Hamiltonian Lattice Formulation of Compact Maxwell-Chern-Simons Theory

Lena Funcke

UNIVERSITÄT BONN

TRA MATTER TRANSDISCIPLINARY RESEARCH AREA

HISKP

ML4Q MATTER AND LIGHT FOR QUANTUM COMPUTING

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Why Maxwell-Chern-Simons (MCS) theory?

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3+1D: Topological θ-Term

Relevance
Strong CP problem, Grand Unified Theories, …

2+1D: Topological Chern-Simons Term

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Problem	State-of-the-Art

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Our goal: Derive *compact* MCS lattice Hamiltonian
Paves the way for simulations on (quantum) computers

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Result: Compact MCS Lattice Hamiltonian

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2+1D QED Lattice Hamiltonian + Extension	Compact Gauge Fields

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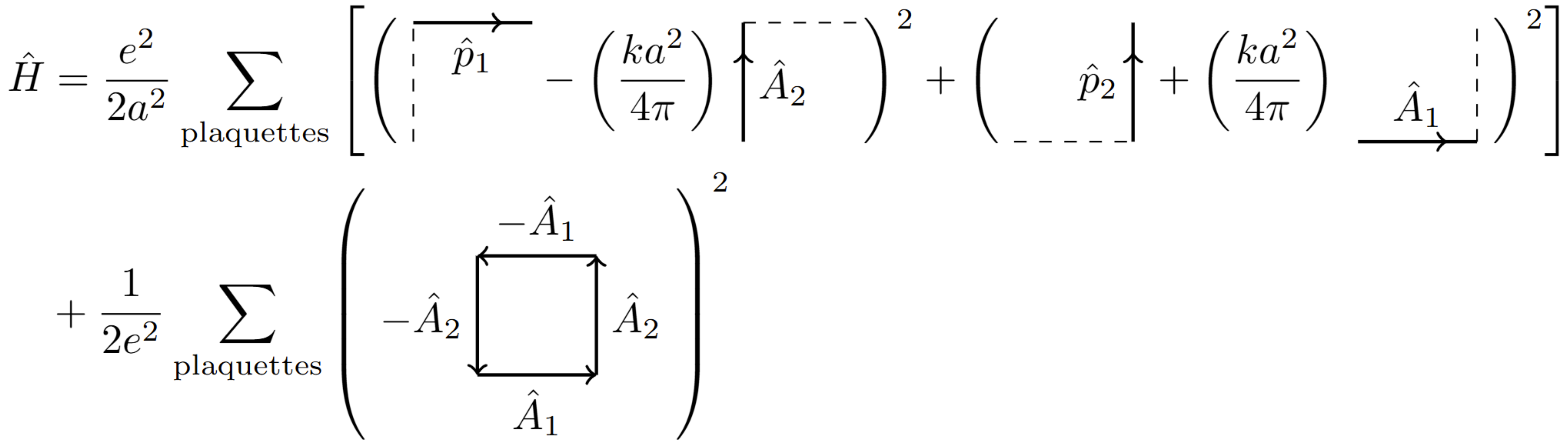
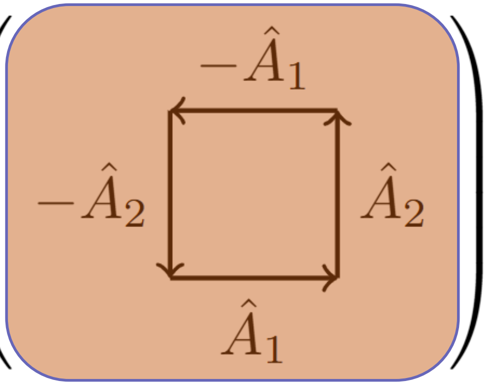


Result: Compact MCS Lattice Hamiltonian

Magnetic field term: similar to QED (without monopoles)

$$\hat{H} = \frac{e^2}{2a^2} \sum_{\text{plaquettes}} \left[\left(\hat{p}_1 - \left(\frac{ka^2}{4\pi}\right) \hat{A}_2 \right)^2 + \left(\hat{p}_2 + \left(\frac{ka^2}{4\pi}\right) \hat{A}_1 \right)^2 \right]$$

$$+ \frac{1}{2e^2} \sum_{\text{plaquettes}} \left(\quad \right)^2$$



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$$+ \frac{1}{2e^2} \sum_{\text{plaquettes}} \left(\hat{A}_1 + \hat{A}_2 - \hat{A}_1 - \hat{A}_2 \right)^2$$

