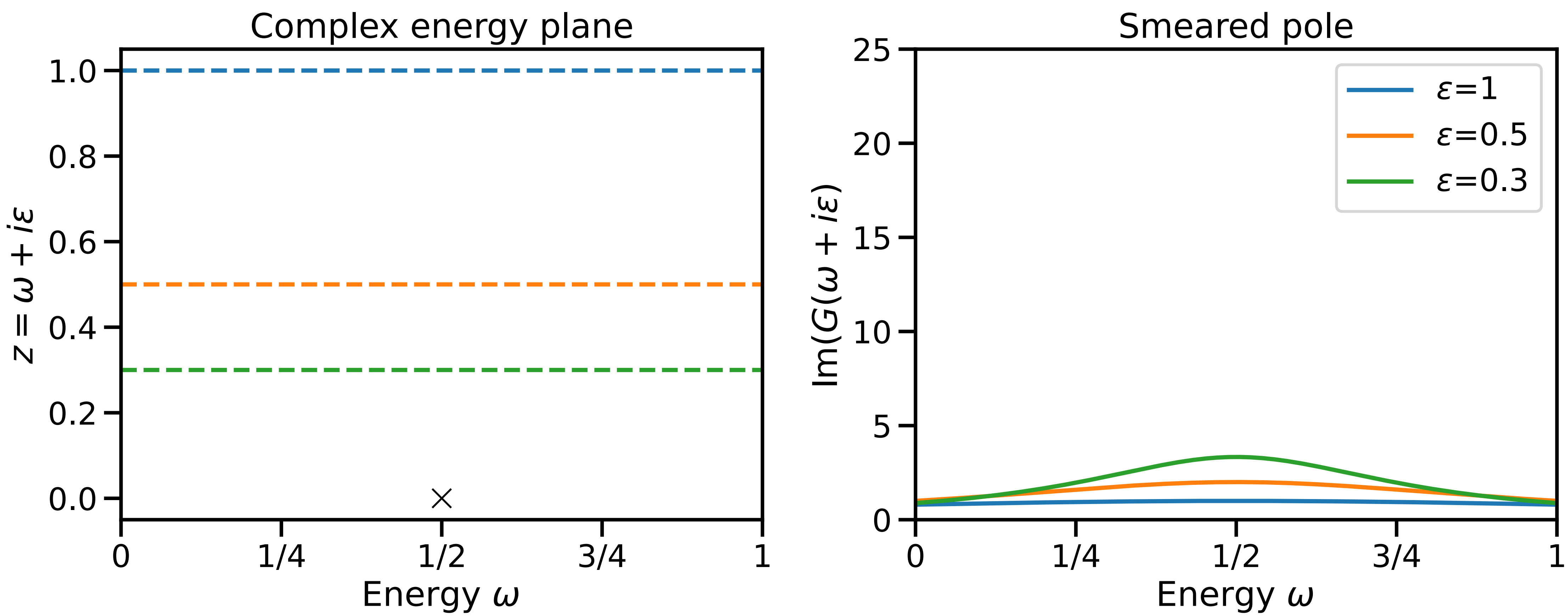


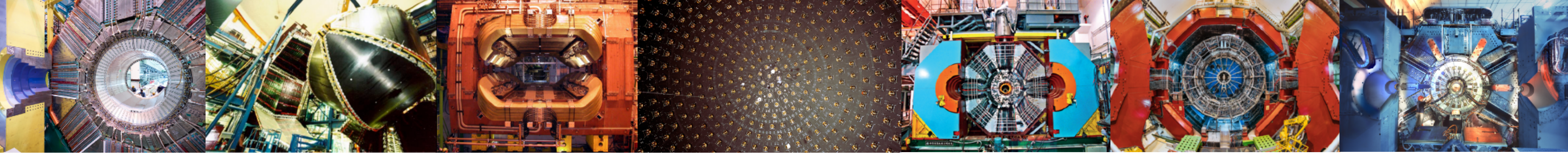
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.



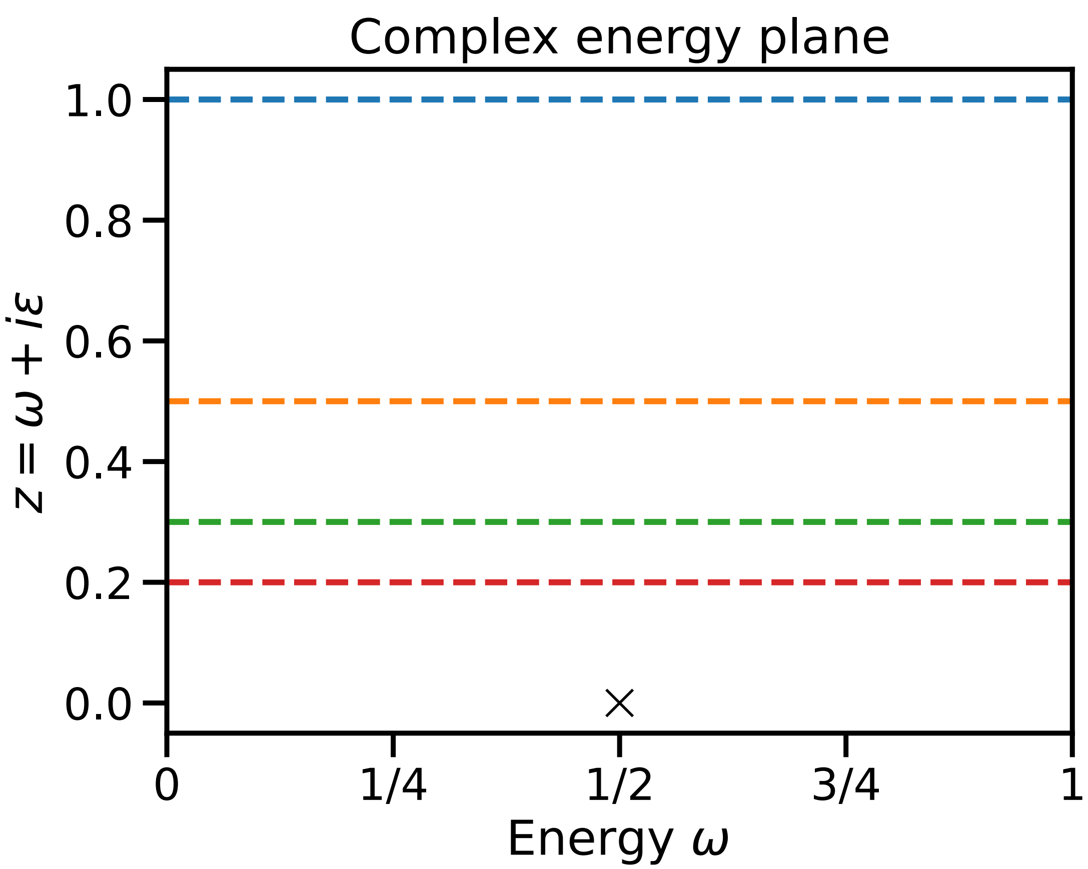


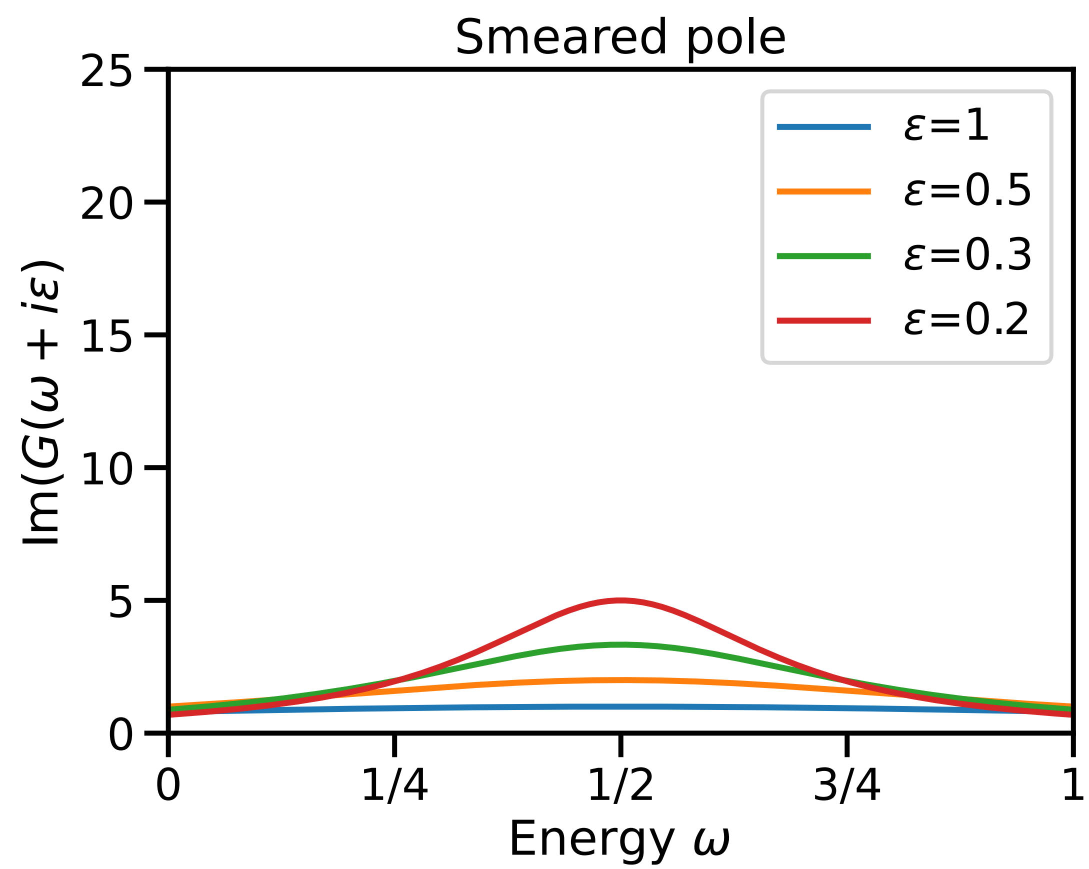
Spectral Densities

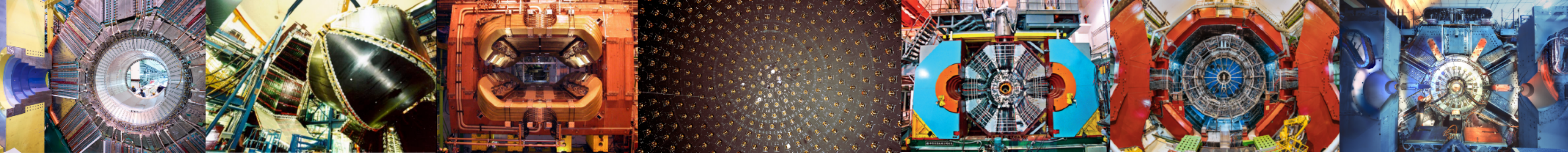
Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.





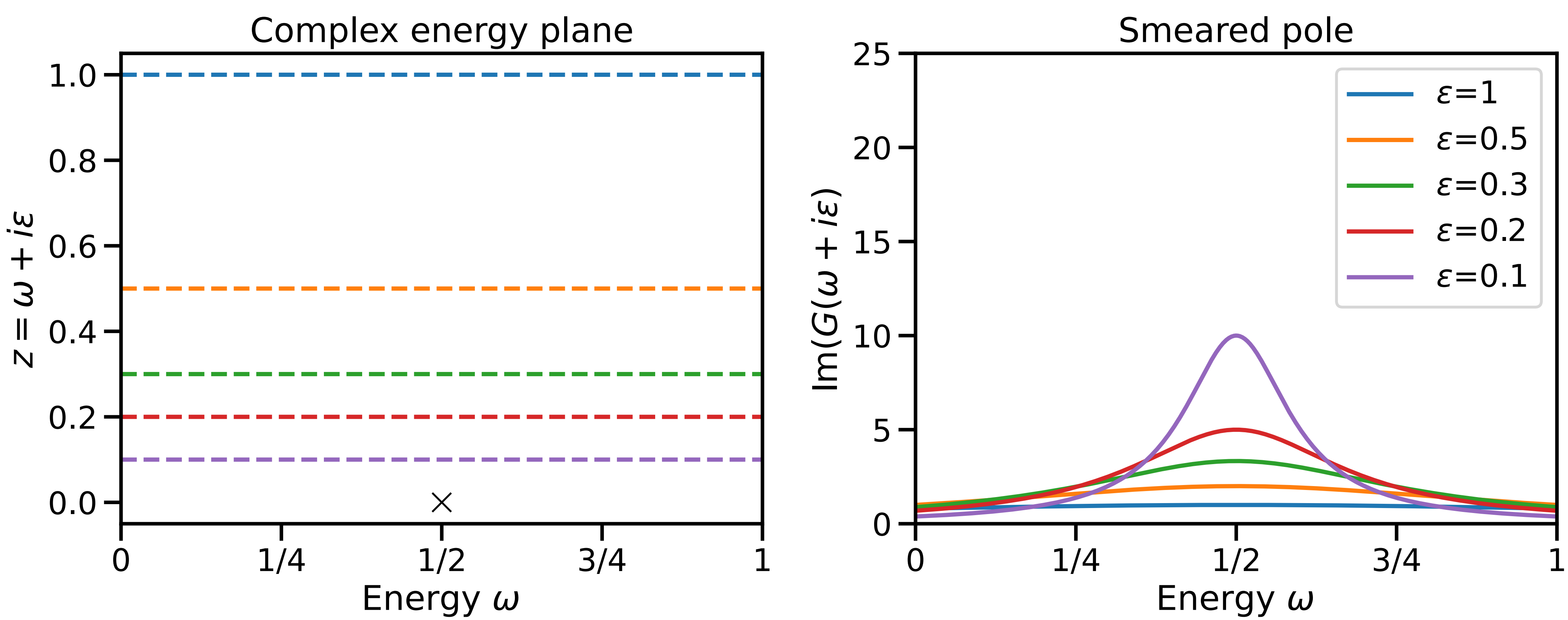


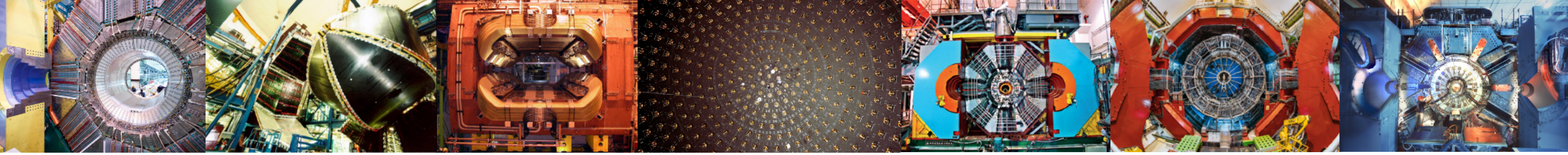
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.



