Equation of state of isospin asymmetric QCD with small baryon chemical potentials

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CRC-TR 211
Strong-interaction matter under extreme conditions

UNIVERSITÄT BIELEFELD
Faculty of Physics

NRW-FAIR

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QCD at non-zero isospin chemical potential

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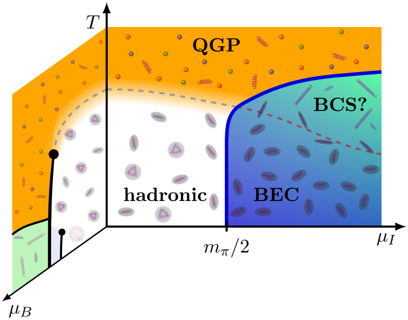
- this session:
  focus on isospin chemical potential μ_I

  important part of QCD parameter space

  - no sign problem



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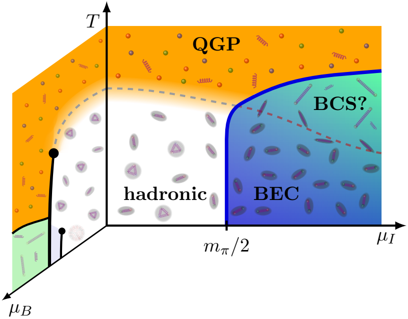

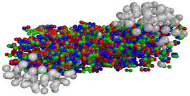

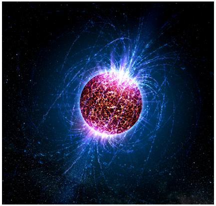

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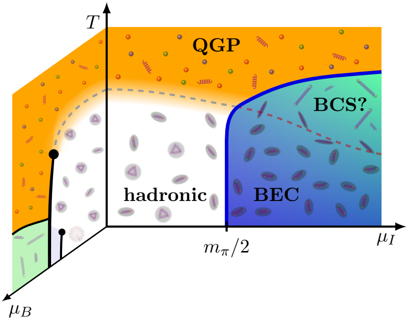
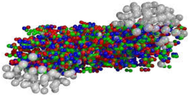
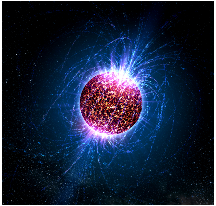
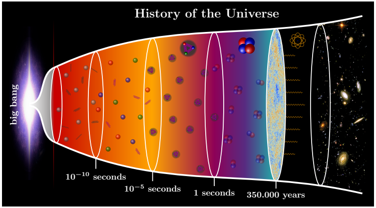






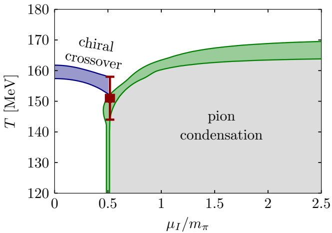


QCD at non-zero isospin chemical potential

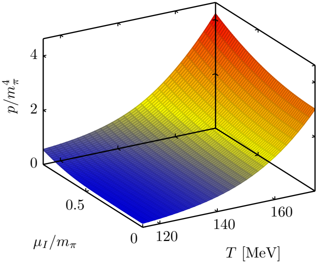
Convenient chemical potential basis for simulations: ("isospin" basis)

$$\mu_u = \mu_L + \mu_I \qquad \mu_d = \mu_L - \mu_I \qquad \mu_s$$

$\mu_I \neq 0, \quad \mu_L = \mu_s = 0$ pure isospin chemical pot. – no sign problem



[Brandt, Endrődi, Schmalzbauer '18]



[Brandt, Cuteri, Endrődi '22]

- improved actions (staggered)
- physical pion masses

EoS from canonical @ $T = 0$ [Detmold *et al* '12, '23, '24]

However: **all systems also feature $\mu_L \neq 0 \neq \mu_s$**

Contents

I Taylor expansion and λ-extrapolations

Taylor expansion around $\mu_I \neq 0$

Extension to $\mu_L,\ \mu_s \neq 0$:

$$p(T,\vec{\mu}) = p(T,\mu_I,0,0) + \sum_{n,m=1}^{\infty} \frac{1}{n!\,m!} \left.\frac{\partial^n \partial^m [p(T,\vec{\mu})]}{\partial \mu_L^n\, \partial \mu_s^m}\right|_{\mu_L,\mu_s=0} \big(\mu_L\big)^n \big(\mu_s\big)^m$$

