Constraints on the Dirac spectrum from chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$ symmetry

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QCD in the chiral limit

Up and down quark very light ⇒ QCD close to $N_f = 2$ chiral limit $m \to 0$, chiral symmetry $\mathrm{U}(2)_L \times \mathrm{U}(2)_R = \underbrace{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}_{\to \mathrm{SU}(2)_V \text{ @low T}} \times \underbrace{\mathrm{U}(1)_A}_{\text{anomalous}} \times \mathrm{U}(1)_V$

Related open questions about the chiral limit:

- nature of finite-temperature transition
- fate of anomalous $\mathrm{U}(1)_A$ in the symmetric phase

$\mathrm{U}(1)_A$ remains broken	⇒	second order, $\mathrm{O}(4)$ class
$\mathrm{U}(1)_A$ restored	⇒	first order, or
		second order, $\mathrm{U}(2)_L \times \mathrm{U}(2)_R / \mathrm{U}(2)_V$ class

[Pisarski, Wilczek (1984), Pelissetto, Vicari (2013)]

Spectrum/eigenvectors of the Dirac operator encode quark dynamics

- How does chiral symmetry restoration constrain them?
- What do constraints tell us about $\mathrm{U}(1)_A$?

Focus on the spectrum, study the scalar and pseudoscalar sector

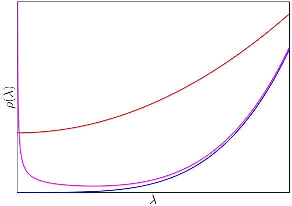
Chiral symmetry restoration and the Dirac spectrum

$\mathrm{SU}(2)_A$ broken if density of near-zero Dirac modes $\rho(0^+;0) \neq 0$

[Banks, Casher (1980)]

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty d\lambda \, \frac{2m}{\lambda^2+m^2}\rho(\lambda;m)$$

$$\rho(\lambda;m) = \lim_{\mathrm{V}\to\infty}\frac{\mathrm{T}}{\mathrm{V}}\left\langle \sum_{n,\lambda_n\neq 0}\delta(\lambda-\lambda_n)\right\rangle$$



@low T: $\langle\bar{\psi}\psi\rangle_{m\to 0} \neq 0 \Rightarrow$ expect $\rho(0^+;m) \neq 0$

@high T: $\langle\bar{\psi}\psi\rangle_{m\to 0} = 0 \Rightarrow$ expect $\rho(0^+;m) = 0$

...instead singular peak, fate as $m \to 0$ unclear

[Edwards *et al.* (1992), Cossu *et al.* (2013), Alexandru, Horváth (2015), Dick *et al.* (2015), Brandt *et al.* (2016), Tomiya *et al.* (2017), Ding *et al.* (2019), Aoki *et al.* (2021), Vig, Kovács (2021), Kaczmarek *et al.* (2021), Meng *et al.* (2023), Alexandru *et al.* (2024)]

- How does a singular peak fit with chiral symmetry restoration?
- What does it do to $\mathrm{U}(1)_A$?

Chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$

How does $\mathrm{SU}(2)_A$ restoration affect $\mathrm{U}(1)_A$?

assumptions	conclusions
• observables analytic in m^2 • ρ power series near $\lambda = 0$ (or $\sim \lambda^\alpha$, $\alpha > 0$)	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration [Cohen (1996), Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]
• thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\not\Rightarrow$ $\mathrm{U}(1)_A$ restoration, $\mathrm{U}(1)_A$ broken by topological effects [Evans, Hsu, Schwetz (1996), Lee, Hatsuda (1996)]
• observables analytic in m^2 • thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration, unless $\rho \sim m^2\delta(\lambda)$ [Azcoiti (2023)]

What are the correct assumptions?

Symmetry restoration condition(s)

Symmetry restoration "levels":

0. Local field theory: symmetry restored

   ⇒ correlators of local operators related by symmetry become equal

1. No massless excitations expected in restored phase

   ⇒ also spacetime integrals of connected correlators of local operators (susceptibilities) related by symmetry become equal

2. Gauge fields unaffected by chiral transformations

   ⇒ also correlators involving nonlocal functionals of gauge fields only (e.g., spectral density) become equal if related by symmetry

Ginsparg-Wilson fermions

Ginsparg-Wilson Dirac operator D obeys [Ginsparg, Wilson (1982)]

$$\{D, \gamma_5\} = 2D\gamma_5 RD$$

Exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry [Lüscher (1998)]