Commutation relations: $[\hat{A}_x, \hat{p}_y] = i\delta_{x,y}$

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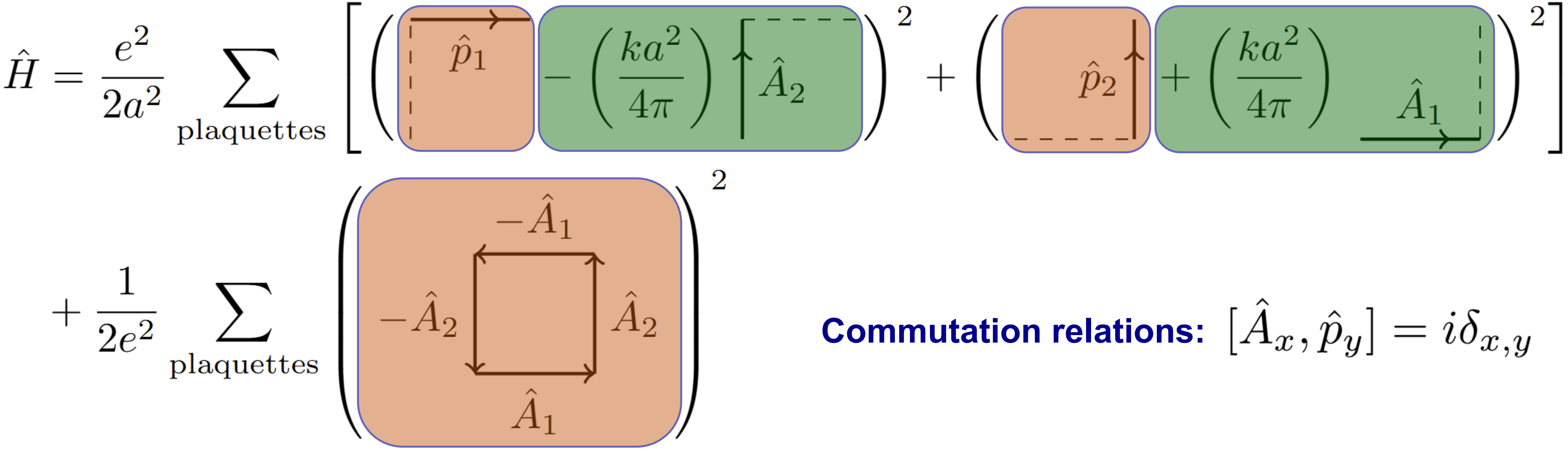
Gauge configurations: can take values in $(-\infty, +\infty)$



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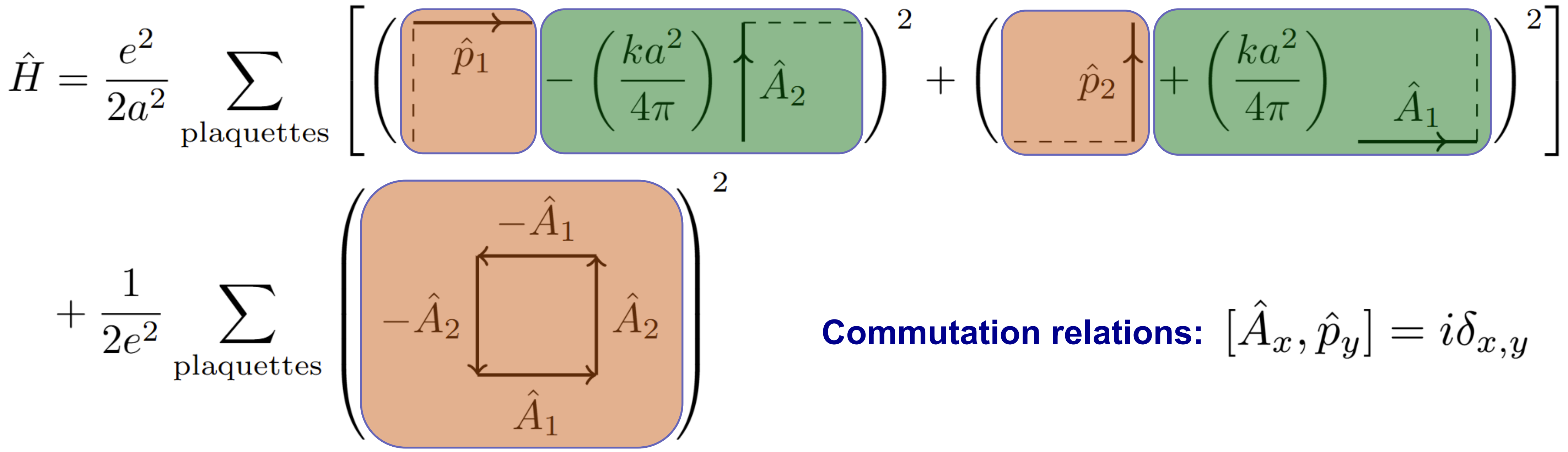
Compactness: ensured by constraints on Hilbert space



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Quadratic Hamiltonian: can be solved analytically!

Exact Solution of Lattice Hamiltonian

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Pure Maxwell Theory	Maxwell-Chern-Simons Theory