- Leading order: only non-zero coefficients are

$$\chi_2^L(T,\mu_I) \equiv \left.\frac{\partial^2[p(T,\vec{\mu})]}{\partial \mu_L^2}\right|_{\mu_L,\mu_s=0} \quad \& \quad \chi_2^s(T,\mu_I) \equiv \left.\frac{\partial^2[p(T,\vec{\mu})]}{\partial \mu_s^2}\right|_{\mu_L,\mu_s=0}$$

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- Generically: $\chi_2^X = \frac{T}{V}\Big[\underbrace{\big\langle c_{XX}\big\rangle}_{\text{connected}} + \underbrace{\big\langle (c_X)^2\big\rangle - \big\langle c_X\big\rangle^2}_{\text{disconnected}}\Big]$

with $c_X = \mathrm{Tr}\Big[M^{-1}\frac{\partial M}{\partial \mu_X}\Big]$ $\qquad M$: fermion matrix

$$c_{XX} = \frac{\partial c_X}{\partial \mu_X} = \mathrm{Tr}\Big[M^{-1}\frac{\partial^2 M}{(\partial \mu_X)^2}\Big] + \mathrm{Tr}\Big[M^{-1}\frac{\partial M}{\partial \mu_X}M^{-1}\frac{\partial M}{\partial \mu_X}\Big]$$

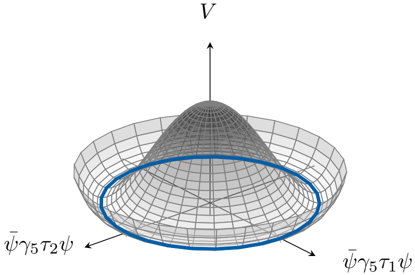
Necessity of λ-extrapolations

$$D = \gamma_\mu D_\mu + m_{ud} + \gamma_0 \tau_3\, \mu_I$$

$$SU_V(2) \longrightarrow \mathrm{U}_Q(1) \longrightarrow \varnothing$$

explicit $\mu_I \neq 0$ | spontaneous $\mu_I \geq m_\pi/2$

- cannot observe spontaneous symmetry breaking in finite V
- low mode in simulations



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explicit $\mu_I \neq 0$ — explicit pionic source λ

- need to break symmetry explicitly
  - ⟶ introduce regulator: $\sim \lambda$ pionic source [Kogut, Sinclair '02]

physical results: extrapolate $\lambda \to 0$

main task for analysis

- cannot observe spontaneous symmetry breaking in finite V
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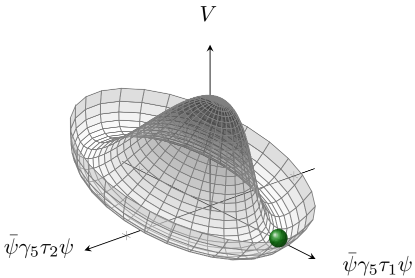

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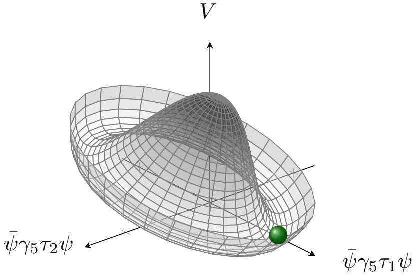

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- facilitated by improvement program:
  [Brandt, Endrődi, Schmalzbauer '18]
  - leading order reweighting
  - valence quark improvement for light quarks

Valence quark improvement for densities

Up to now: consider 1st derivatives – densities (e.g.: $n_I = c_I$)

$$O \equiv \mathrm{Tr}\left[M^{-1}\hat{O}\right] \qquad c_X: \ \hat{O} = \frac{\partial M}{\partial \mu_X} \quad \text{with} \quad M = D^\dagger(\mu)D(\mu) + \lambda^2$$

- typically: compute the trace stochastically

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$$O = \sum_{n=0}^{N_{\rm lat}} \frac{\varphi_n^\dagger \hat{O}\varphi_n}{\xi_n^2 + \lambda^2} \approx \sum_{n=0}^{N} \frac{\mathcal{O}_{nn}}{\xi_n^2 + \lambda^2} \qquad \text{with} \quad \mathcal{O}_{nm} = \varphi_n^\dagger \hat{O}\varphi_m$$

