An update on the determination of the sphaleron rate in finite-temperature QCD

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Based on the work in collaboration with:
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The strong sphaleron rate

Strong sphaleron rate → real time topological transitions in finite-T QCD

$$\Gamma_{\mathrm{Sphal}} = \lim_{\substack{V_s \to \infty \\ t_{\mathrm{M}} \to \infty}} \frac{1}{V_s t_{\mathrm{M}}} \left\langle \left[\int_0^{t_{\mathrm{M}}} dt'_{\mathrm{M}} \int_{V_s} d^3x \, q(t'_{\mathrm{M}}, \vec{x}) \right]^2 \right\rangle = \int dt_{\mathrm{M}} d^3x \, \langle q(t_{\mathrm{M}}, \vec{x}) q(0, \vec{0}) \rangle$$

$q(x)$ is the QCD *topological charge density* operator

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \, \mathrm{Tr}\{G^{\mu\nu}(x) G^{\rho\sigma}(x)\}$$

Phenomenological role of Γ_{Sphal}

- axion thermal production in the early Universe [A. Notari, F. Rompineve, G. Villadoro; PRL 131, 011004 (2023)]
- local imbalances in the number of left/right-handed quark species in the quark-gluon plasma → Chiral Magnetic Effect [K. Fukushima, D. E. Kharzeev, H. J. Warringa; PRD 78, 074033 (2008)]

Computation of Γ_{Sphal} on the lattice

Euclidean space-time on the lattice $\rightarrow$ real time definition of Γ_{Sphal} can not be directly used

How to extract Γ_{Sphal} from lattice simulations?

Γ_{Sphal} is related to the *spectral density* $\rho(\omega)$ of the Euclidean topological charge density time correlator $G(t)$ via the Kubo formula

$$\Gamma_{\mathrm{Sphal}} = 2T \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

$$G(t) \equiv \int d^3x \langle q(t,\vec{x}) q(0,\vec{0}) \rangle = -\int_0^\infty \frac{d\omega}{\pi} \rho(\omega) \frac{\cosh\left[\frac{\omega}{2T} - \omega t\right]}{\sinh\left[\frac{\omega}{2T}\right]}$$

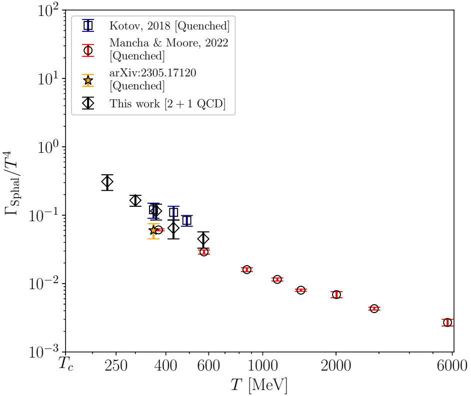
Compute $G(t)$ on the lattice $\rightarrow$ invert the correlator to obtain $\rho(\omega)$

The correlator $G(t)$ is inverted by using a HLT modification of the Backus–Gilbert method [M. Hansen, A. Lupo, N. Tantalo; PRD 99, 094508 (2019)], which allows to estimate the $g_t(0)$ coefficients

$$\frac{\Gamma_{\mathrm{Sphal}}}{2T} = \left[\frac{\bar{\rho}(\bar{\omega})}{\bar{\omega}}\right]_{\bar{\omega}=0} = -\pi \sum_{t=0}^{1/T} g_t(0) G(t)$$

First computation of Γ_{Sphal} in $N_f = 2+1$ QCD at physical point

T [MeV]	$\Gamma_{\mathrm{Sphal}}/T^4$
230	0.310(80)
300	0.165(40)
365	0.115(30)
430	0.065(20)
570	0.045(12)



Full QCD results larger than quenched ones, but fall in similar ballpark

Higher temperatures needed to clarify the functional dependence of $\Gamma_{\mathrm{Sphal}}(T)$

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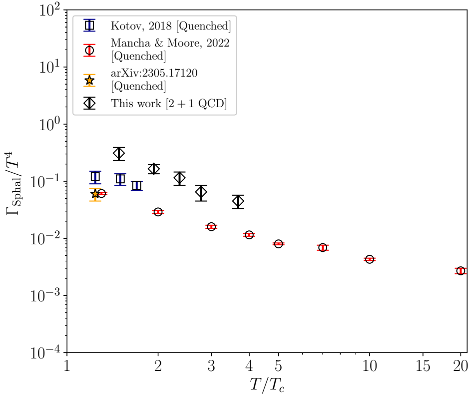


$$T_c^{\text{QCD}} \simeq 155 \text{ MeV}$$
$$T_c^{\text{SU}(3)} \simeq 287 \text{ MeV}$$