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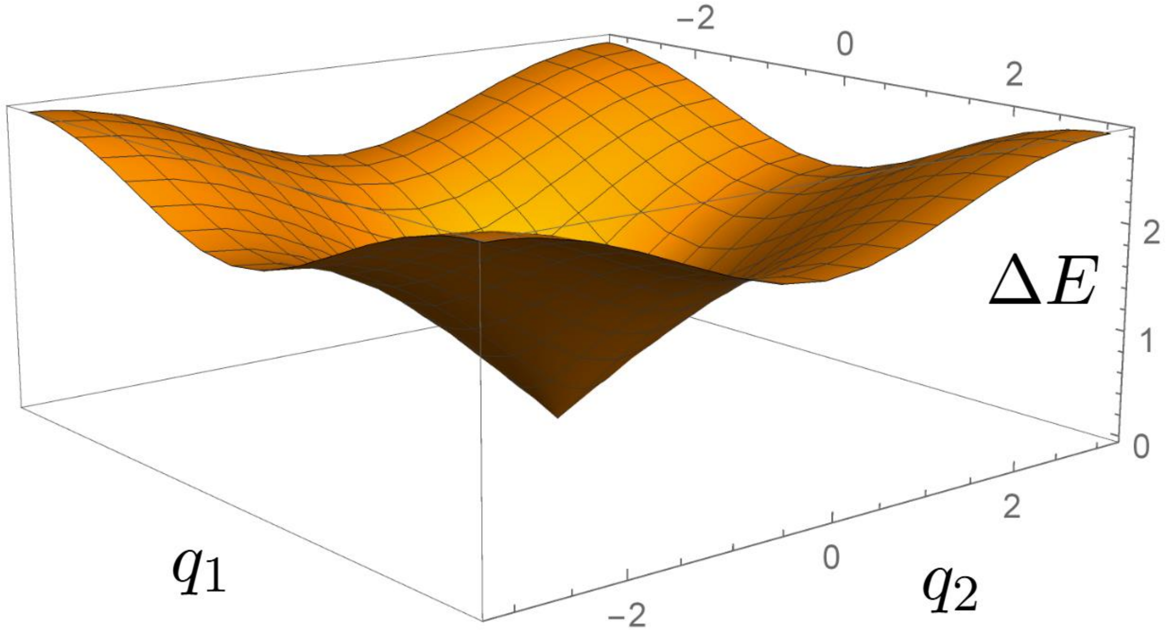
No gap: Linear dispersion relation of gapless photon



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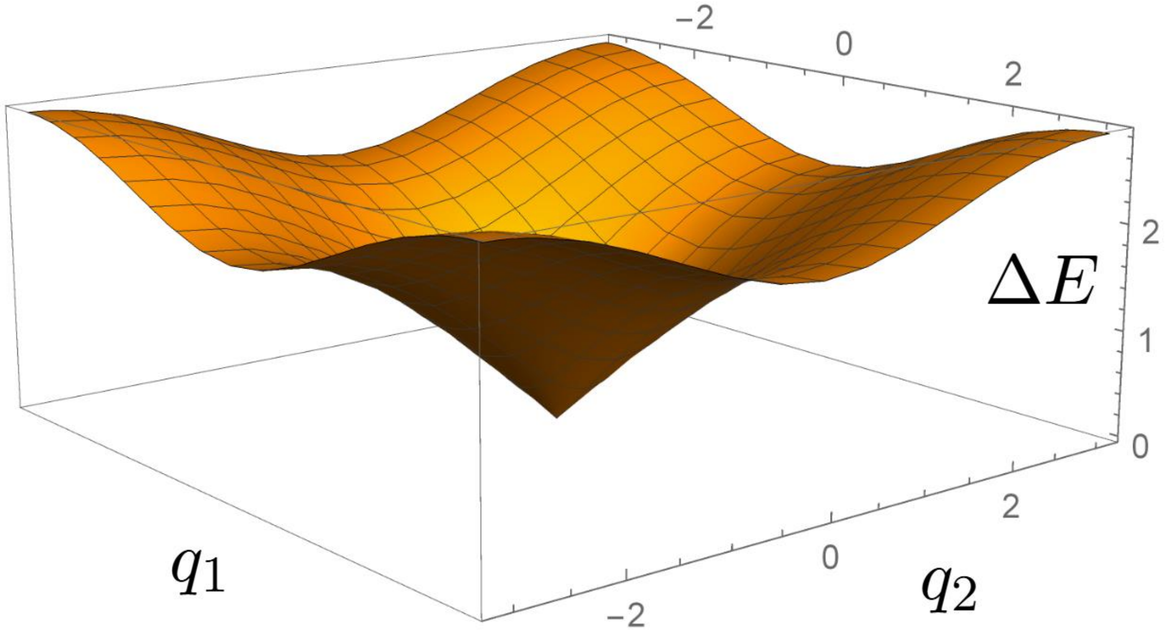

Plot of dispersion relation: $$\Delta E = \omega = \sqrt{\frac{1}{a^2}\left[2(1-\cos q_1) + 2(1-\cos q_2)\right] + \left(\frac{ke^2}{4\pi}\right)^2 \left[2 + 2\cos(q_1+q_2)\right]}$$