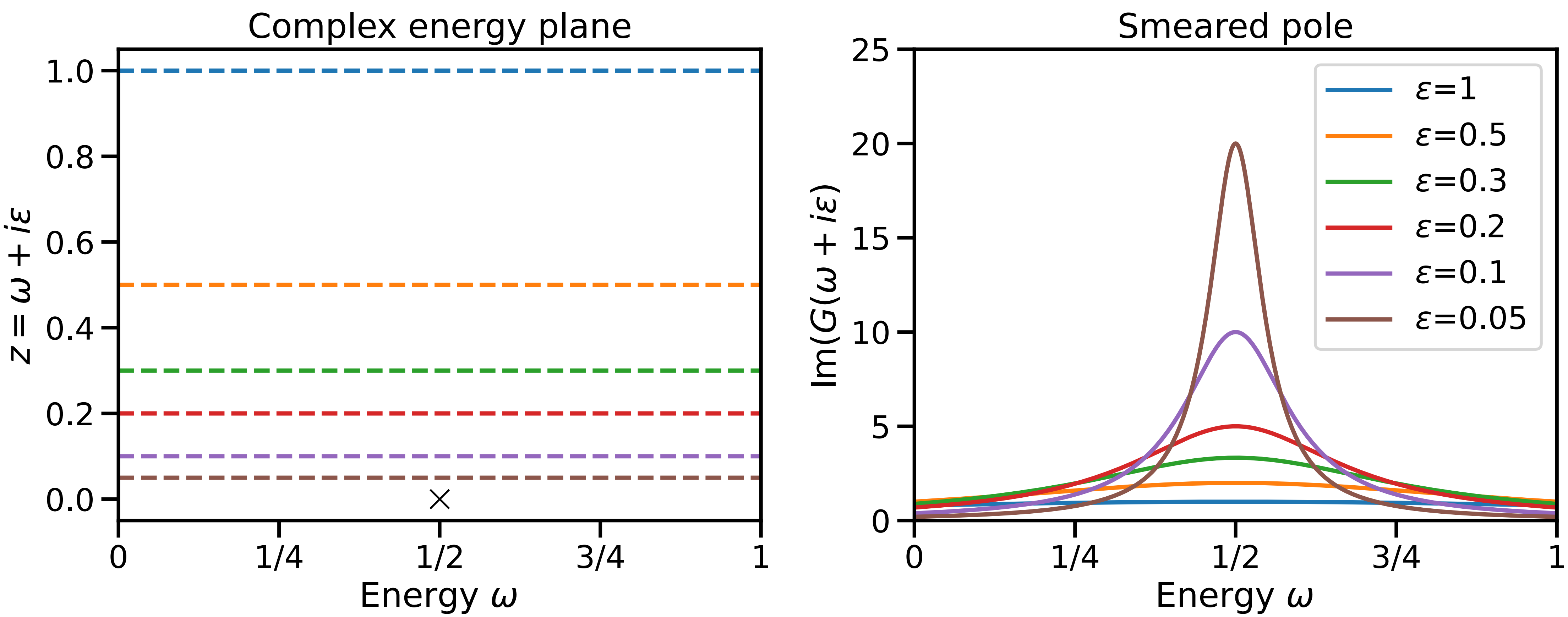


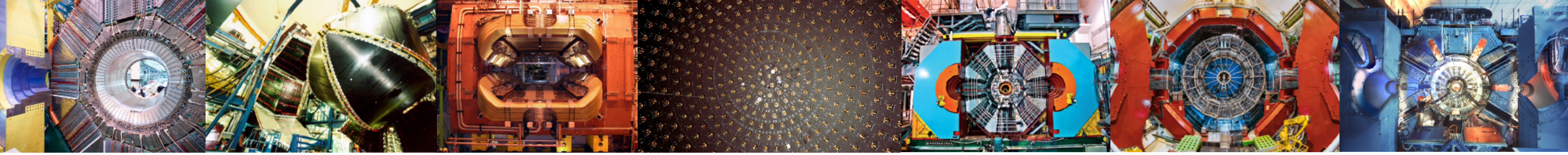
Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.





Spectral Densities

Another perspective on smearing

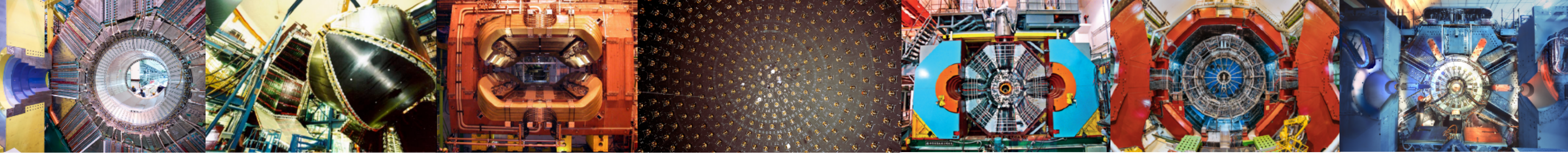
Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.

Motivates *defining*

$$\rho_\epsilon(\omega) \equiv \frac{1}{\pi}\mathrm{Im}G(\omega + i\epsilon)$$

$$\int d\omega' \delta_\epsilon(\omega - \omega')\rho(\omega')$$



Spectral Densities

Poggio, Quinn, and Weinberg
PRD 13 (1976) 1958

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.

Motivates *defining*

$$\rho_\epsilon(\omega) \equiv \frac{1}{\pi}\mathrm{Im}G(\omega + i\epsilon)$$

$$\int d\omega' \delta_\epsilon(\omega - \omega')\rho(\omega')$$

PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

Smearing method in the quark model*

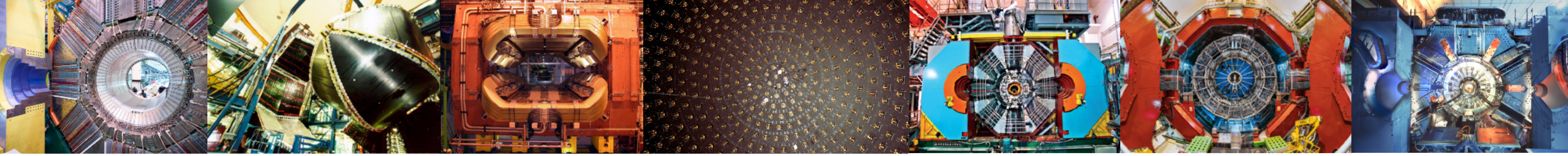
E. C. Poggio, H. R. Quinn,† and S. Weinberg
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 GeV^2 in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

The smeared ratio is defined as

$$\overline{R}(s,\Delta) = \frac{\Delta}{\pi}\int_0^\infty \frac{ds' R(s')}{(s'-s)^2+\Delta^2}. \quad (3)$$

$$2i\overline{R}(s,\Delta) = \Pi(s+i\Delta) - \Pi(s-i\Delta). \quad (5)$$



Spectral Densities

Another perspective on smearing

Consider a Green function $G(z) = 1/(z - E_0)$ with $E_0 = \frac{1}{2}$.

Look at $\mathrm{Im}G(\omega + i\epsilon)$ for various distances ϵ above real line.

Motivates *defining*

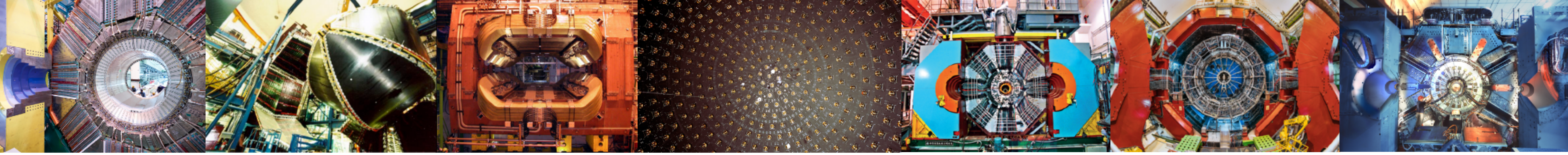
$$\rho_\epsilon(\omega) \equiv \frac{1}{\pi}\mathrm{Im}G(\omega + i\epsilon)$$

$$\int d\omega' \delta_\epsilon(\omega - \omega')\rho(\omega')$$

HLT framing of problem: this is a "Cauchy" kernel:

$$\delta_\epsilon(\omega - \omega') \equiv \frac{1}{\pi}\frac{\epsilon}{(\omega - \omega')^2 + \epsilon^2}$$

✓Becomes delta-function as $\epsilon \to 0$



The Inverse Problem

$$G(\tau) = \int \frac{d\omega}{2\pi} \rho(\omega) e^{-\omega\tau}$$

Intertwined conceptual/technical challenges:

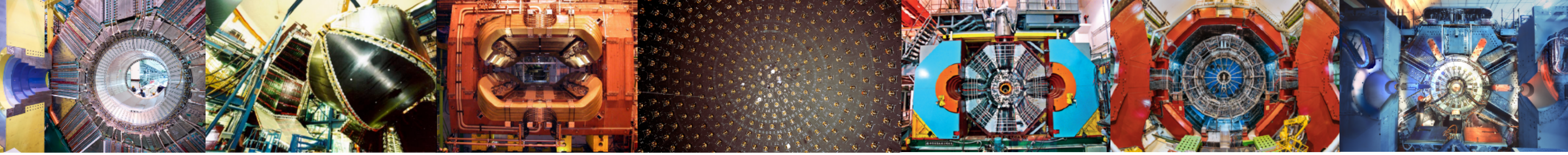
✓ Calculation in finite volume deforms the spectrum.

$$\rho(\omega) = \lim_{\epsilon \to 0} \lim_{L \to \infty} \rho_\epsilon(\omega, L)$$

2. Euclidean data is available at a finite set of points.

3. Statistical uncertainty is present.

"How much analytic information is contained in this set of points?"



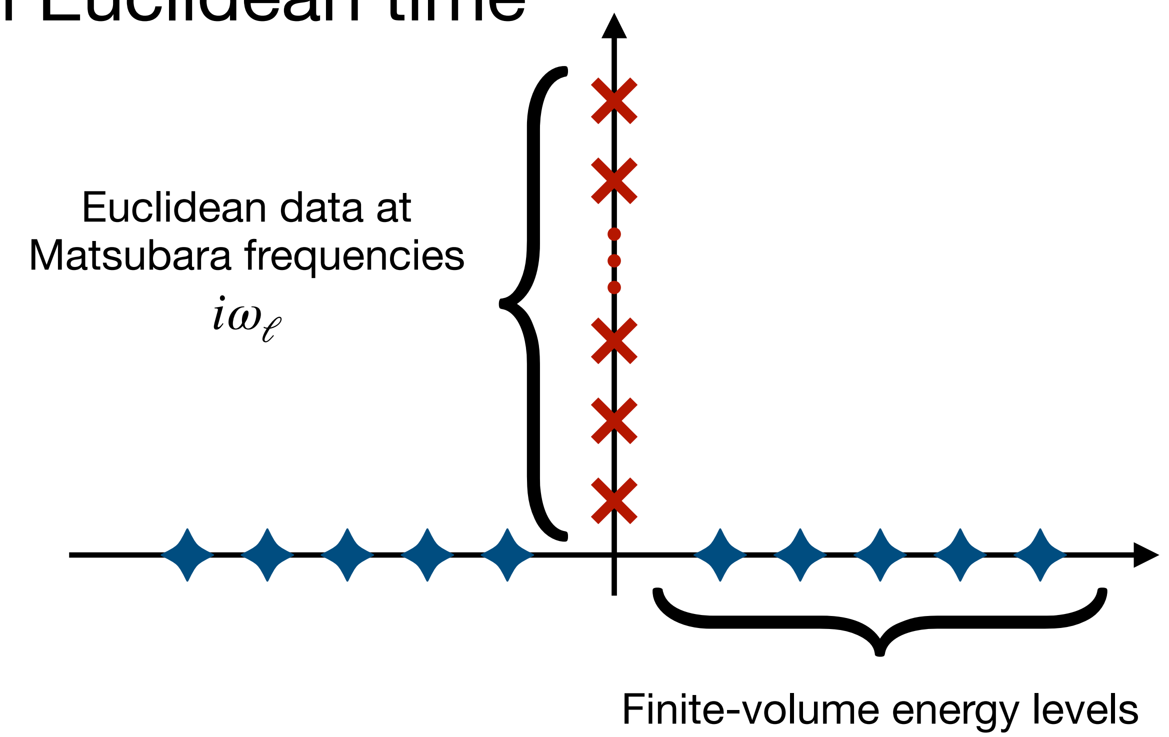
The Inverse Problem

Analytic continuation from a finite set of points

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

- Lattice QCD calculations furnish data in Euclidean time

$$G(\tau) = \sum_n \left| \langle 0 | \mathcal{O} | n \rangle \right|^2 \left(e^{-E_n \tau} + e^{-E_n(\beta - \tau)} \right)$$

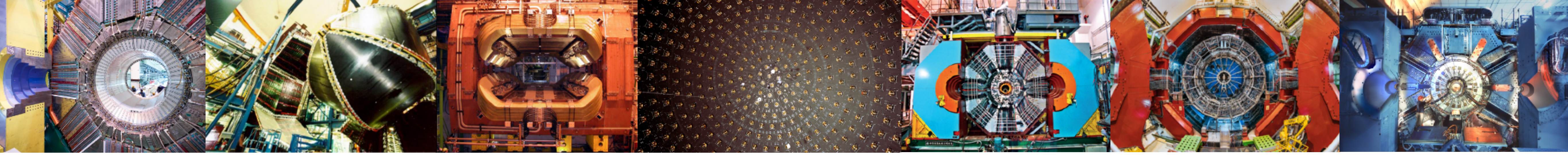


- In frequency space (take $a \ll 1$):

$$G(i\omega_\ell) = \int d\tau \, e^{i\omega_\ell \tau} G(\tau)$$

$$= \sum_n \left| \langle 0 | \mathcal{O} | n \rangle \right|^2 \left(\frac{1}{E_n + i\omega_\ell} + \frac{1}{E_n - i\omega_\ell} \right)$$

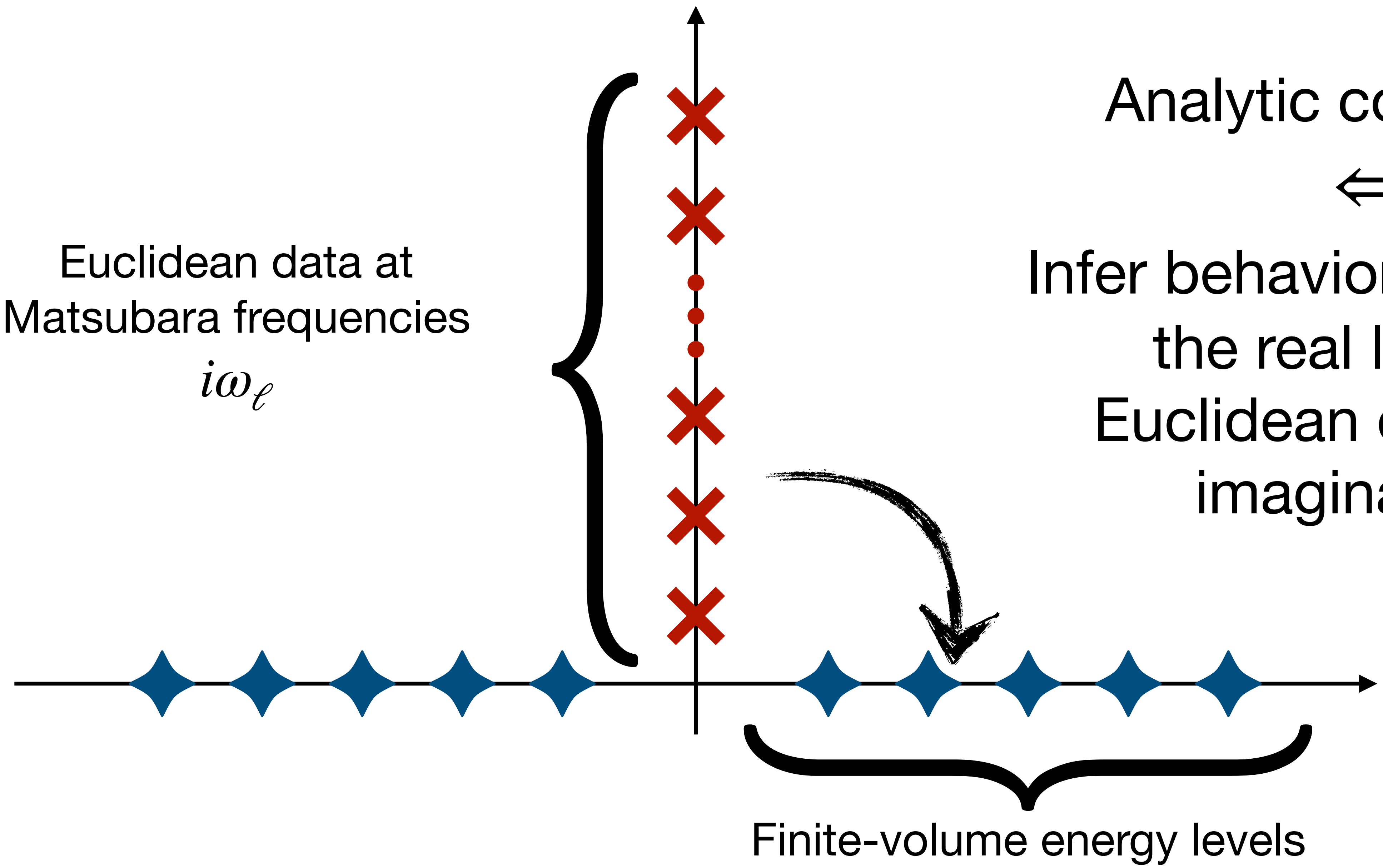
Spectral weight $\Longleftrightarrow$ Residue of pole(s)