Scalar and pseudoscalar densities form irreducible chiral multiplets

$$O_V = (S, i\vec{P}) \qquad\qquad O_W = (iP, -\vec{S})$$
$$S = \bar{\psi}(1 - DR)\psi \qquad\qquad P = \bar{\psi}(1 - DR)\gamma_5\psi$$
$$\vec{P} = \bar{\psi}(1 - DR)\vec{\sigma}\gamma_5\psi \qquad\qquad \vec{S} = \bar{\psi}(1 - DR)\vec{\sigma}\psi$$

Under chiral transformations

$$O_{V,W} \to \mathcal{R}^T O_{V,W} \qquad \mathcal{R} \in \mathrm{SO}(4)$$

$\mathrm{SO}(4)$ double cover of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$

Corresponding susceptibilities expressible in terms of the spectrum of D only, constraints result from symmetry restoration

Generating function of scalar/pseudoscalar susceptibilities

Include source terms for integrated densities in the partition function

$$\mathcal{Z}(V, W; m) = \int DUD\psi D\bar{\psi}\, e^{-S(U)-\bar{\psi}D_m(U)\psi-K(\psi,\bar{\psi},U;V,W)}$$

$$S = \text{gauge and massive fermion contributions}$$

$$D_m = D + m(1 - DR)$$

$$K(\psi, \bar{\psi}, U; V, W) = j_S S + i\vec{\jmath}_P \cdot \vec{P} + ij_P P - \vec{\jmath}_S \cdot \vec{S} = V \cdot O_V + W \cdot O_W$$

Generating function of scalar and pseudoscalar susceptibilities

$$\mathcal{W}(V, W; m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \ln \mathcal{Z}(V, W; m)$$

Under a chiral transformation

$$\mathcal{W}(V, W; m) \to \mathcal{W}(\mathcal{R}V, \mathcal{R}W; m)$$

Symmetry restoration condition (level 1):

$$\lim_{m\to 0} \mathcal{W}(\mathcal{R}V, \mathcal{R}W; m) = \lim_{m\to 0} \mathcal{W}(V, W; m)$$

Symmetry restoration in scalar/pseudoscalar sector

Chiral symmetry of exactly massless theory + $\mathcal{Z}$ function of $j_S + m$ only $\Rightarrow$

$$\mathcal{W}(V, W; m) = \hat{\mathcal{W}}(m^2 + \overbrace{2mj_S + V^2}^{u}, \overbrace{W^2}^{w}, \overbrace{2(mj_P + V \cdot W)}^{\tilde{u}})$$
$$= \sum_{n_u, n_w, n_{\tilde{u}}} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2)$$

Necessary and sufficient conditions for chiral symmetry restoration: ▸ details

$\boxed{\mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2) \text{ finite (non-divergent) in the chiral limit}}$

Since $\frac{\partial}{\partial m^2} \mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2) = \mathcal{A}_{n_u+1, n_w, n_{\tilde{u}}}(m^2)$

$\boxed{\mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2) \text{ power series in } m^2}$

If we assume χSR also for *nonlocal* gauge functionals (level 2)

$\boxed{\rho(\lambda; m) \text{ power series in } m^2}$

Also via argument using local operators in PQ theory

Lowest-order constraints

Requirement of finiteness of $\mathcal{A}_{1,0,0}$, $\mathcal{A}_{0,1,0}$, $\mathcal{A}_{0,0,2}$ reduces to ▸ details

$$\lim_{m\to 0}\chi_\pi = \lim_{m\to 0}\int_0^2 d\lambda\, \frac{2h(\lambda)\rho(\lambda;m)}{\lambda^2+m^2h(\lambda)} < \infty \qquad \left[h(\lambda) = 1-\tfrac{\lambda^2}{4}\right]$$

$$\Delta \equiv \lim_{m\to 0}\frac{\chi_\pi-\chi_\delta}{4} = \lim_{m\to 0}\int_0^2 d\lambda\, \frac{2m^2h(\lambda)^2\rho(\lambda;m)}{[\lambda^2+m^2h(\lambda)]^2} = \lim_{m\to 0}\frac{\chi_t}{m^2} < \infty$$