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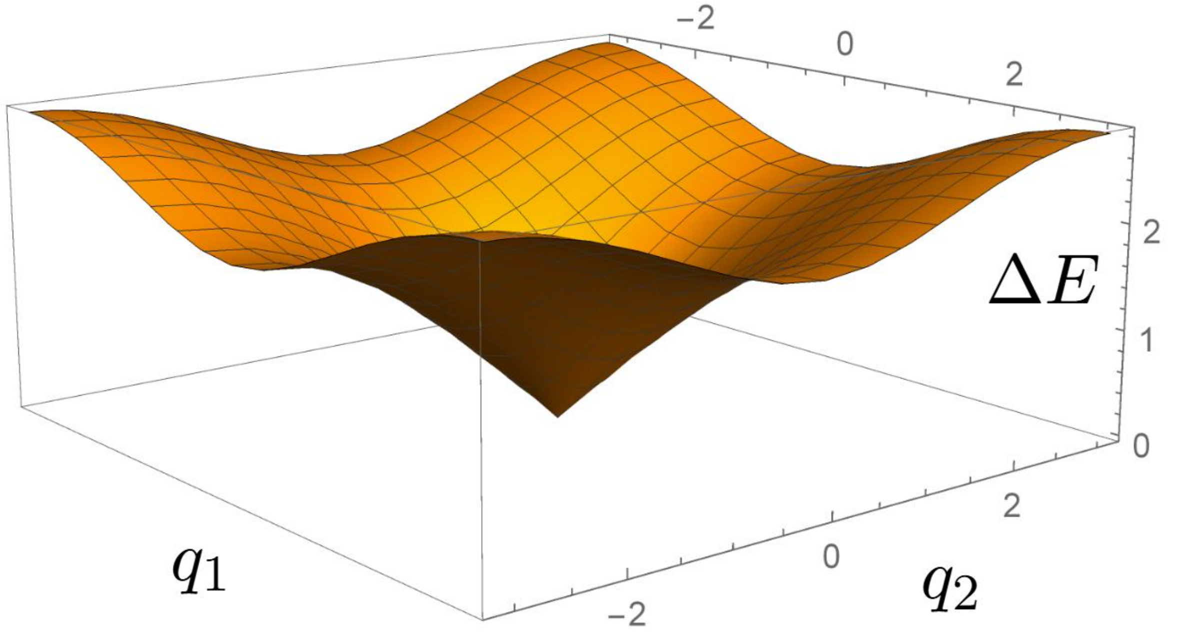
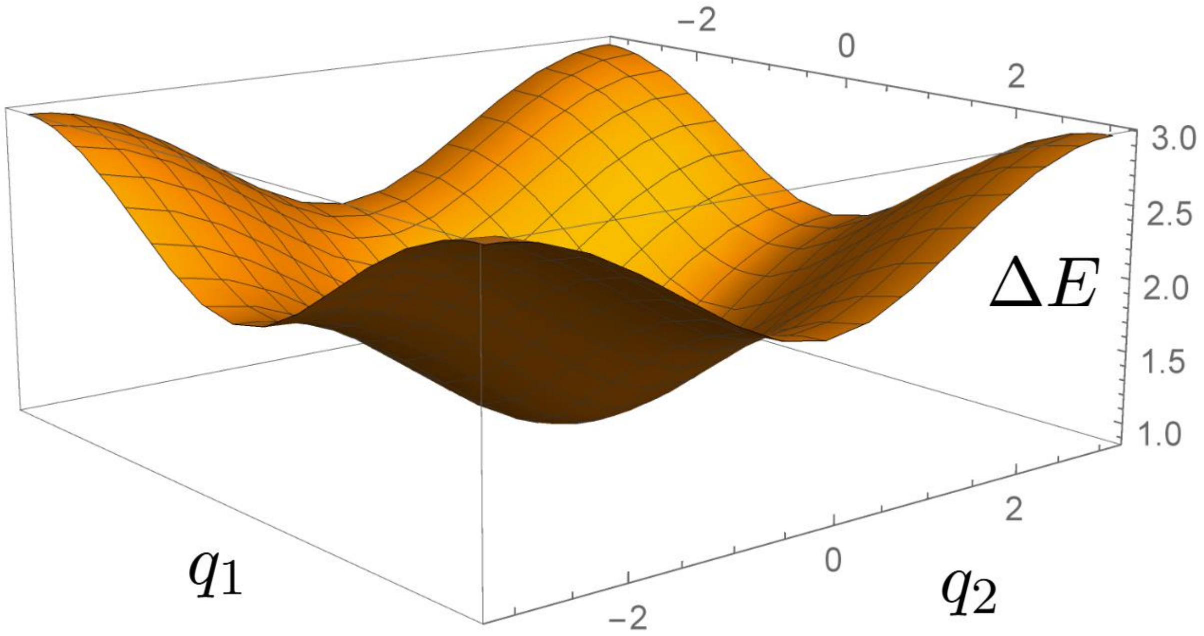


Gap: Chern-Simons term gives mass to photon



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Photon Mass & Quantized Coupling	Ground-State Degeneracy

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Continuum limit of dispersion relation

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$$\omega^2 \to |\tilde{q}|^2 + \left(\frac{ke^2}{2\pi}\right)^2$$

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Quantization of Chern-Simons level

Constraint from large gauge transformations:

$$e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = e^{2\pi i k} e^{2\pi i \hat{L}_2} e^{2\pi i \hat{L}_1}$$

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$$e^{2\pi i k} = 1 \qquad \Longrightarrow \quad k \in \mathbb{Z}$$

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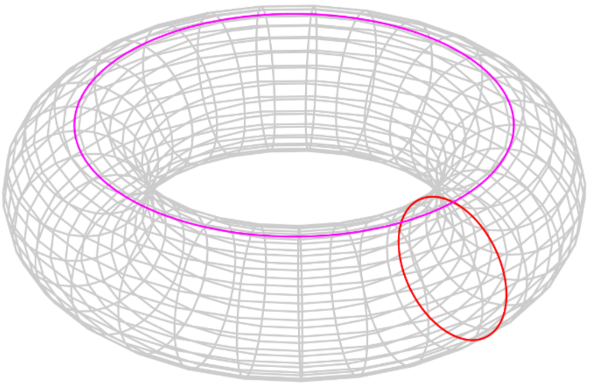
Torus: periodic boundary conditions → non-trivial topology