The Inverse Problem

Analytic continuation from a finite set of points

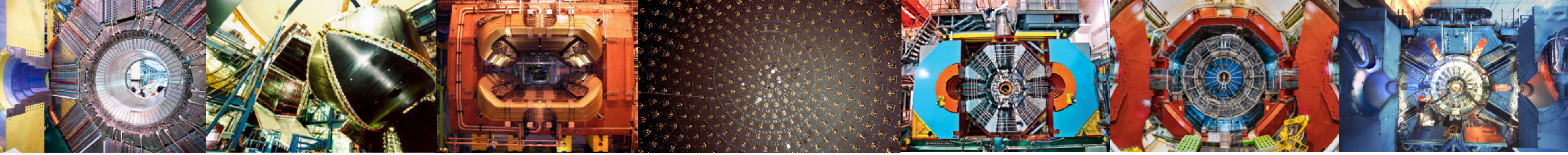
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190



Analytic continuation

$\Longleftrightarrow$

Infer behavior of $G(z)$ near the real line given Euclidean data on the imaginary axis

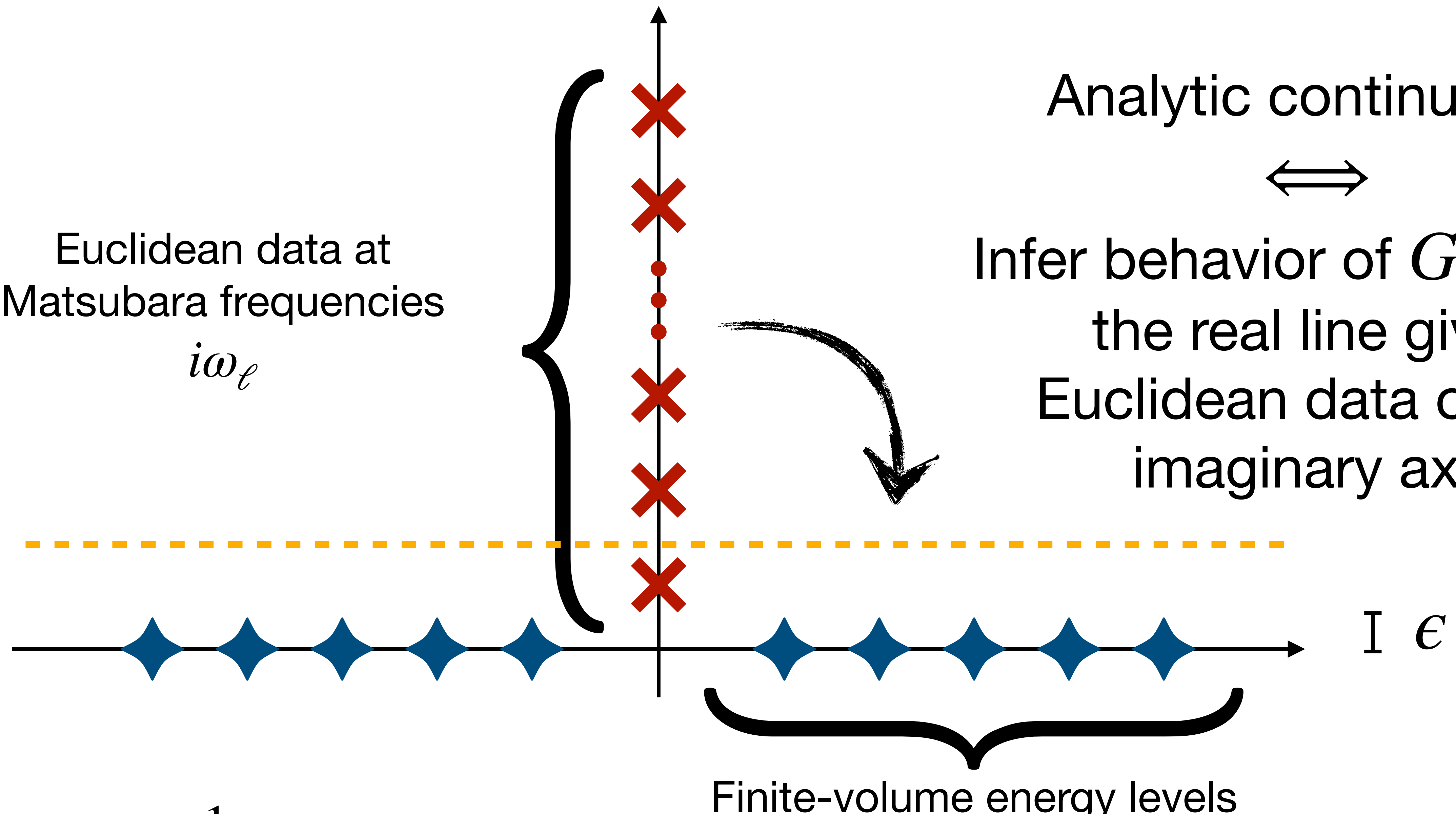


The Inverse Problem

Analytic continuation from a finite set of points

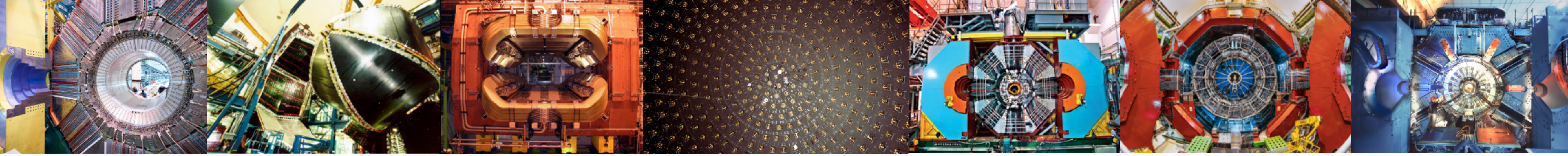
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

Euclidean data at
Matsubara frequencies
$i\omega_\ell$



Analytic continuation
$\Longleftrightarrow$
Infer behavior of $G(z)$ near the real line given Euclidean data on the imaginary axis

$\rho^{\epsilon}(\omega) \equiv \frac{1}{\pi}\text{Im}\, G(\omega + i\epsilon)$ can be viewed as a smeared spectral function.

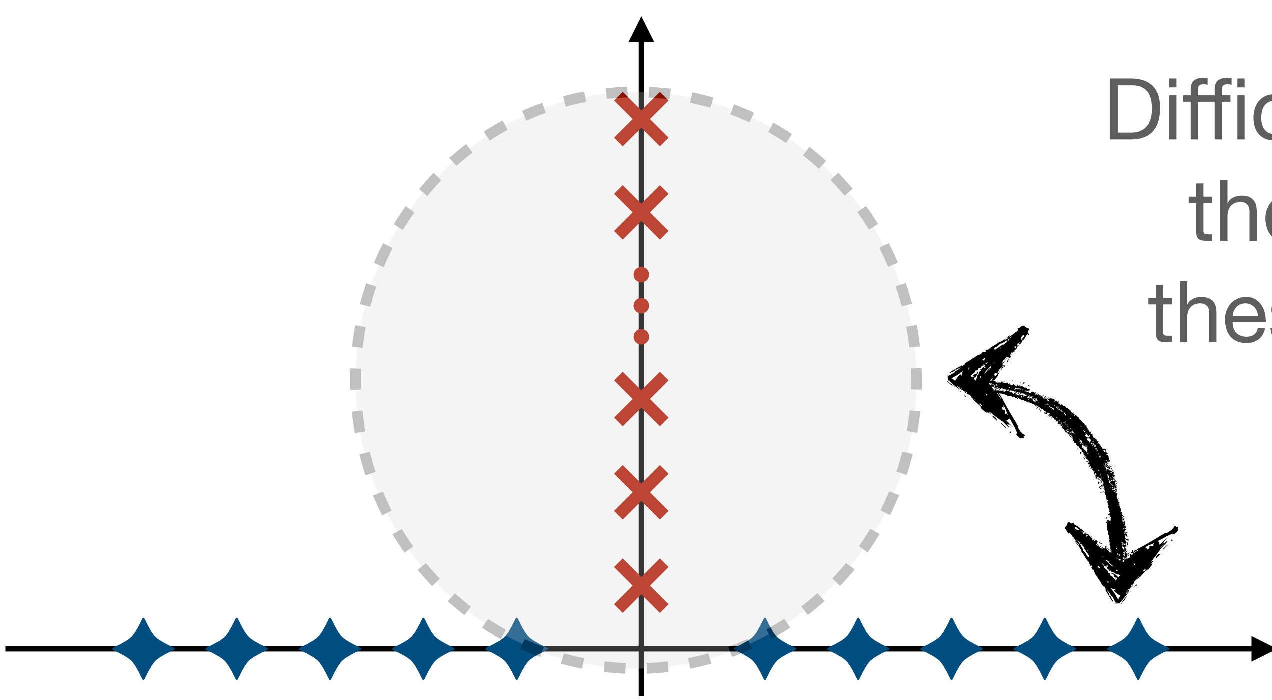


The Inverse Problem

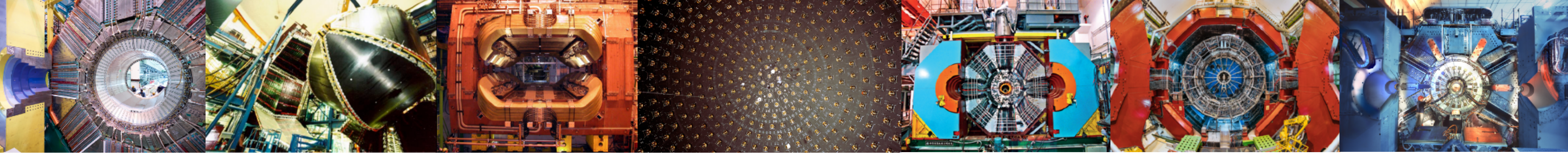
The role of conformal maps

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

- Recall: analytic functions are defined by convergent power series in an open set around each nonsingular point
- Radius of convergence is determined by the location of the nearest pole



Difficult to “see past the first pole” in these coordinates



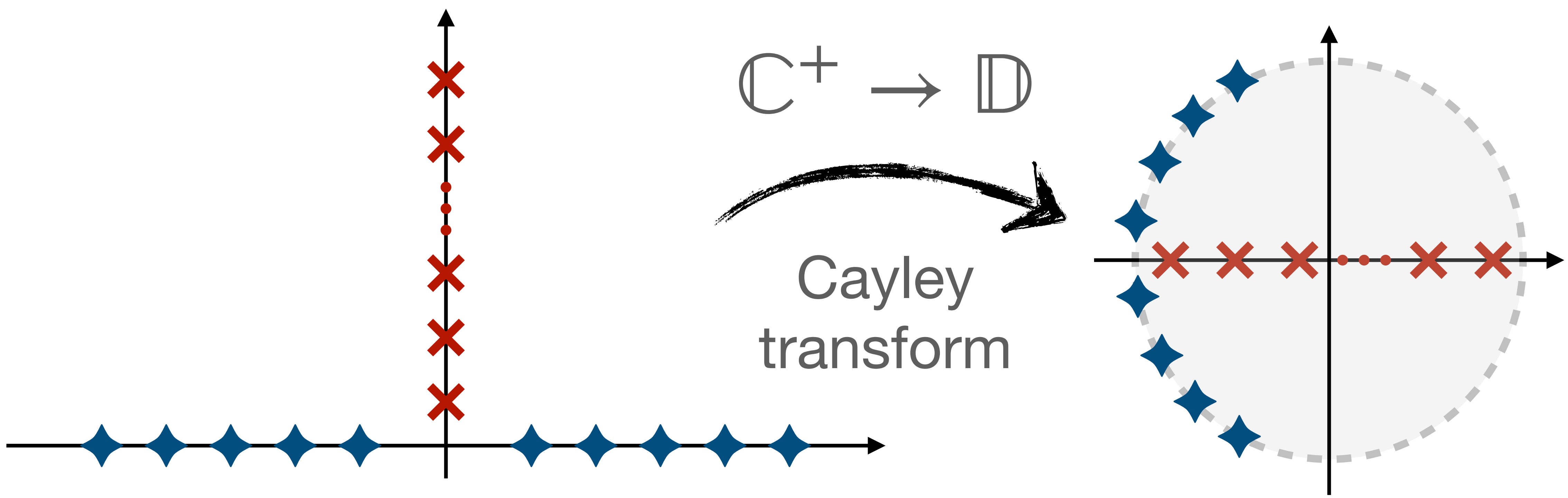
The Inverse Problem

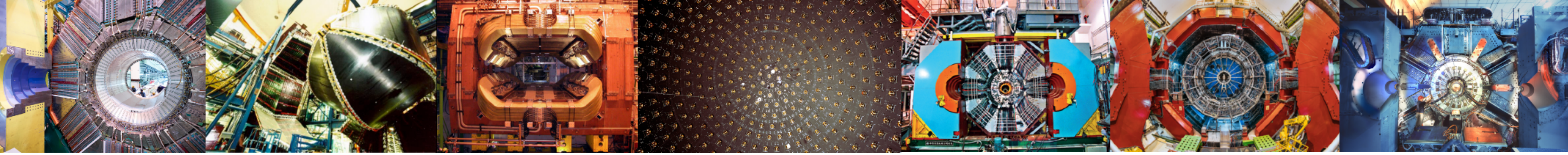
The role of conformal maps

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

- Recall: analytic functions are defined by convergent power series in an open set around each nonsingular point
- Radius of convergence is determined by the location of the nearest pole

So change coordinates!





The Inverse Problem

The sharp technical problem

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

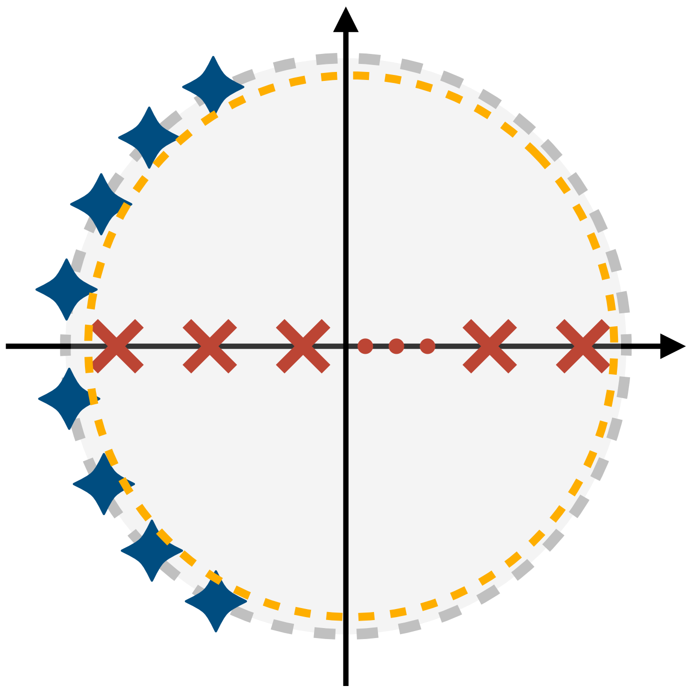
- **Given Euclidean data** $\{\zeta_l\}, \{w_l\}$

$$\{i\omega_\ell\} \to \zeta_\ell \subset \mathbb{D},$$

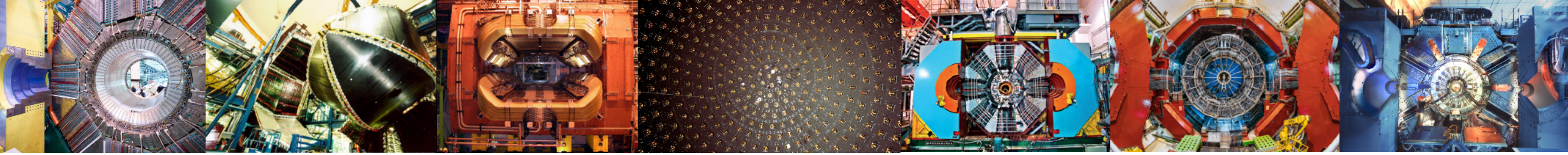
$$\{G(i\omega_\ell)\} \mapsto w_\ell \subset \mathbb{D},$$

construct an analytic function $f(\zeta)$

on the disk that interpolates these points: $f(\zeta_\ell) = w_l$.