... and $\frac{\chi_\pi-\chi_\delta}{4} - \frac{\chi_t}{m^2} = c_0 m^2 + o(m^2)$

$R=\frac{1}{2}$, $D^\dagger = \gamma_5 D \gamma_5$ for simplicity

$\mathcal{W}$ even in $\tilde{u}$ due to CP symmetry

- Constraints not new, but *all* the direct constraints on ρ and χ_t: higher-order coefficients involve higher-point eigenvalue correlators
- $\mathrm{SU}(2)_A$ restoration and $\mathrm{U}(1)_A$ breaking compatible at this stage, but
  - $\mathrm{U}(1)_A$ restored if $\rho(\lambda;m) = \sum_n \rho_n(m^2)\lambda^n$ or $\rho(\lambda;m) \simeq C(m)\lambda^\alpha$, $\alpha > 0$, confirming [Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]
  - $\mathrm{U}(1)_A$ broken only by $\rho \sim m^2\delta(\lambda)$ if $\mathrm{V}\to\infty / m\to 0$ commute [Azcoiti (2023)]

How can one possibly get $\mathrm{U}(1)_A$ breaking?

$\mathrm{U}(1)_A$ breaking by singular peak

Assume that near $\lambda = 0$

$$\rho(\lambda; m) \simeq C(m)\lambda^{\alpha(m)}$$

with $|\alpha(m)| < 1$ for $m \neq 0$, $\alpha(0) \neq 1$, allowing also $\alpha(m) < 0$

- $\mathrm{SU}(2)_A$ restoration (level 1) requires only

$$C(m) = \frac{\cos\left(\alpha(m)\frac{\pi}{2}\right)}{(1-\alpha(0))\frac{\pi}{2}}|m|^{1-\alpha(0)}\hat{C}(m) \qquad |\hat{C}(0)| < \infty$$

- $\mathrm{U}(1)_A$ order parameter: $\Delta = \hat{C}(0)$, symmetry broken if $\hat{C}(0) \neq 0$
  $\Rightarrow$ thermodynamic and chiral limit *do not* commute
- ρ power series in m^2 (level 2) strongly restricts possibilities:

$$\boxed{\alpha(0) = -1}$$

with $\alpha(m)$ and $\hat{C}(m)$ power series in m^2

$C(m) = \frac{\hat{C}(m)}{\ln(2/|m|)}$ if $\alpha(0) = 1$, no $\mathrm{U}(1)_A$ breaking

Singular peak and topology

Singular peak compatible with χSR if

$$\rho_{\text{peak}}(\lambda; m) \underset{m\to 0}{\to} \left[\Delta + O(m^2)\right] \frac{m^2}{2} \frac{\gamma m^2}{\lambda^{1-\gamma m^2}}$$

- Peak reproduced in weakly interacting, dilute (density $n_{\text{inst}} = \chi_t \propto m^2$) instanton gas model: peak modes $\sim$ instanton zero modes [Kovács (2023)]
  - m-dependent power α, peak "height" both decreasing as $m \to 0$
  - $\alpha \to -1$ as disorder ($\sim 1/n_{\text{inst}}$) increases in a similar cond-mat model [Evangelou and Katsanos (2003)]

(see Kovács' plenary, 03/08)

- Ideal-gas-like behaviour with $n_{\text{inst}} = \chi_t$ of topological charge distribution *required* by chiral symmetry restoration [Kanazawa, Yamamoto (2015)]
- Density of peak modes n_{peak} matches the required instanton density

$$\frac{n_{\text{peak}}}{m^2} = \frac{2}{m^2} \int_0^2 d\lambda\, \rho_{\text{peak}}(\lambda; m) \underset{m\to 0}{\to} \Delta = \lim_{m\to 0} \frac{\chi_t}{m^2} = \lim_{m\to 0} \frac{n_{\text{inst}}}{m^2}$$

Summary and outlook

- Chiral symmetry is restored in the scalar/pseudoscalar sector in the $N_f = 2$ massless limit *if and only if* $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2)$ are non-divergent

$$\mathcal{W}(V, W; m) = \sum_{n_u,n_w,n_{\tilde{u}}} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2)$$

- $\mathrm{U}(1)_A$ breaking compatible with $\mathrm{SU}(2)_A$ restoration but requires singular near-zero spectral density $\rho \sim m^4/\lambda$ as $m \to 0$
- Requires also singular two-point function, near-zero modes *not* localised (near-zero mobility edge?) from second-order constraints
- Features required for $\mathrm{U}(1)_A$ breaking occur naturally if topology of gauge field configurations includes ideal instanton gas contribution