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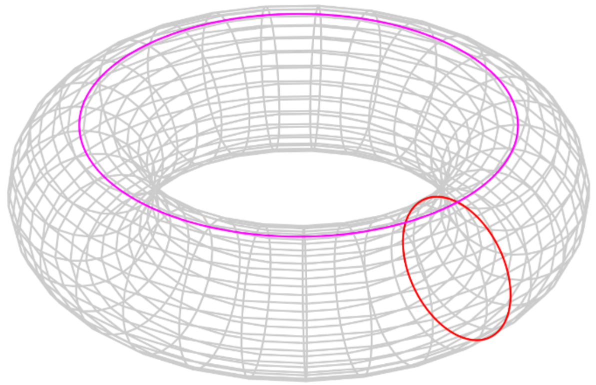

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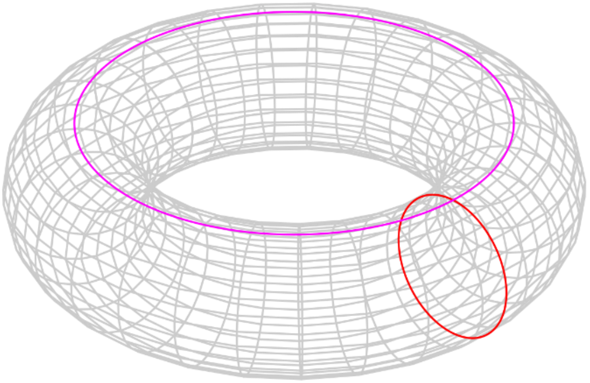

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→ good cross-check / benchmark for numerical methods!

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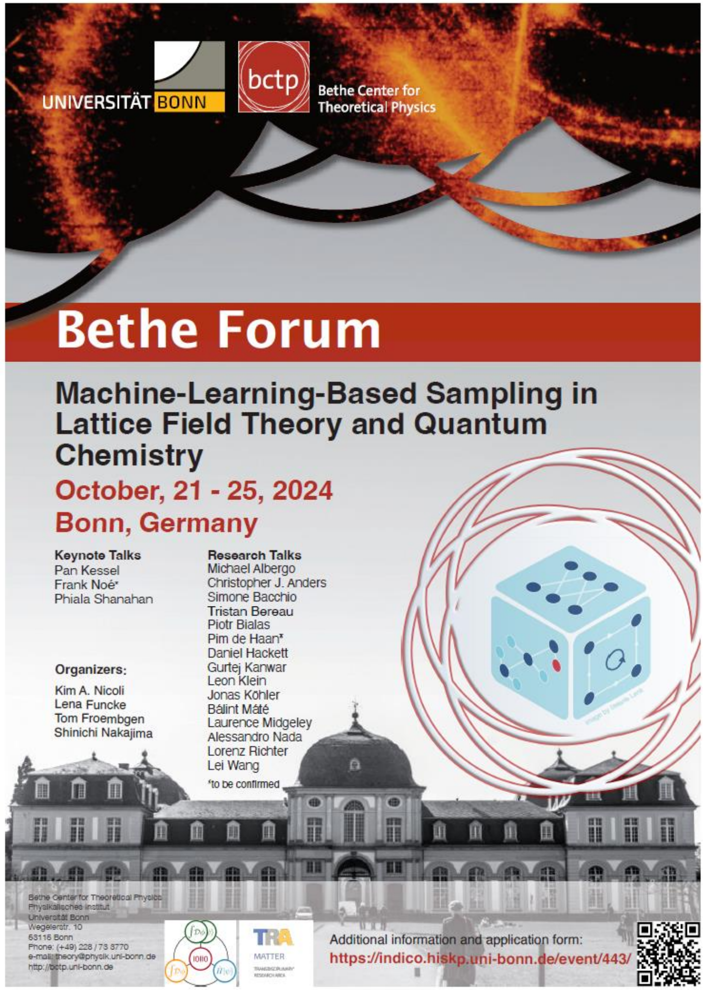
For more details, see arXiv:

Peng, Diamantini, **LF**, Hassan, Jansen, Kühn, Luo, Naredi, "Hamiltonian Lattice Formulation of Compact Maxwell-Chern-Simons Theory", arXiv:2407.20225

Workshop Advertisement

New algorithms: machine learning
Applications outside of particle physics: lattice ↔ chemistry



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(DESY)

Stefan Kühn
(DESY)

Pranay Naredi
(DESY &
Cyprus Inst.)

Syed Muhammad
Ali Hassan
(Cyprus Inst.)

Thanks to you for listening! Questions?