- Evaluating this function near the boundary gives $\rho_\epsilon(\omega)$



Nevanlinna-Pick Interpolation

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

The big idea: “factor out what you know”

- **Basic fact (maximum modulus principle $\Longrightarrow$):**

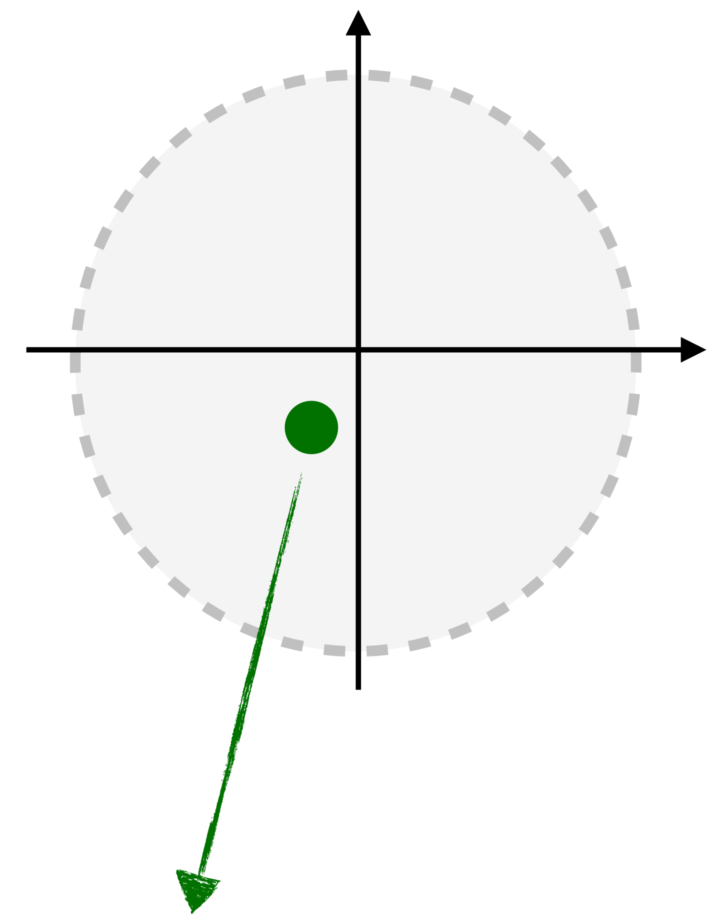
Let $g(\zeta) : \mathbb{D} \to \mathbb{D}$ be an analytic function.

Suppose $g(\zeta)$ has a zero at $\mathbf{a} \in \mathbb{D}$: $g(\mathbf{a}) = 0$.

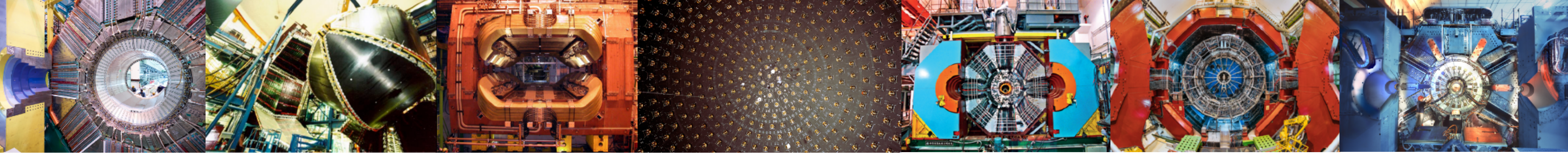
Then $g(\zeta) = b_a(\zeta)\tilde{g}(\zeta)$.

Blaschke factor

“Remainder” (analytic in $\mathbb{D}$)



$g(\mathbf{a}) = 0$



Nevanlinna-Pick Interpolation

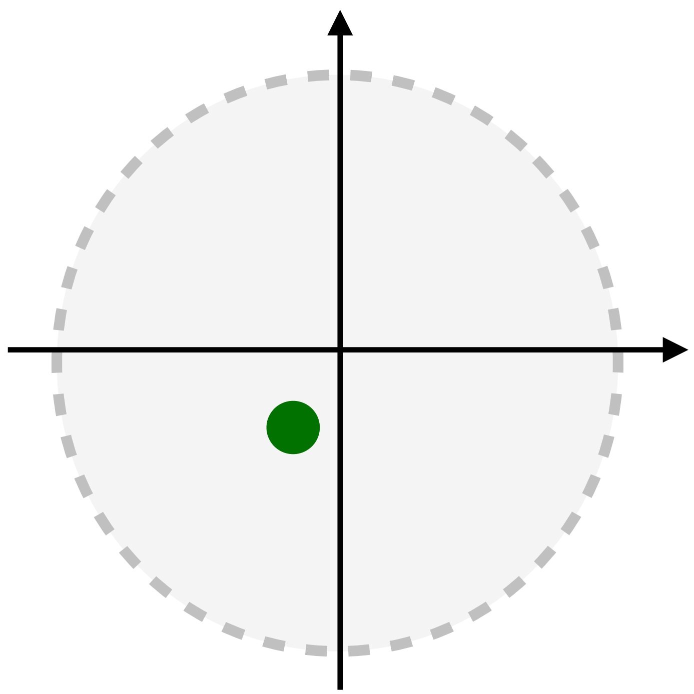
The big idea: “factor out what you know”

- **Basic fact (maximum modulus principle $\Longrightarrow$):**

  Let $g(\zeta) : \mathbb{D} \to \mathbb{D}$ be an analytic function.

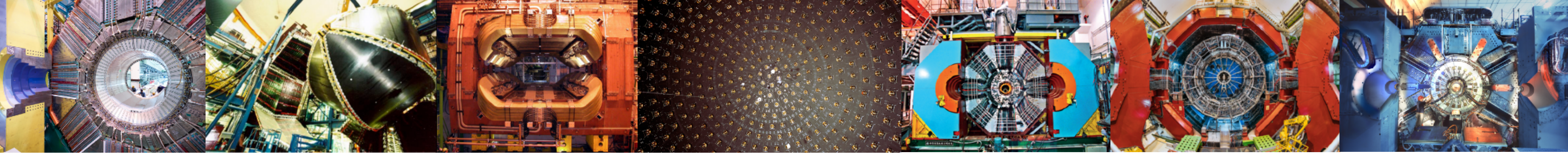
  Suppose $g(\zeta)$ has a zero at $\mathbf{a} \in \mathbb{D}$: $g(\mathbf{a}) = 0$.

  Then $g(\zeta) = b_a(\zeta)\tilde{g}(\zeta)$.



- Note: Setup familiar in quark-flavor physics from *z-expansion* of form factors
  - Blaschke factors "factor out” known analytic structure, e.g., sub-threshold poles.

Boyd, Grinstein, Lebed
Nucl.Phys.B **461 (1996) 493-511**
Phys.Rev.D **56 (1997) 6895-6911**
Caprini, Lellouch, Neubert
Nucl.Phys.B 530 (1998) 153-181



Analytic Continuation

Repeated application of “factoring”

R. Nevanlinna
Ann. Acad. Sci. Fenn. Ser. A 13 (1919)
Ann. Acad. Sci. Fenn. Ser. A 32 (1929)

A. Nicolau
Proc. Summer School in Complex and Harmonic analysis… (2016)
[LINK]

First application in QFT
(Condensed Matter Physics)
J. Fei, C.-N. Yeh, E. Gull,
PRL 126, 056402 (2021)
arXiv:2010.04572

Theorem (Nevanlinna, 1919/1929):

- Any solution to the interpolation problem with N points can be written in the form

$$f(\zeta) = \frac{P_N(\zeta) f_N(\zeta) + Q_N(\zeta)}{R_N(\zeta) f_N(\zeta) + S_N(\zeta)}.$$

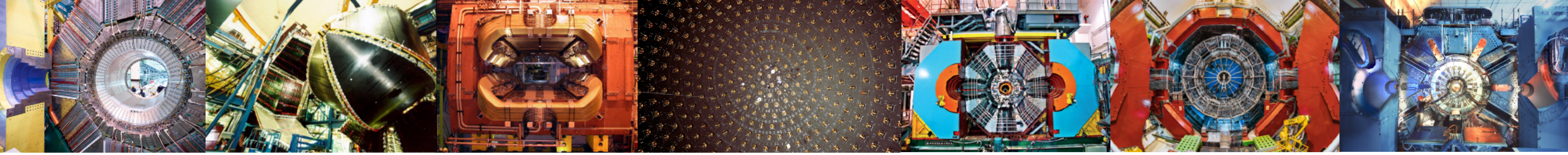
- “*Nevanlinna coefficients*” P_N, Q_N, R_N, S_N

  $\Longleftrightarrow$ Known / calculable from input data

- Arbitrary function analytic function $f_N(\zeta) : \mathbb{D} \to \mathbb{D}$

  $\Longleftrightarrow$ Freedom to specify further Euclidean data to constrain the interpolating function

  $\Longleftrightarrow$ Plays role of the “remainder” function on the previous slide



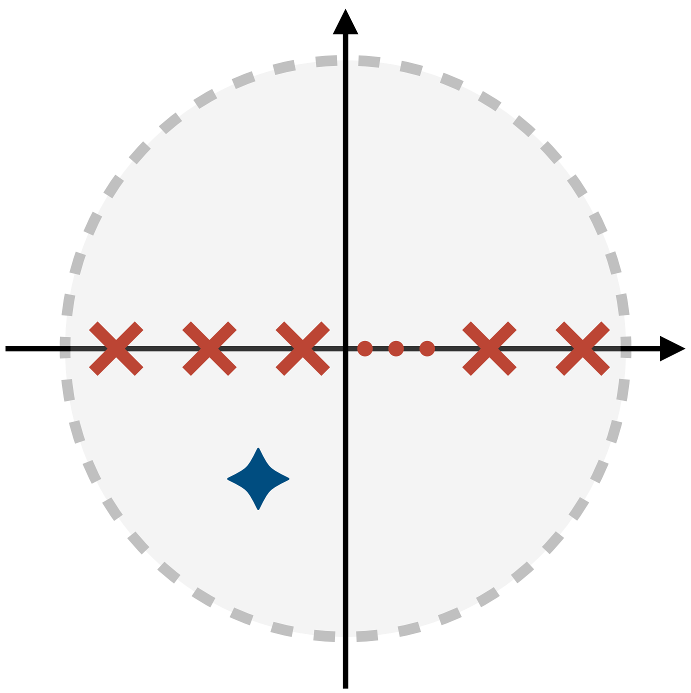
Analytic Continuation

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

The full space of solutions

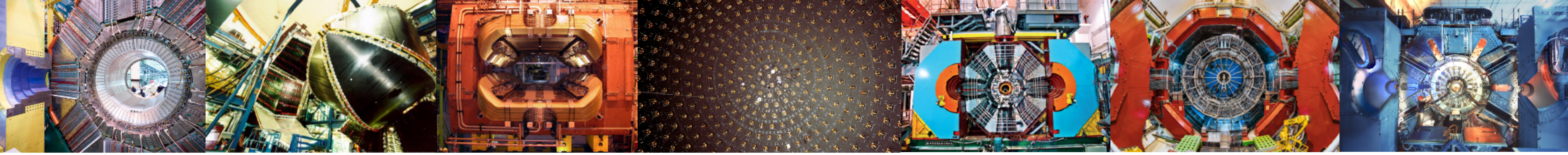
- Key point: The freedom and influence of the "remainder" is constrained, since $f_N(\zeta) \in \mathbb{D}$.

- **Question**: What possible values can the interpolating function $f(\zeta)$ can take when extrapolated to arbitrary points "✦"?
    - Remarkably, this set can be parameterized explicitly for each N and each point "✦".
    - Size of this set $\Longleftrightarrow$ ambiguity in the analytic continuation



× = given

$f(✦) = ?$



Analytic Continuation

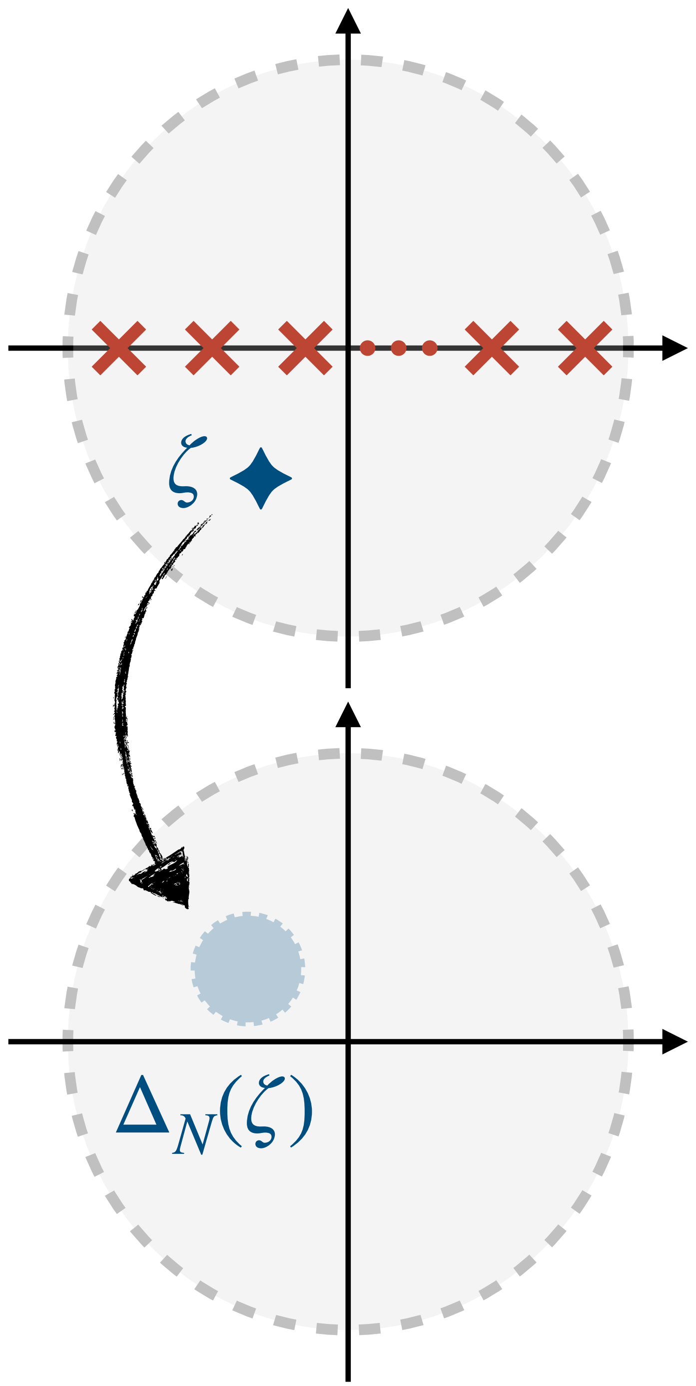
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

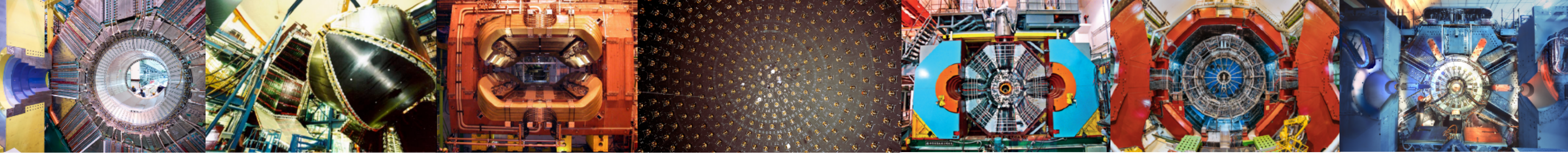
The full space of solutions

- **Answer**: The space of possible values is a disk of radius $r_N(\zeta)$ centered at $c_N(\zeta)$. This disk called the *Wertevorrat* $\Delta_N(\zeta)$,.

$$c_N = \frac{P_N\overline{(-R_N/S_N)} + Q_N}{R_N\overline{(-R_N/S_N)} + S_N} \qquad r_N = \frac{|P_N S_N - Q_N R_N|}{|S_N|^2 - |R_N|^2}$$

- Given N interpolation points, the *Wertevorrat* $\Delta_N(\zeta)$ rigorously contains all possible analytic continuations at each extrapolation point $\zeta \in \mathbb{D}$.
  - Complete characterization of systematic uncertainty
  - No "regularization" beyond smearing
  - No model assumptions — just analyticity!