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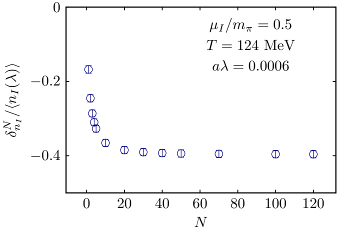

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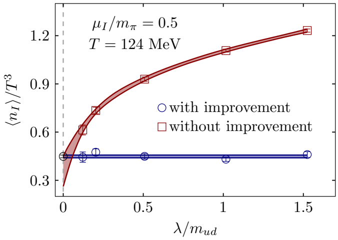

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$$\chi_2^X = \frac{T}{V}\Big[\big\langle c_{XX}\big\rangle + \big\langle (c_X)^2\big\rangle - \big\langle c_X\big\rangle^2\Big]$$

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CRC-TR 211

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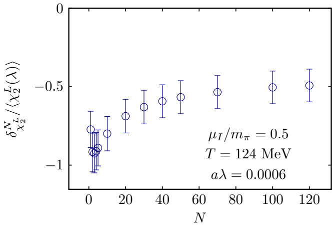

→ improvement term:

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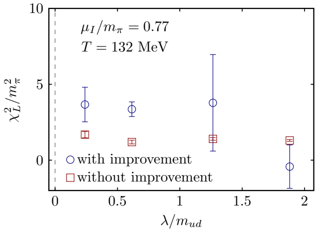

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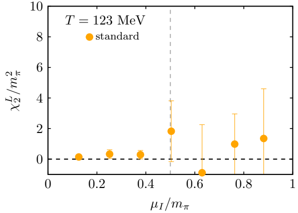
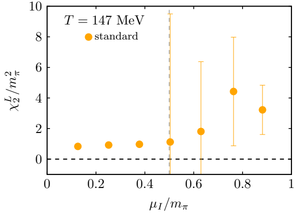
λ-extrapolated Taylor coefficents @ $N_t = 8$

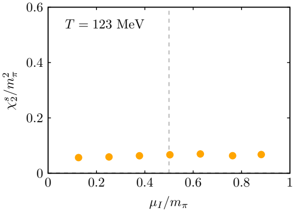
- results for $\chi_2^L(T, \mu_I)$ using standard λ-extrapolations:

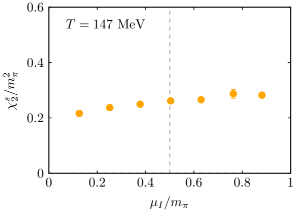




Large uncertainties for $\chi_2^L(T, \mu_I)$ in the BEC phase!

- $\chi_2^s(T, \mu_I)$ not affected (no source parameter)





II Improved computation of $\chi_2^L(T, \mu_I)$

Improved computation method

Observation: equal connected parts in μ_L and μ_I derivatives

$$c_{LL} = c_{II}$$

→ Can we use that somehow?

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- for EoS computation: model independent spline interpolation of n_I

[Brandt, Cuteri, Endrődi '22]

→ know μ_I dependence of n_I

compute $\frac{\partial n_I}{\partial \mu_I} = \chi_2^I(T, \mu_I)$ analytically from spline interpolation

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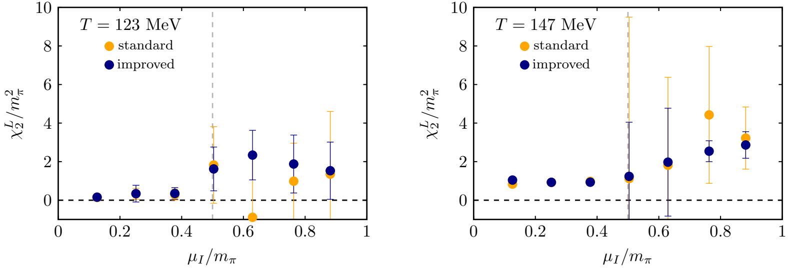
- finally compute $\chi_2^L(T, \mu_I)$ using

$$\chi_2^L(T,\mu_I) = \underbrace{\chi_2^I(T,\mu_I)}_{\lambda=0} + \frac{T}{V}\Big[\underbrace{\big\langle (c_L)^2 \big\rangle - \big\langle c_L \big\rangle^2 - \big\{ \big\langle (c_I)^2 \big\rangle - \big\langle c_I \big\rangle^2 \big\}}_{\lambda \neq 0}\Big]$$

(λ-extrapolation for disconnected typically better behaved)

λ-extrapolated Taylor coefficents @ $N_t = 8$

- comparison of the methods for $\chi_2^L(T,\mu_I)$ in BEC:



→ reduced uncertainties for $\chi_2^L(T,\mu_I)$ in the BEC phase!