Backup: Hamiltonian Simulations of Topological Terms

State-of-the-Art

Markov Chain Monte Carlo
Sign problem: No simulation of topological terms

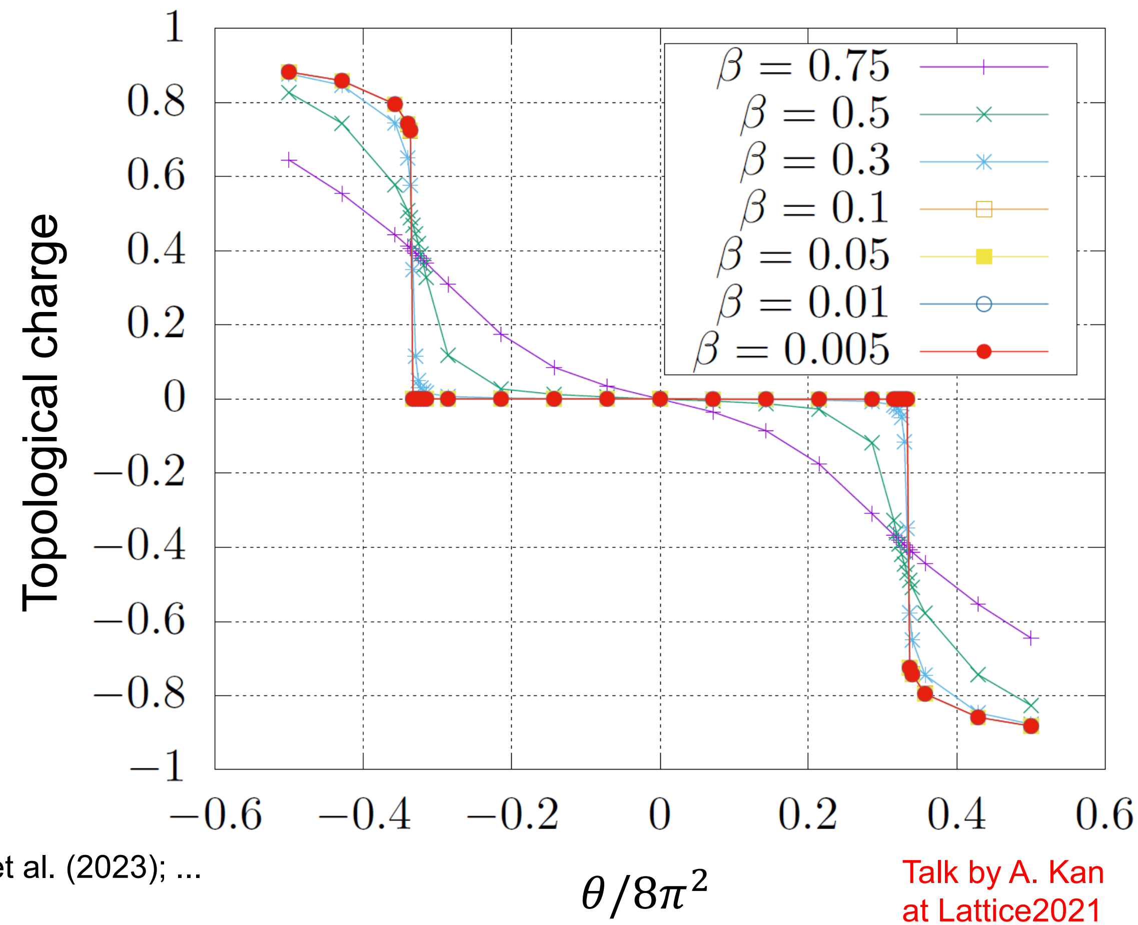
Tensor networks [1]
Simulation of θ-term for 1+1D compact QED, CP(1), …

Quantum computing [2]
Simulation of θ-term for 1+1D compact QED

Exact diagonalization [3]
Simulation of θ-term for 3+1D compact U(1)

Why no simulations of topological terms in 2+1D?
So far, no suitable lattice formulation available!

Numerical Results for 3+1D θ-Term



Talk by A. Kan at Lattice2021

[1] Byrnes et al. (2002), Buyens et al. (2017), LF, et al. (2020); Nakayama, LF, et al. (2022); LF, et al. (2023); ...
[2] Thomson, Siopsis (2022); Angelides et al. (2023); [3] Kan, LF, et al. (2021); …

Backup: Monopole Problem of Compact MCS Theory

1+1D Lattice Theories | **Modified Villain Approach**

Compact Maxwell Theory
Existence of quantized magnetic fluxes:

$$\int_\Sigma dA \in 2\pi\mathbb{Z}$$

If closed surface is contractible, **monopole** inside

Compact MCS Theory
Monopole configuration: large gauge transformation → changes Chern-Simons action → boundary terms at spatial infinity → action **violates gauge invariance**

Conventional Villain Approach
Add discrete plaquette variables n, encode magnetic flux
Interpret n as discrete gauge fields for shift symmetry:

$$A_0 \to A_0 + \frac{2\pi}{d\tau}, \quad A_i \to A_i + \frac{2\pi}{a}, \quad i = 1, 2$$

Gauge the discrete shifts → study compact gauge theory:

$$U(1) = \mathbb{R}/2\pi\mathbb{Z}$$

Modified Villain Approach
Eliminate monopoles with Lagrange multiplier

Backup: 2+1D QED Hamiltonian With(out) Monopoles

2+1D Compact QED …

… including monopoles / instantons

$$H = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{1}{e^2 a^2} \sum_{\text{plaquettes}} \left(1 - \cos a^2 B\right)$$

… with monopoles / instantons removed

$$H = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{a^2}{2e^2} \sum_{\text{plaquettes}} B^2$$

Villain Approximation

Compact variables
For compact θ_i, action contains terms $\cos(\sum_i c_i \theta_i)$

Conventional Villain approach
Replace cosine terms by periodic Gaussian potential:

$$e^{\beta \cos(\theta)} \approx \sum_{n=-\infty}^{\infty} e^{-\frac{\tilde{\beta}}{2}(\theta + 2\pi n)^2}$$

→ discrete gauge fields n can take values in $(-\infty, +\infty)$
→ compactness ensured by constraints on Hilbert space

Modified Villain approach
Lagrange multiplier constrains gauge fields n to be flat