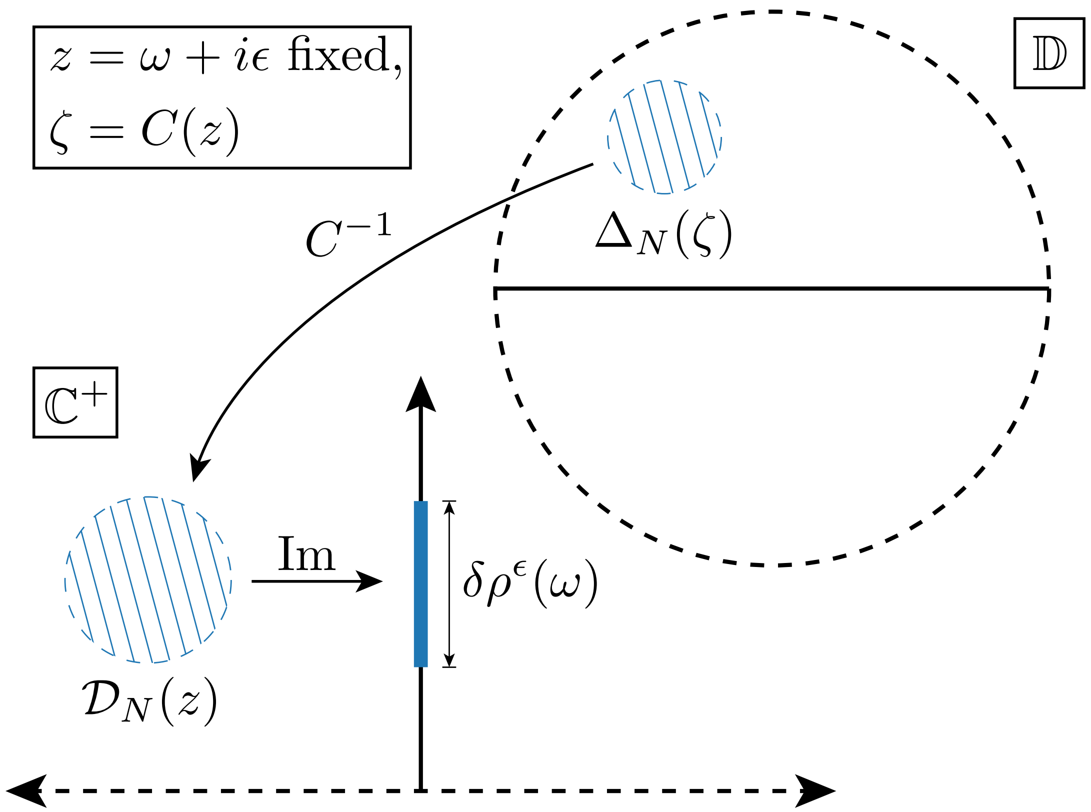


Analytic Continuation

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

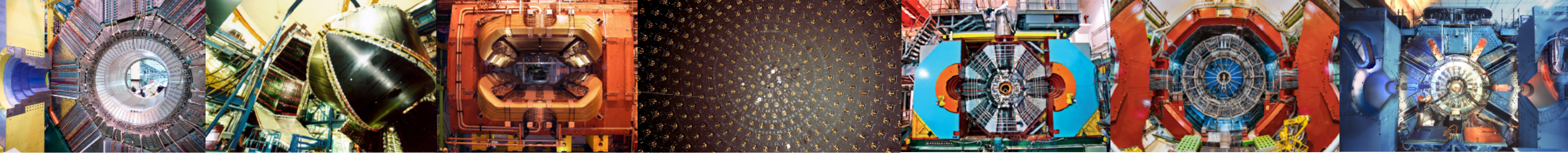
Back to the upper half-plane

- Map the *Wertevorrat* back to the original coordinates



$$\rho^\epsilon(\omega) = \frac{1}{\pi} \text{Im}\, G(\omega + i\epsilon)$$

$$\delta\rho^\epsilon(\omega) = \frac{1}{\pi} \Big[\max \text{Im}\, \partial D_N(\omega + i\epsilon) - \min \text{Im}\, \partial D_N(\omega + i\epsilon)\Big]$$



Numerical Example

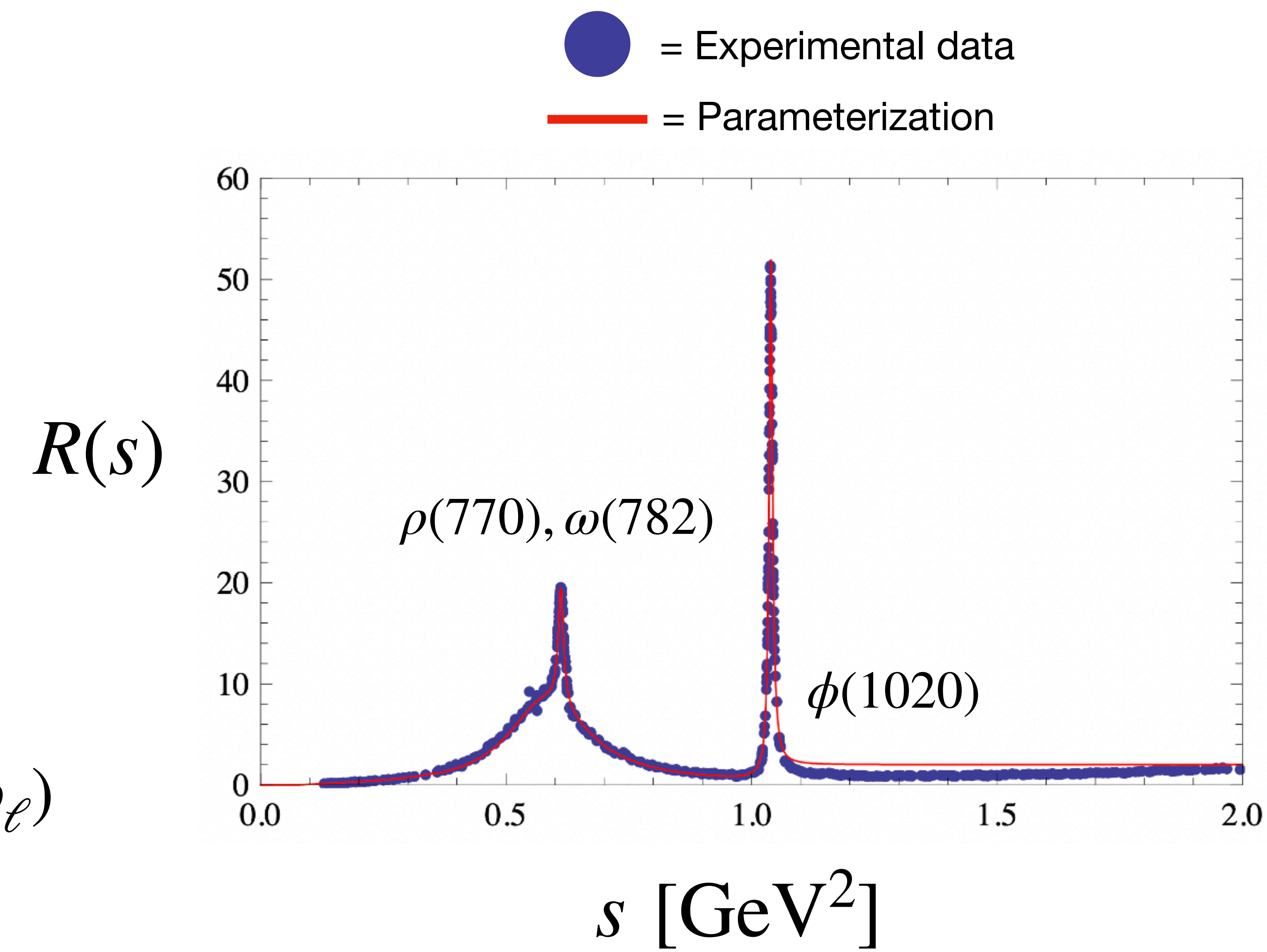
Bernecker and Meyer
Eur.Phys.J.A 47 (2011) 148
arXiv:1107.4388

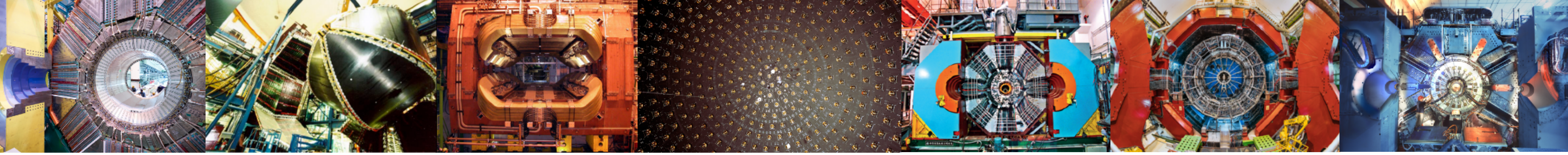
The R-ratio — reconstructing a parameterization

- Bernecker and Meyer give a useful parameterization of R-ratio data
- This parameterization can serve as input for a spectral reconstruction
- Can easily convert:
  $R(s) \iff \rho(\omega) \iff G(i\omega_\ell)$

Formula from beginning of talk

“Laplace transform”



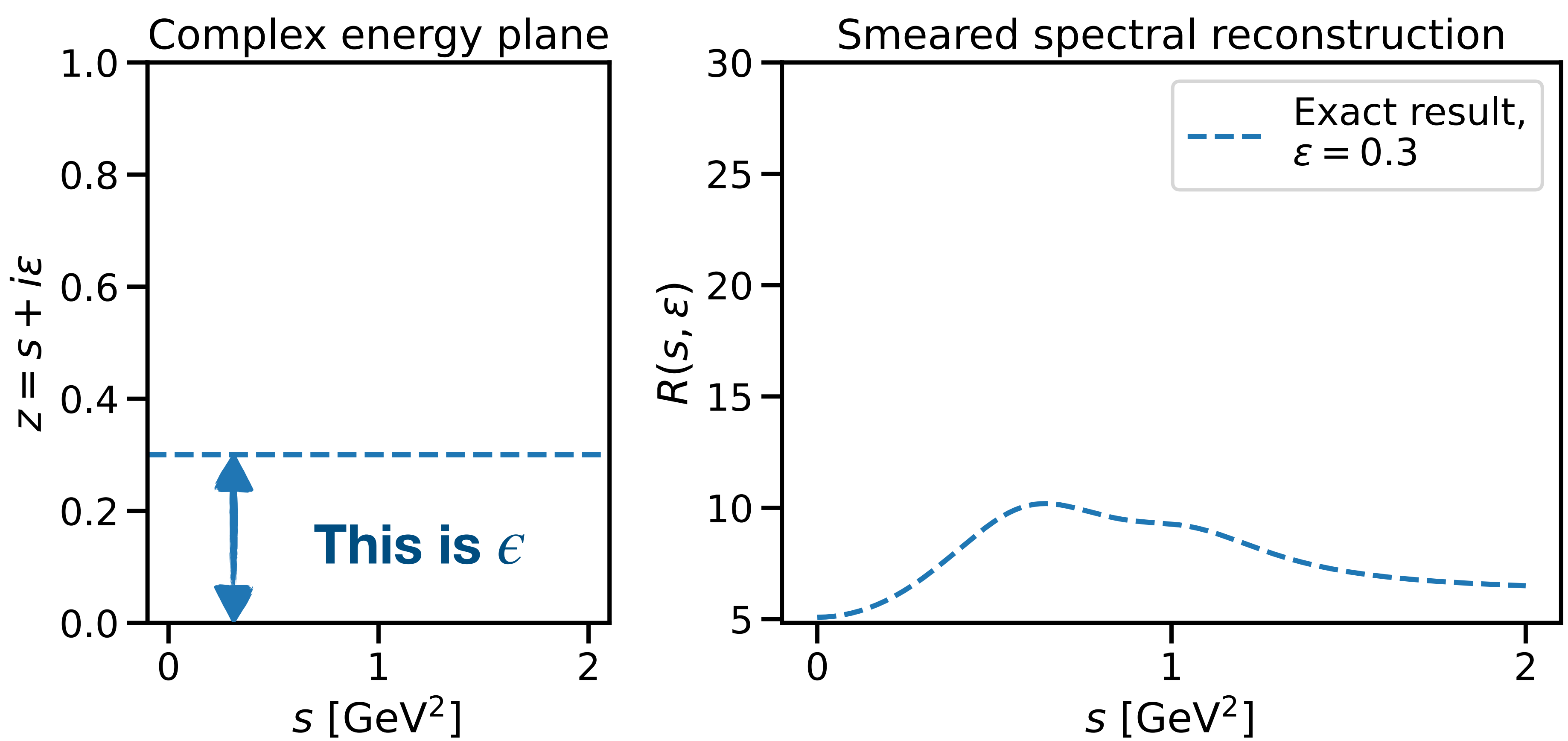


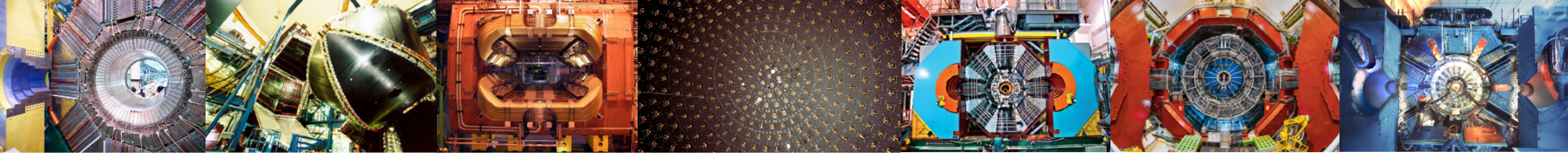
Numerical Example

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis
- Run reconstruction for different smearing widths ϵ