Open issues:

- other sectors
- larger N_f
- test against numerical results

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Symmetry restoration in scalar/pseudoscalar sector - details

$$\text{Chiral symmetry restored} \Leftrightarrow \left|\lim_{m\to 0} \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2)\right| < \infty$$

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$\Leftarrow$ Sufficiency: obvious

$\Rightarrow$ Necessity: symmetry restoration condition implies

$$0 = \lim_{m\to 0} \left(\partial_{j_S} \hat{\mathcal{W}}\right)_{V^2,W^2,V\cdot W} = \lim_{m\to 0} \left(2m\partial_{V^2} \hat{\mathcal{W}}\right)_{j_{S,P},W^2,V\cdot W}$$

Mass derivative becomes

$$\lim_{m\to 0} \partial_m \hat{\mathcal{W}} = 2 \lim_{m\to 0} \left(j_S \partial_{V^2} + j_P \partial_{2V\cdot W}\right) \hat{\mathcal{W}}$$

$$\lim_{m\to 0} \partial_m \hat{\mathcal{W}}(\vec{\jmath}_P^{\,2}, \vec{\jmath}_S^{\,2}, \vec{\jmath}_P \cdot \vec{\jmath}_S) = 0 \Rightarrow \lim_{m\to 0} \partial_m \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2) = 0$$

$\Rightarrow \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(0)$ is finite

Corollary:

$\lim_{m\to 0} (\partial_{m^2})^k \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2) = \lim_{m\to 0} \mathcal{A}_{n_u+k,n_w,n_{\tilde{u}}}(m^2)$ are finite

Lowest-order constraints - details

$$\mathcal{A}_{1,0,0} = \frac{n_0}{m^2} + 2\int_0^2 d\lambda\, \frac{h(\lambda)\rho(\lambda;m)}{\lambda^2 + m^2 h(\lambda)} = \frac{\chi_\pi}{2} = \lim_{\mathrm{V}\to\infty} \frac{\langle (iP_1)^2\rangle}{2\mathrm{V}/\mathrm{T}}$$

$$\mathcal{A}_{0,1,0} = -\frac{n_0}{m^2} + 2\int_0^2 d\lambda\, \frac{h(\lambda)[\lambda^2 - m^2 h(\lambda)]\rho(\lambda;m)}{[\lambda^2 + m^2 h(\lambda)]^2} = \frac{\chi_\delta}{2} = \lim_{\mathrm{V}\to\infty} \frac{\langle {S_1}^2\rangle}{2\mathrm{V}/\mathrm{T}}$$

$$\tfrac{1}{2}\mathcal{A}_{0,0,2} = \frac{n_0 - \chi_t}{2m^4} + \int_0^2 d\lambda\, \frac{h(\lambda)^2\rho(\lambda;m)}{[\lambda^2 + m^2 h(\lambda)]^2} = \lim_{\mathrm{V}\to\infty} \frac{\langle (iP_1)S_1(iP_2)S_2\rangle}{8\mathrm{V}/\mathrm{T}}$$

$$h(\lambda) \equiv 1 - \frac{\lambda^2}{4} \qquad n_0 = \lim_{\mathrm{V}\to\infty} \frac{\langle N_+ + N_-\rangle}{\mathrm{V}/\mathrm{T}} \qquad \chi_t = \lim_{\mathrm{V}\to\infty} \frac{\langle (N_+ - N_-)^2\rangle}{\mathrm{V}/\mathrm{T}}$$

$\lambda_n^2/2 \pm i\lambda_n\sqrt{1-\lambda_n^2/4}$: complex eigenvalues of D

$N_\pm$: n. of chiral zero modes

$n_0 = 0$, $|\chi_\delta| \le |\chi_\pi| \Rightarrow$ constraints simplify to

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$$\frac{\chi_\pi}{2} = \int_0^2 d\lambda\, \frac{h(\lambda)\rho(\lambda;m)}{\lambda^2 + m^2 h(\lambda)} \quad \text{finite as } m \to 0$$

$$\frac{\chi_\pi - \chi_\delta}{4} - \frac{\chi_t}{m^2} = \int_0^2 d\lambda\, \frac{2m^2 h(\lambda)^2\rho(\lambda;m)}{[\lambda^2 + m^2 h(\lambda)]^2} - \frac{\chi_t}{m^2} = c_0 m^2 + o(m^2)$$