Backup: Constraints on Hilbert Space

Local Gauge Transformations

Gauss' law

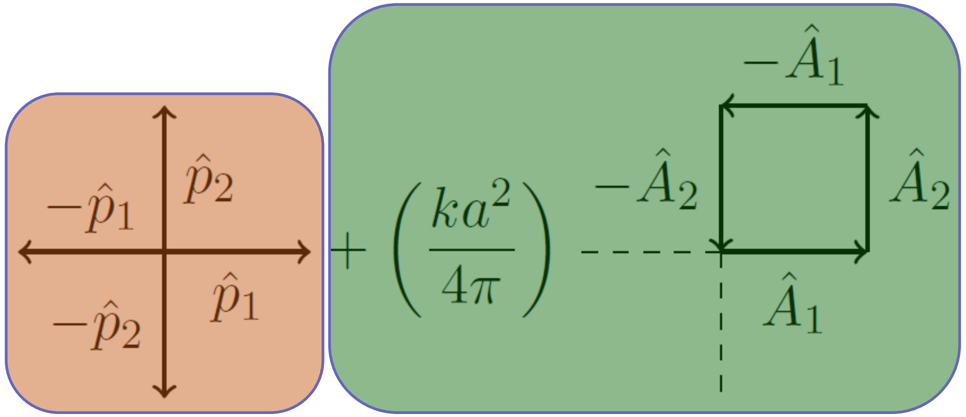
Constraints on physical states in Hilbert space:

$$e^{i\lambda\hat{G}}|\psi\rangle = |\psi\rangle, \quad \forall\lambda \in \mathbb{R}, \qquad [\hat{H}, \hat{G}] = 0$$

Generator for local gauge transformations

Similar as for QED, but modified by Chern-Simons term:

$\hat{G} =$



Large Gauge Transformations

Two additional constraints

$$e^{2\pi i\hat{L}_1}|\psi\rangle = e^{i\theta_1}|\psi\rangle, \quad e^{2\pi i\hat{L}_2}|\psi\rangle = e^{i\theta_2}|\psi\rangle$$

→ compactify gauge field configurations

→ similar as for QED (without monopoles)

Generators for large gauge transformations

Similar as for QED, but modified by Chern-Simons term

→ enforce quantized Chern-Simons coupling

Backup: Compactification Through Constraints

Large Gauge Transformations

Constraints on Hilbert space …

$$e^{2\pi i \hat{L}_i} |\psi\rangle = e^{i\theta_i} |\psi\rangle$$

… allow only certain gauge field configurations

$$\hat{L}_i |\psi\rangle = \left(\frac{\theta_i}{2\pi} + m_i \right) |\psi\rangle, \; m_i \in \mathbb{Z}, \; i = 1, 2$$

Different topological sectors of theory

Transformation $|\psi\rangle \to e^{2\pi i \hat{L}_j} |\psi\rangle$

… changes sector $m_i \to m_i + k\, \epsilon_{ij}$

… due to non-zero commutator $[\hat{L}_1, \hat{L}_2] = -\frac{k}{2\pi} i$

Invariance of Partition Function

Partition function

Sum over all sectors / values of m_i with equal weights
→ stays invariant under large gauge transformations

Similarity to Villain approximation:

$$e^{\beta \cos(\theta)} \approx \sum_{n=-\infty}^{\infty} e^{-\frac{\tilde{\beta}}{2}(\theta + 2\pi n)^2}$$

→ obtain periodic function (i.e., compact gauge field) by summing over multiple non-periodic functions

Numerical simulation

Truncation of infinite sum, neglect terms with large n

Backup: Large Gauge Transformation Constraint #1

2+1D Compact QED … | **… With Chern-Simons Term**

Constraint on Hilbert space

$$e^{2\pi i \hat{L}_1} |\psi\rangle = e^{i\theta_1} |\psi\rangle$$

Generator for large gauge transformation

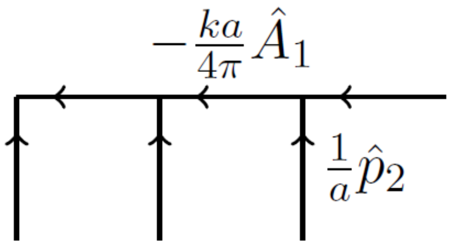
$\hat{L}_1 = \cdots$

$\cdots$

$$= \frac{1}{a} \sum_{x_1=0}^{N_1-1} \hat{p}_{(x_1, N_2-1);2}$$

Modified generator for large gauge transformation

$\hat{L}_1 = \cdots$



$\cdots$

$$= \sum_{x_1 \in \{0,1,\ldots,N_1-1\}} \left(\frac{1}{a} \hat{p}_{(x_1,x_2);2} - \frac{ka}{4\pi} \hat{A}_{(x_1,x_2+1);1} \right)$$

Backup: Large Gauge Transformation Constraint #2

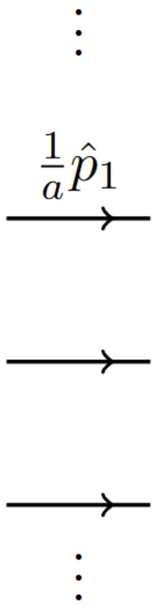
2+1D Compact QED … / … With Chern-Simons Term