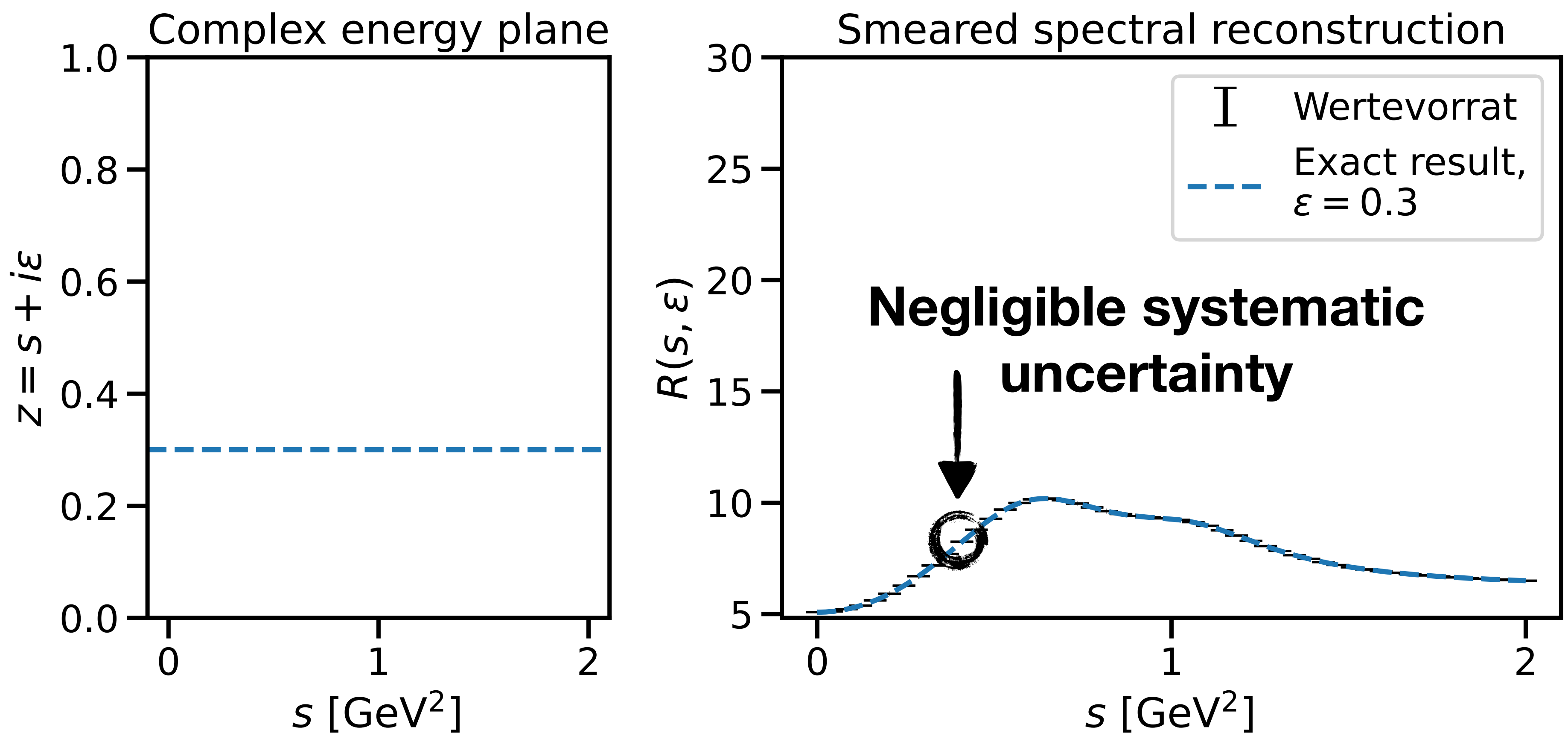
Numerical Example

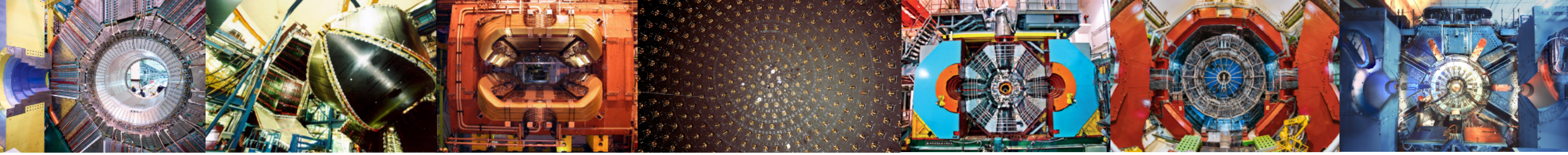
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis
- Run reconstruction for different smearing widths ϵ

✓ Exact answer is contained within the bounding envelope of the Wertevorrat





Numerical Example

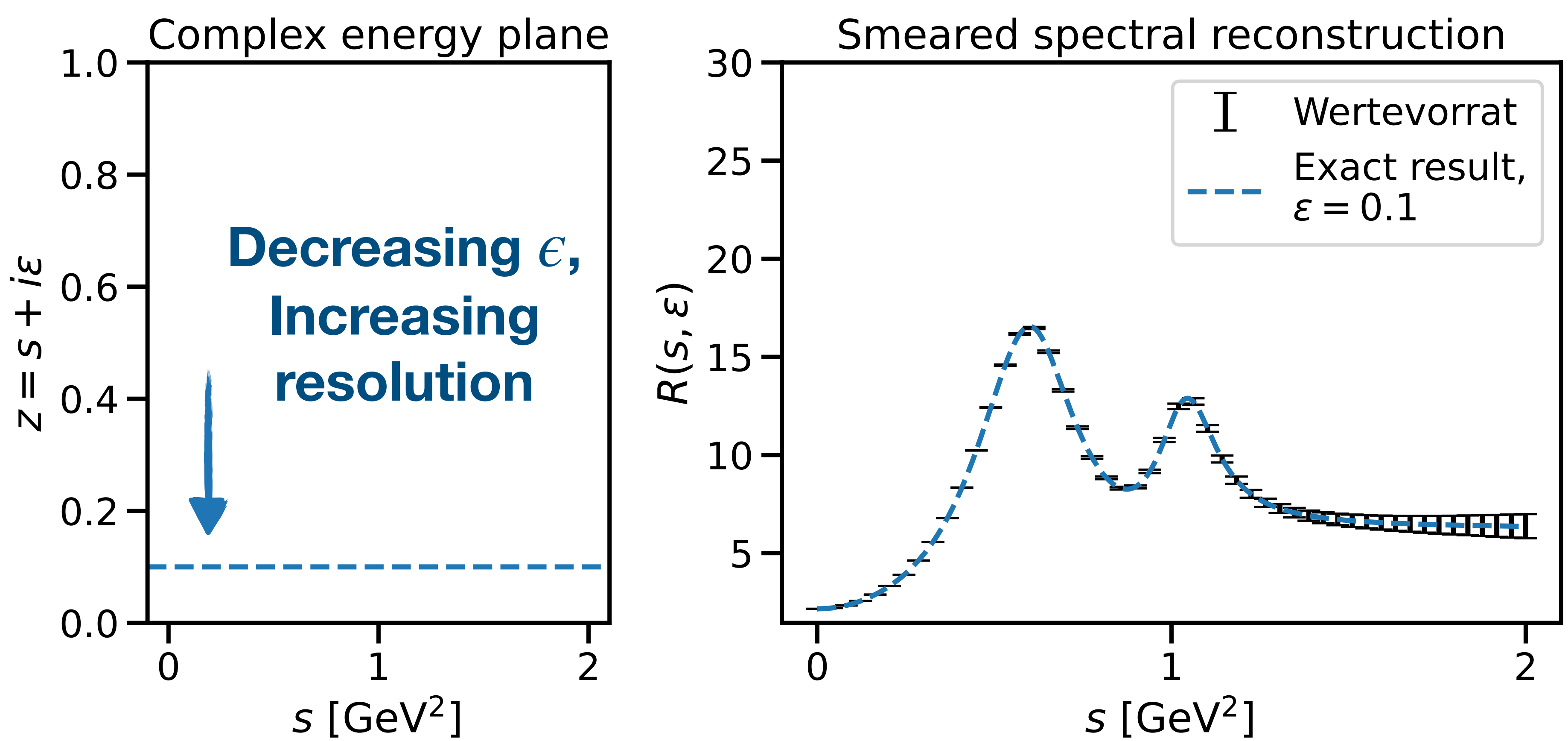
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

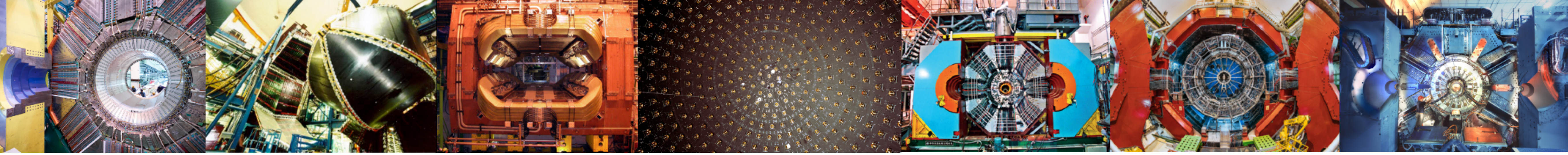
The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis
- Run reconstruction for different smearing widths ϵ

✓ Exact answer is contained within the bounding envelope of the Wertevorrat

✓ Spectral peaks from $\rho(770)/\omega(782)$ and $\phi(1020)$ clearly visible in reconstructions





Numerical Example

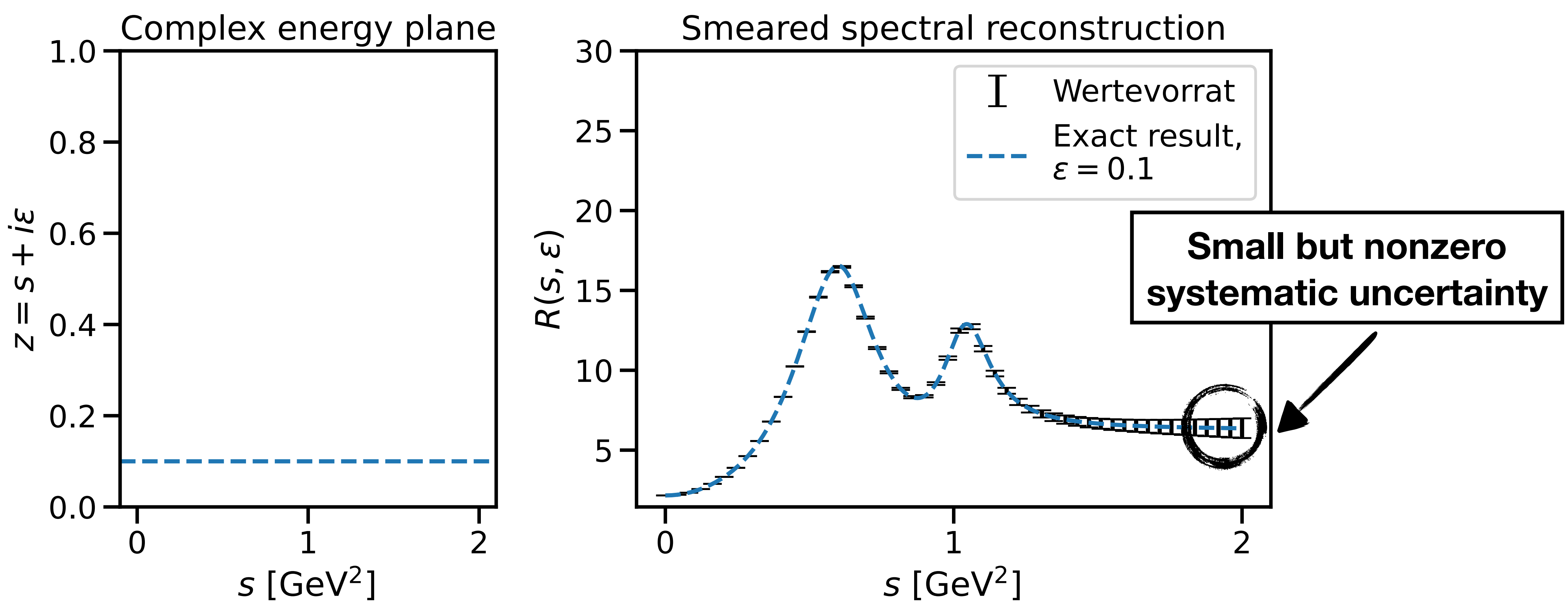
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

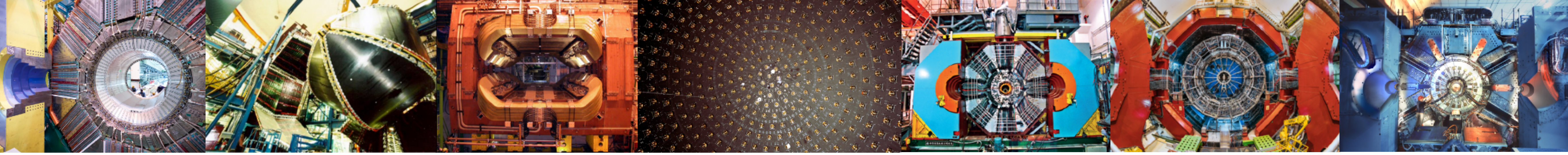
The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis
- Run reconstruction for different smearing widths ϵ

✓ Exact answer is contained within the bounding envelope of the Wertevorrat

✓ Spectral peaks from $\rho(770)/\omega(782)$ and $\phi(1020)$ clearly visible in reconstructions





Numerical Example

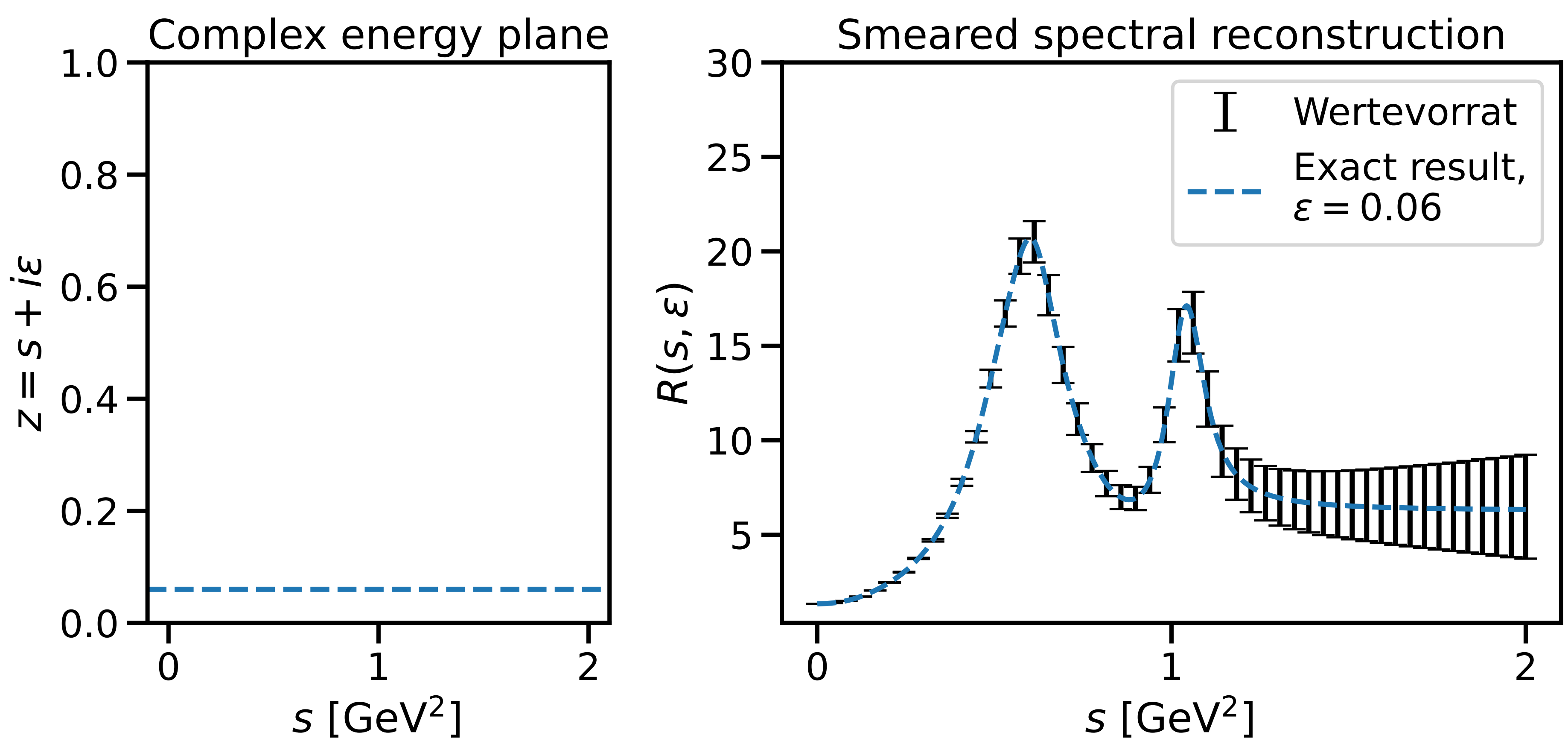
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