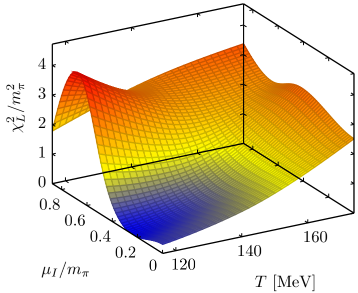
Results for Taylor coefficients @ $N_t = 8$

To extend the EoS: interpolate Taylor expansion coefficients

as for EoS: [Brandt, Cuteri, Endrődi '22] [Brandt, Endrődi '16]

use spline interpolations with Monte-Carlo generated nodepoints

- constraint: $\chi_2^L(T,\mu_I)$ & $\chi_2^s(T,\mu_I)$ vanish at $T = 0$



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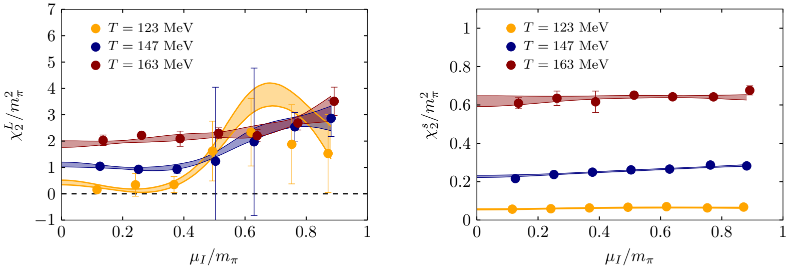


III An application: EoS at non-zero charge chemical potential

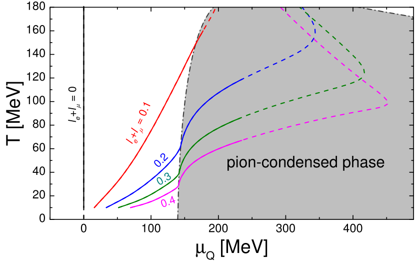
EoS for early Universe @ large l_ℓ

early Universe: sum of lepton flavour asymmetries is constrained

$|l_e + l_\mu + l_\tau| < 0.012$ [Oldengott, Schwarz '17]

but: for large individual l_ℓ

→ large μ_Q and small μ_B & μ_S along trajectories



- model equation of state – matched to lattice at pure μ_I

  [Vovchenko, *et al* '20]

- Taylor expansion from $\vec{\mu} = 0$ – no pion condensation

  [Middeldorf-Wygas, *et al* '20]

EoS for early Universe @ large l_ℓ

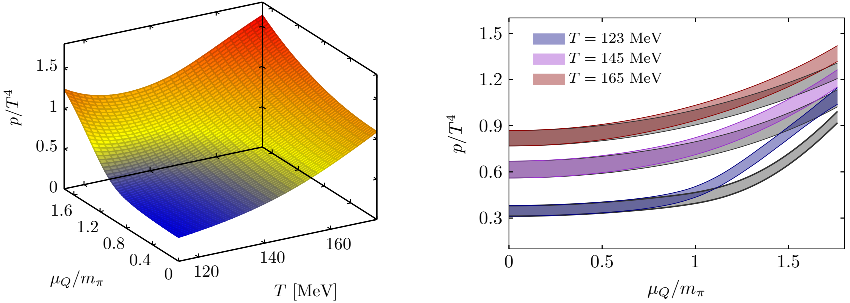
early Universe: sum of lepton flavour asymmetries is constrained

$|l_e + l_\mu + l_\tau| < 0.012$ [Oldengott, Schwarz '17]

but: for large individual l_ℓ

→ large μ_Q and small μ_B & μ_S along trajectories

- can provide a full lattice EoS:



assumption: BEC phase boundary does not change drastically

Conclusions

- Simulations at $\mu_I \neq 0$:
  offer a novel expansion point to explore (μ_B, μ_Q, μ_S) space
- λ-extrapolations necessary:
  → large uncertainties for $\chi_2^L(T, \mu_I)$ in BEC phase
- alternative: computation via $\chi_2^I(T, \mu_I)$
  obtained from spline interpolation of n_I
  → improves uncertainties
  some details still to be understod
- application and outlook:
  compute EoS at non-zero μ_Q and in its vicinity