Constraint on Hilbert space

$$e^{2\pi i\hat{L}_2}|\psi\rangle = e^{i\theta_2}|\psi\rangle$$

Generator for large gauge transformation

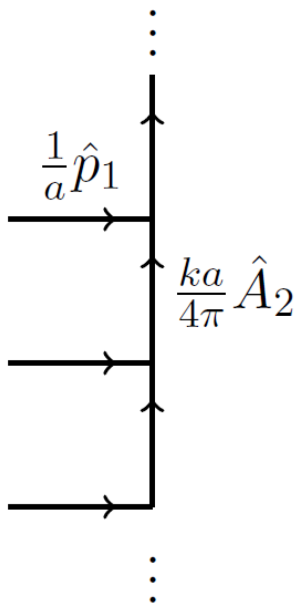
$\hat{L}_2 =$



$$= \frac{1}{a}\sum_{x_2=0}^{N_2-1}\hat{p}_{(N_1-1,x_2);1}$$

Modified generator for large gauge transformation

$\hat{L}_2 =$



$$= \sum_{x_2\in\{0,1,\dots,N_2-1\}}\left(\frac{1}{a}\hat{p}_{(x_1,x_2);1} + \frac{ka}{4\pi}\hat{A}_{(x_1+1,x_2);2}\right)$$

Backup: Compatibility of Constraints

Commutators

Constraints need to commute with Hamiltonian

$$[\hat{H}, \hat{G}] = 0$$

$$[\hat{H}, \hat{L}_i] = 0, \quad i = 1, 2$$

Constraints need to commute with Gauss' law

$$[\hat{G}, \hat{L}_i] = 0, \quad i = 1, 2$$

But: constraints do not commute with each other!

$$[\hat{L}_1, \hat{L}_2] = -\frac{k}{2\pi} i$$

Quantization of Chern-Simons Level

Non-zero commutator yields quantization condition!

Constraint from large gauge transformations:

$$e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = e^{2\pi i k} e^{2\pi i \hat{L}_2} e^{2\pi i \hat{L}_1}$$

Correctly reproduces quantization property: [1]

$$e^{2\pi i k} = 1 \qquad \Longrightarrow \quad k \in \mathbb{Z}$$

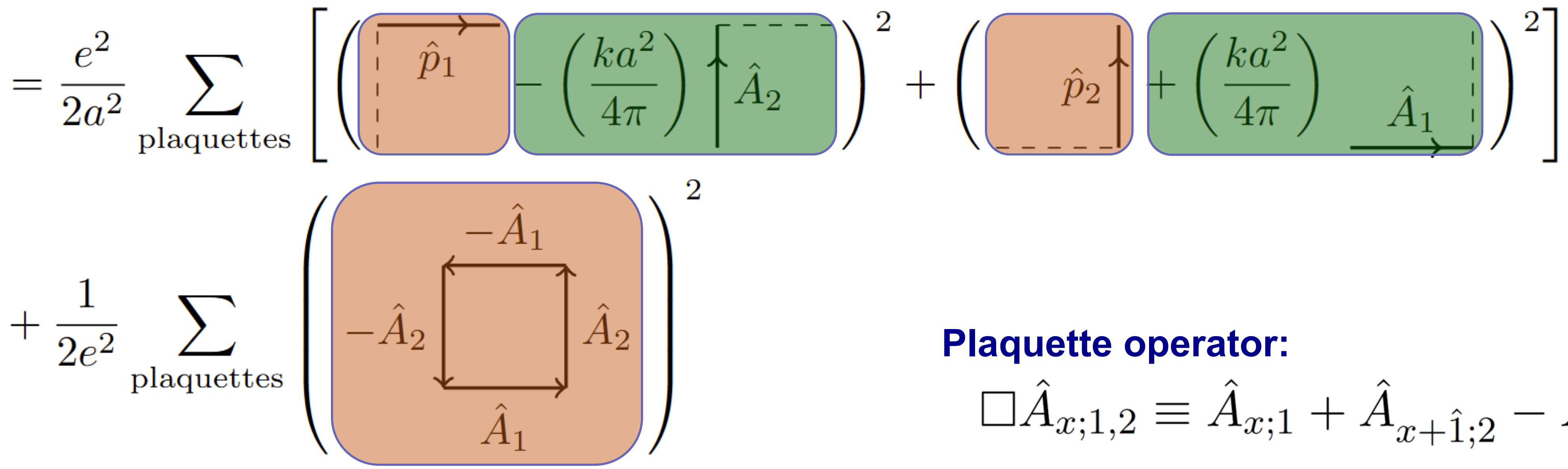
[1] Pisarski (1986)

Backup: Plaquette Visualization of Lattice Hamiltonian

Compact MCS Hamiltonian

Magnetic field term: similar to QED (without monopoles)
Electric field term: modified by Chern-Simons term

$$\hat{H} = \sum_{x \in \text{sites}} \frac{e^2}{2a^2} \left[\left(\hat{p}_{x;1} - \frac{ka^2}{4\pi} \hat{A}_{x-\hat{2};2} \right)^2 + \left(\hat{p}_{x;2} + \frac{ka^2}{4\pi} \hat{A}_{x-\hat{1};1} \right)^2 \right] + \frac{1}{2e^2} \left(\Box \hat{A}_{x;1,2} \right)^2$$



Plaquette operator:

$$\Box \hat{A}_{x;1,2} \equiv \hat{A}_{x;1} + \hat{A}_{x+\hat{1};2} - \hat{A}_{x+\hat{2};1} - \hat{A}_{x;2}$$