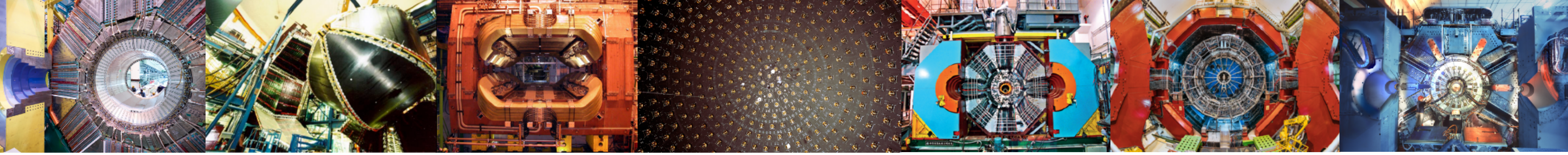
The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis
- Run reconstruction for different smearing widths ϵ

✓ Exact answer is contained within the bounding envelope of the Wertevorrat

✓ Spectral peaks from $\rho(770)/\omega(782)$ and $\phi(1020)$ clearly visible in reconstructions



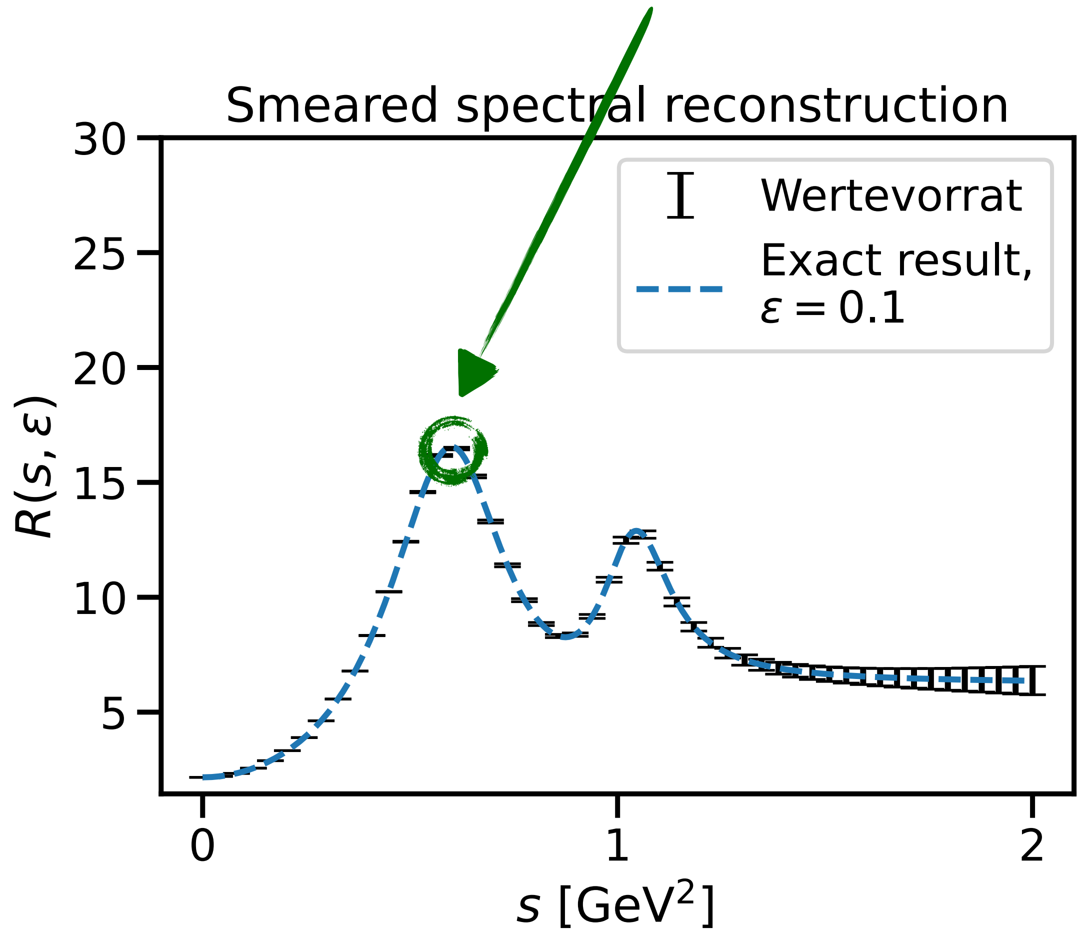


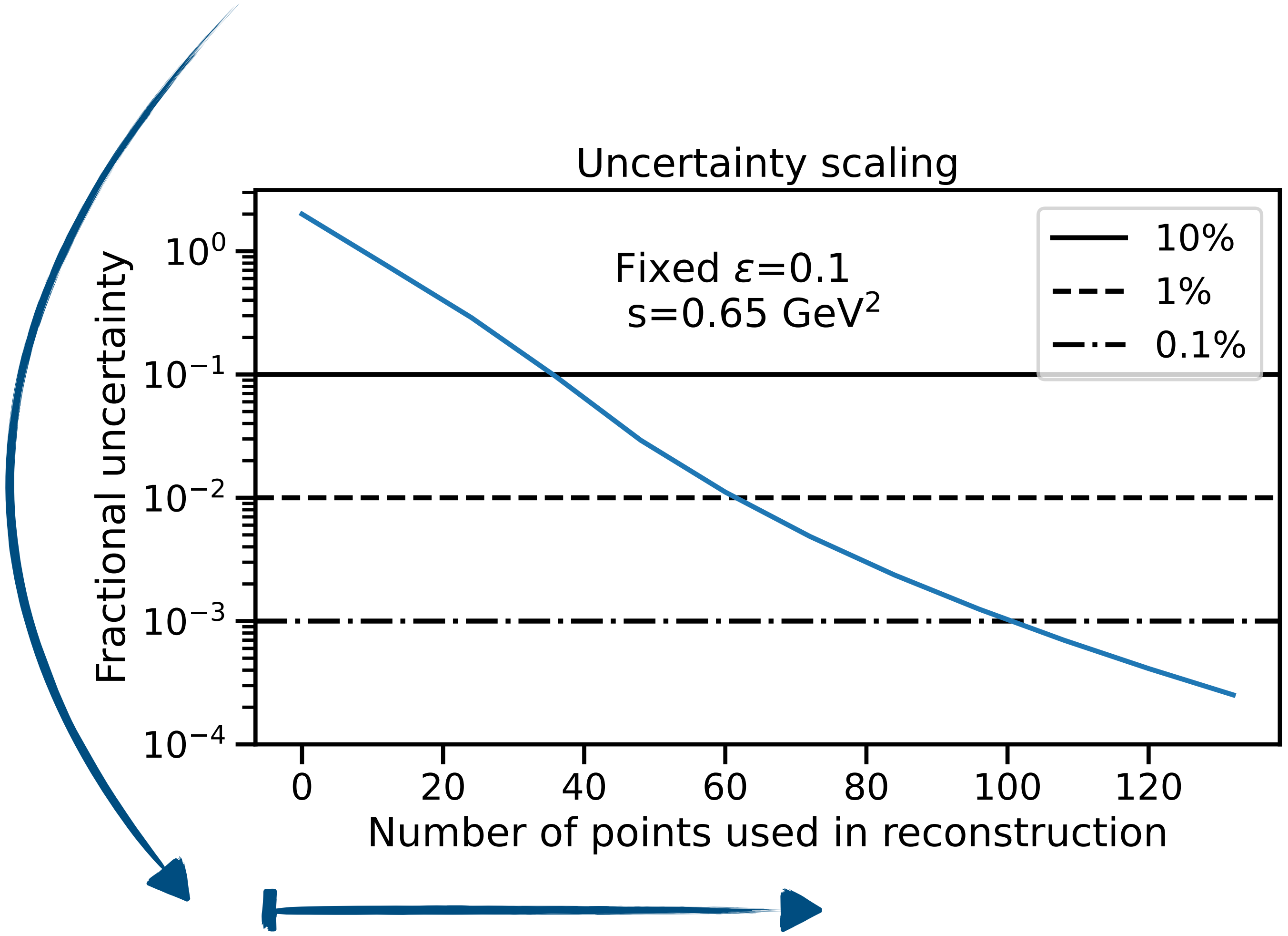
Numerical Example

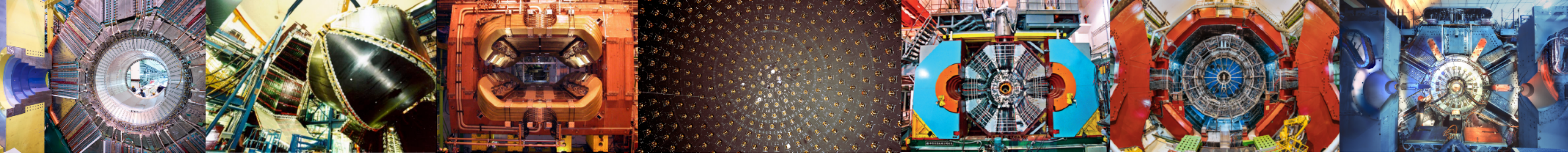
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

The R-ratio — reconstructing a parameterization

- Euclidean data generated for $\beta = 144$ total points on the imaginary-energy axis
- How does the size of the Wertevorrat scale with the number of points?
- Fix reconstruction energy and smearing ϵ. Vary number of points used in reconstruction.







Numerical Example

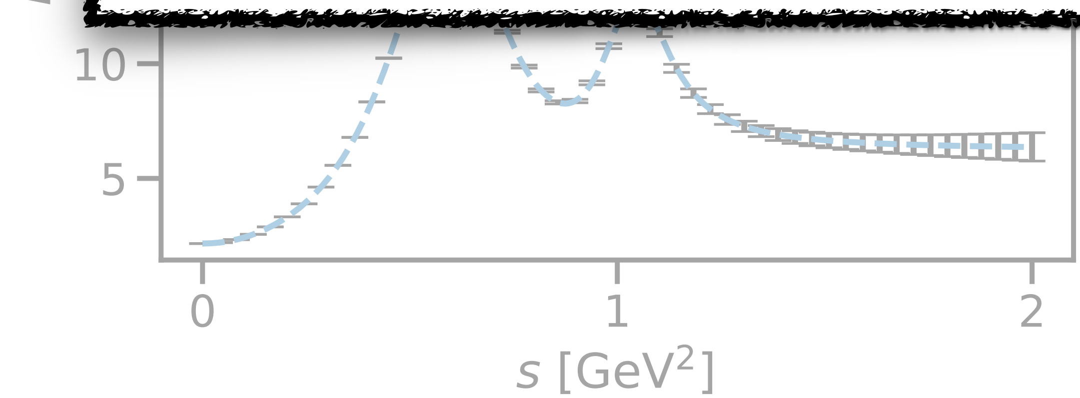
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

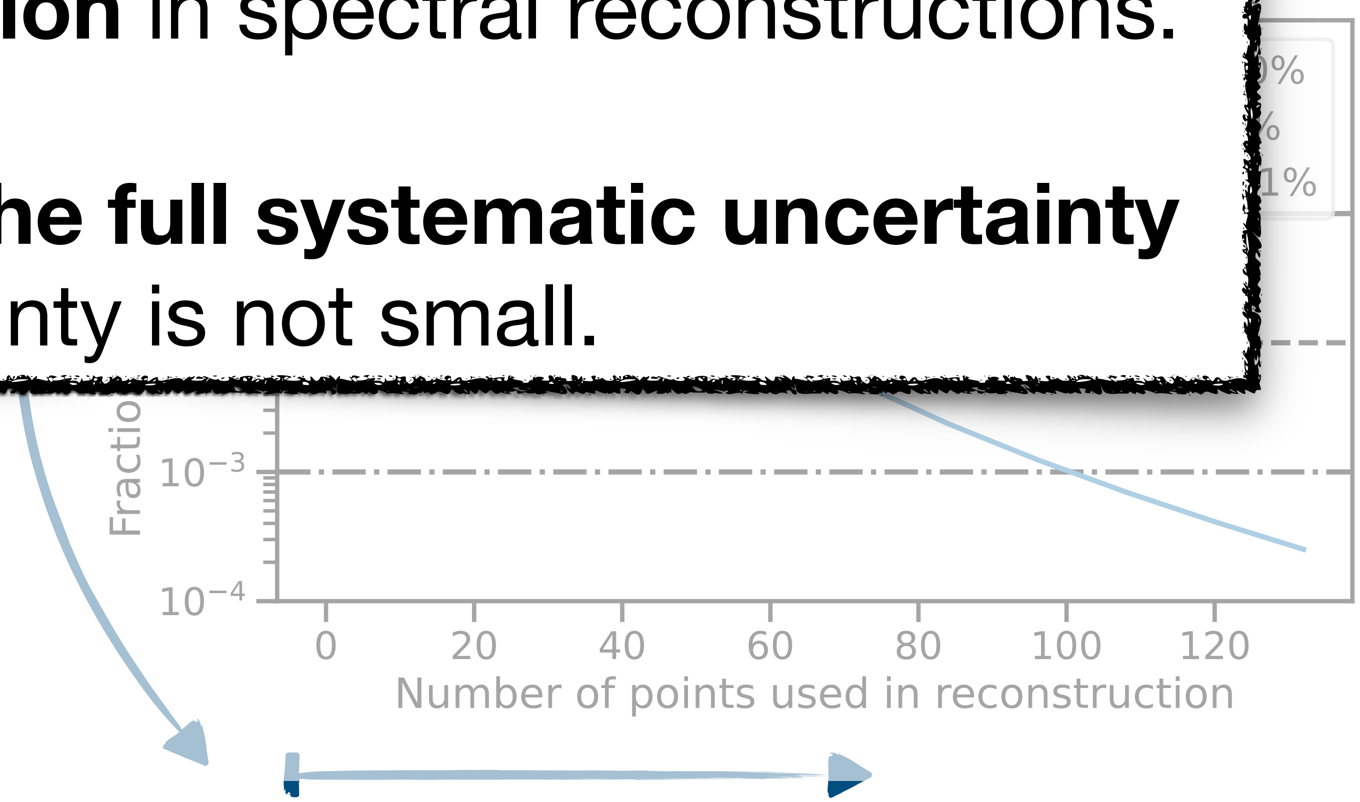
The R-ratio — reconstructing a parameterization

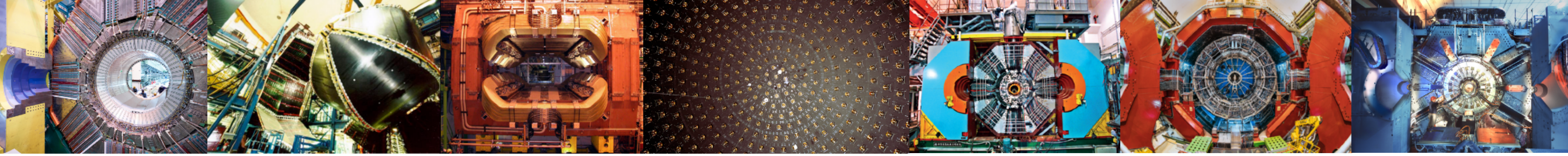
- Euclidean data generated for $\beta = 144$ total points on the imaginary-energy axis
- How does the size of the Wertevorrat scale with the number of points?

The Wertevorrat offers a **systematically improvable approach** for **increased energy resolution** in spectral reconstructions.

The Wertevorrat **bounds the full systematic uncertainty** — even when this uncertainty is not small.







The Inverse Problem

$$G(\tau) = \int \frac{d\omega}{2\pi} \rho(\omega) e^{-\omega\tau}$$

Intertwined conceptual/technical challenges:

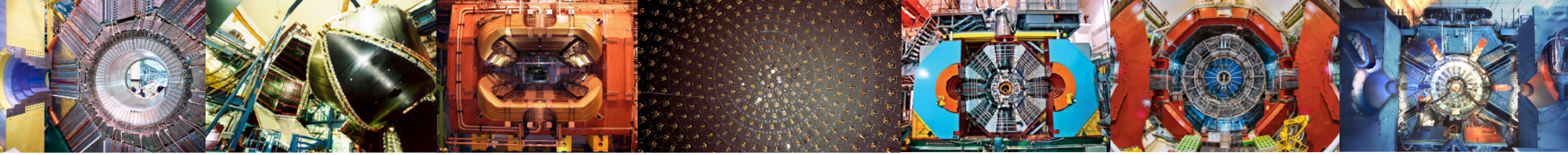
✓ Calculation in finite volume deforms the spectrum. $\rho(\omega) = \lim_{\epsilon \to 0} \lim_{L \to \infty} \rho_\epsilon(\omega, L)$

✓ Euclidean data is available at a finite set of points. $\delta\rho^\epsilon(\omega) \sim \mathrm{Im}\, \partial D_N(\omega + i\epsilon)$

3. Statistical uncertainty is present.

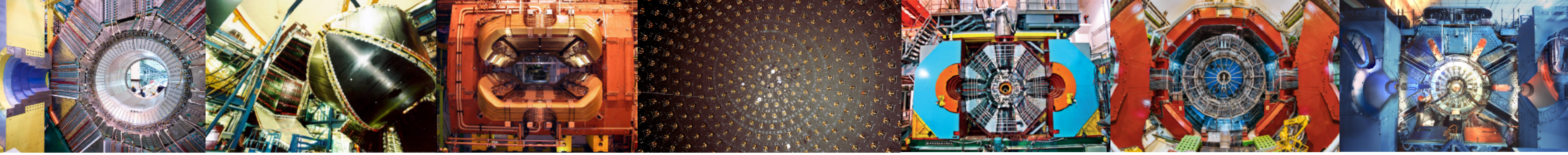
- Regularize, e.g., as with HLT or other familiar methods
- Impose analytic self-consistency conditions on statistical noise.

Abbott, WJ, Oare
In progress
arXiv: 240?.XXXXX

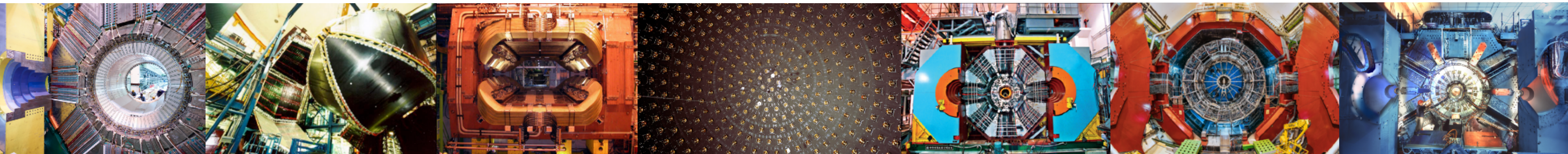


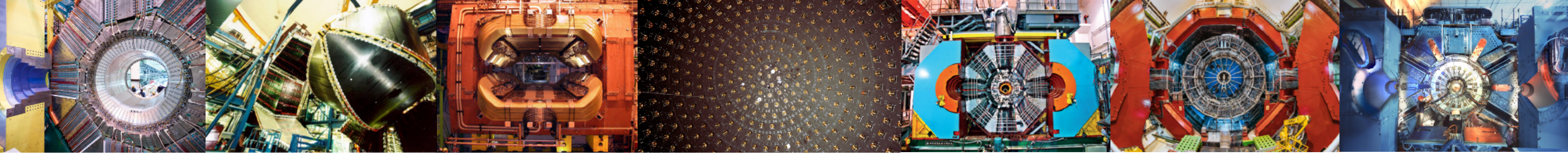
Summary

- Inclusive quantities contain a wealth of hadronic information
- A fresh look at these observables is timely:
  - Muon (g-2) and the R-ratio
  - $|V_{ud}|$, $|V_{us}|$, and the “Cabibbo anomaly”
  - Inclusive versus exclusive determinations of $|V_{cb}|$ and $|V_{ub}|$
  - Hadronic structure needed for upcoming DUNE and the EIC experiments
- Recent improved practical and formal understanding of the challenges associated with spectral reconstruction
- Exciting calculations have appeared over the past few years. I expect the community to see many more in the coming years.
- Lots of important and exciting work is happening in our field that I haven’t had time to discuss!



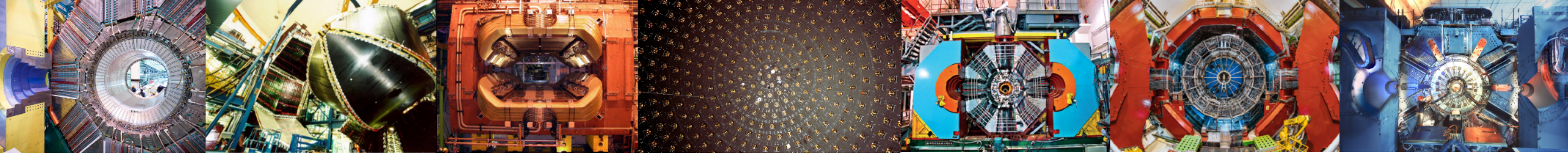
Backup slides





What about $\sin(iz)$ and friends?

- Recall: a green function is a map $G(z) : \mathbb{H} \to \mathbb{H}$ ($\mathbb{H}$=upper half-plane)
  - Functions with this property are called *Nevanlinna functions*
  - Roughly speaking, any Nevanlinna function can be written as an integral of a suitable spectral function.
  - Mapping the problem to the disk to invoke Nevanlinna's theorem invokes these properties in an essential way.
  - In other words, *the interpolating function $f : \mathbb{D} \to \mathbb{D}$ already and automatically has the correct analytic structure*
- The function $\sin(iz)$:
  - Vanishes at infinitely many points, e.g., $z \in i\pi\,\mathbb{N}$
  - Blows up to $\pm\infty \implies$ <u>Not</u> a function $\mathbb{H} \to \mathbb{H}$.
  - Has the wrong singularity structure/asymptotic behavior.
- Constructing an interpolating function $f : \mathbb{D} \to \mathbb{D}$ automatically excludes inconsistent/pathological functions like $\sin(iz)$. This property holds when translated back to $G(z) : \mathbb{H} \to \mathbb{H}$.



What about statistical noise?

The method announces its failure in two ways.

1. The Wertevorrat is expected to decrease monotonically as more information is included. If the radius of the Wertevorrat begins to jitter around some "saturation width," numerical precision has become a limiting factor.

2. Nevanlinna's theorem assumes the data satisfy an analytic self-consistency condition: the Pick matrix P_{ij} must be positive semi-definite.

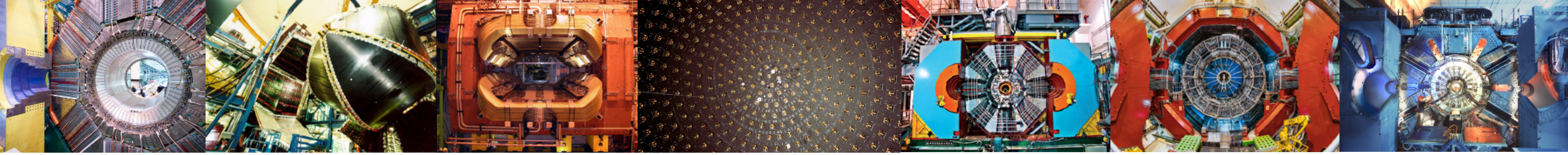
$$P_{ij} = \frac{1 - w_i \bar{w}_j}{1 - \zeta_i \bar{\zeta}_j}$$

Possible Solutions

A. Check this condition and avoid data that violate the hypotheses of the theorem.

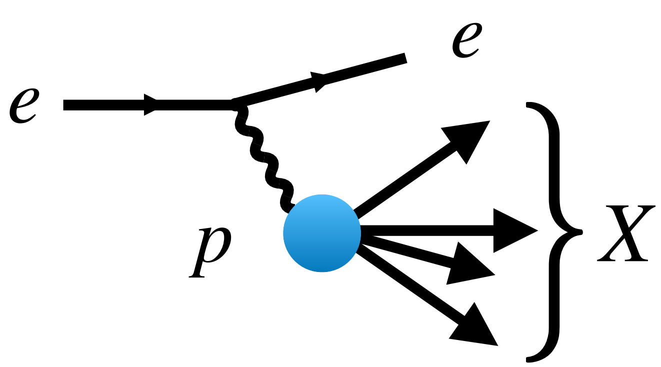
B. Rephrase the difficulty as a statistical pre-denoising problem:

Given a statistical sample of $\mathbf{G} \in \mathbb{R}^N$, project to the closest set of points $\mathbf{G}' \in \mathbb{R}^N$ such that P_{ij} is positive semidefinite. "Closest" is determined by the covariance matrix.



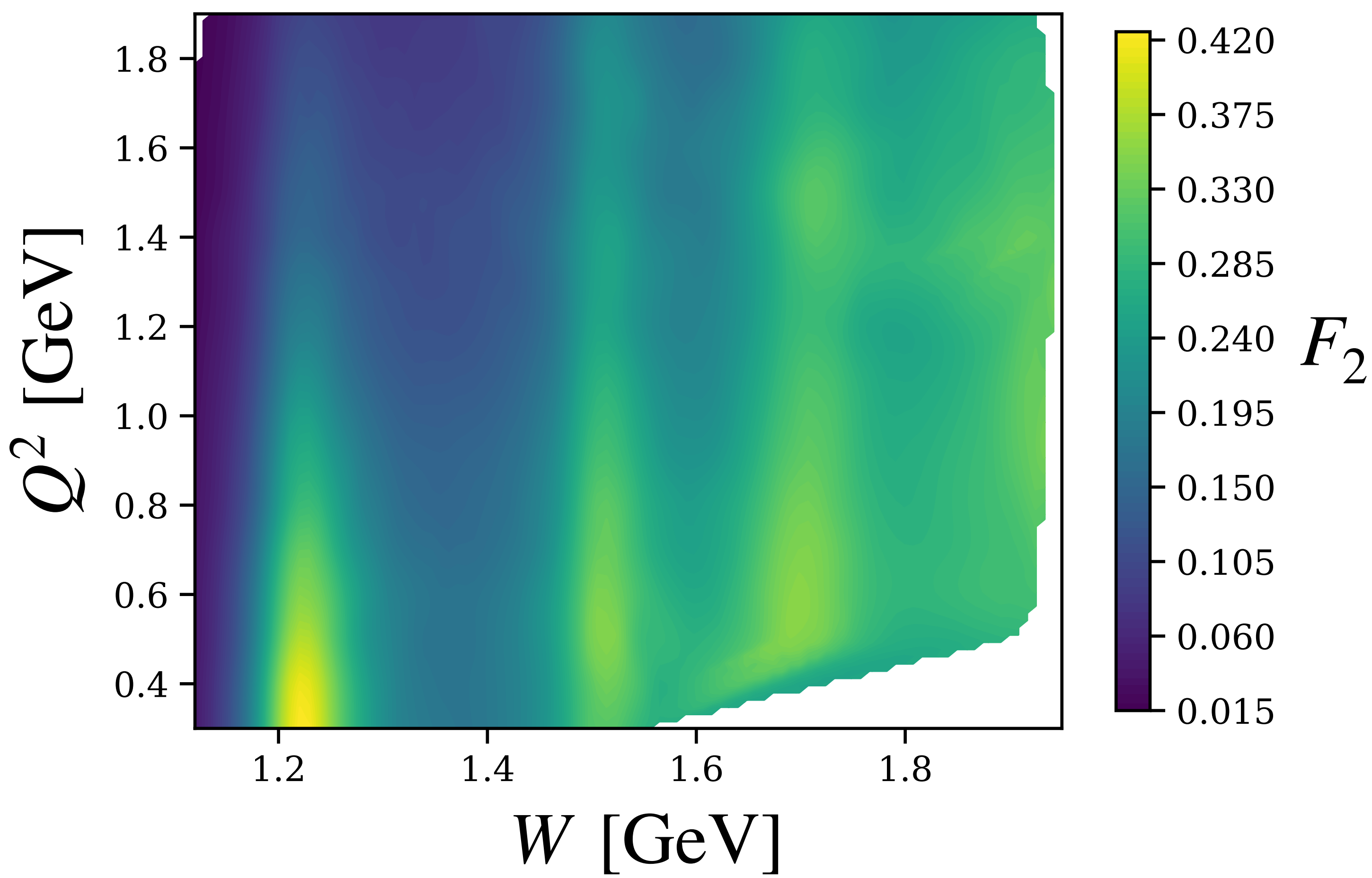
Spectral functions for inclusive observables

Inclusive electron-proton scattering: $ep \to eX$



$$\sigma(e^- p^+ \to e^- X) \sim F_1(Q^2, W) + F_2(Q^2, W)$$

Structure functions



Optical theorem

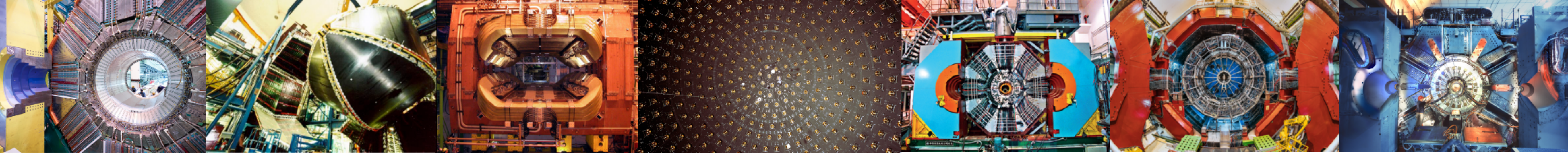
$$\int d^4x\, e^{iq\cdot x} \langle p \,|\, [j_\mu^{\mathrm{EM}}(x), j_\nu^{\mathrm{EM}}(0)] \,|\, p \rangle$$

$= F_1 \times$ (Lorentz projectors)

$+F_2 \times$ (Lorentz projectors)

W: Hadronic invariant mass

Q^2: Momentum transfer



Inclusive decay of the τ lepton and $|V_{us}|$

Spectral reconstruction of $\langle J_{us} J_{us} \rangle$ correlators with HLT

ETMC
PRL 132 (2024) 26, 261901
arXiv:2403.05404

ETMC
PRD 108 (2023) 7, 074513
arXiv:2308.03125

$$R_{us}^{(\tau)} = \frac{\Gamma(\tau \mapsto X_{us}\nu_\tau)}{\Gamma(\tau \mapsto e\bar{\nu}_e \nu_\tau)}$$

$$\frac{R_{us}^{(\tau)}}{|V_{us}|^2} \propto \int dE\, K^\sigma(E/m_\tau)\, E^2 \rho(E^2)$$

Smearing kernel (from phase space/ kinematics)

Spectral function from $\langle J_{us} J_{us} \rangle$

- Results given in continuum limit, with estimate of finite-size effects
- HLT method used, with the step function from kinematic threshold regulated via smearing