Axion rate from strong sphalerons

A. Notari, F. Rompineve, G. Villadoro [PRL 131, 011004 (2023)]

For relativistic axions $p^\mu = (E = |\vec{p}|, \vec{p})$ and the quantity

$$\Gamma^{>}_{\rm top} \equiv \int d^4x \, e^{ip^\mu x_\mu} \left\langle \frac{\alpha_s}{8\pi} G\tilde{G}(x^\mu) \frac{\alpha_s}{8\pi} G\tilde{G}(0) \right\rangle$$

enters the Boltzmann equation for the momentum-dependent distribution function $f_{\vec{p}}$.

Notice that $\Gamma^{>}_{\rm top}(p^\mu = 0)$ is the sphaleron rate.

For $T \lesssim 5$ GeV the sphaleron-like contribution to the rate becomes very important, a non-perturbative determination of $\Gamma^{>}_{\rm top}(\vec{p})$ is needed to compute the total axion production rate.

Expectation

- $\Gamma^{>}_{\rm top}(E = |\vec{p}| < |\vec{p}_s|) \simeq \Gamma_{\rm Sphal}$, where $|\vec{p}_s|$ is the 3-momentum associated to the sphaleron size (of order $1/N_c \alpha_s T$)
- $\Gamma^{>}_{\rm top}$ decays for $E = |\vec{p}| > |\vec{p}_s|$

Computation of $\Gamma^{>}_{\rm top}(p^\mu)$ on the lattice

- We introduce the energy spectral density $\rho(\omega, \vec{p})$ of the Euclidean time-correlator of $q(x)$ at non-zero spatial momentum

$$G^{\vec{p}}(t) \equiv \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \langle q(t,\vec{x})(0,\vec{0})\rangle = -\int \frac{d\omega}{\pi} \rho(\omega,\vec{p}) \frac{\cosh\left[\frac{\omega}{2T} - \omega t\right]}{\sinh\left[\frac{\omega}{2T}\right]}$$

- Compute $G^{\vec{p}}(t)$ on the lattice $\rightarrow$ invert the correlator to obtain $\rho(\omega,\vec{p})$ (HLT Backus–Gilbert)

$$\Gamma^{>}_{\rm top}(|\vec{p}|) = \left[\coth\left(\frac{\bar{\omega}}{2T}\right) \bar{\rho}(\bar{\omega},\vec{p})\right]_{\bar{\omega}=|\vec{p}|} = \coth\left(\frac{|\vec{p}|}{2T}\right) \left[-\pi \sum_{t=0}^{1/T} g_t(\bar{\omega}=|\vec{p}|) G^{\vec{p}}(t)\right]$$

- The inversion has to be performed in $\omega = |\vec{p}|$ because of the axion dispersion relation in the massless approximation

Numerical setup

Monte Carlo simulations of $SU(3)$ gauge theory at $T \simeq 1.24\, T_c$, $N_t = 14, 16, 20$, $N_s/N_t = 4$

Computation of $G^{\vec{p}}(t)$

- $q(x)$ discretized with the standard clover definition $q_L(n_t; \vec{n})$
- we define the 3-momentum dependent time profile:

$$Q_L^{\vec{p}}(n_t) \equiv \sum_{\vec{n}} e^{i\vec{p}\cdot\vec{n}} q_L(n_t; \vec{n}) \ , \qquad \vec{p} = \frac{2\pi}{N_s}(k, 0, 0), \ k \in [0, \dots, N_s - 1]$$

- and finally the spatial Fourier transform:

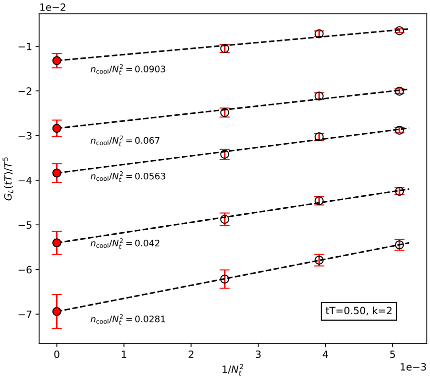
$$\frac{G_L^{\vec{p}}(tT)}{T^5} = \frac{N_t^5}{N_s^3} \left\langle Q_L^{\vec{p}}(n_{t,1}) Q_L^{-\vec{p}}(n_{t,2}) \right\rangle \ , \qquad tT = \min\left\{ \frac{|n_{t,1} - n_{t,2}|}{N_t}; 1 - \frac{|n_{t,1} - n_{t,2}|}{N_t} \right\}$$

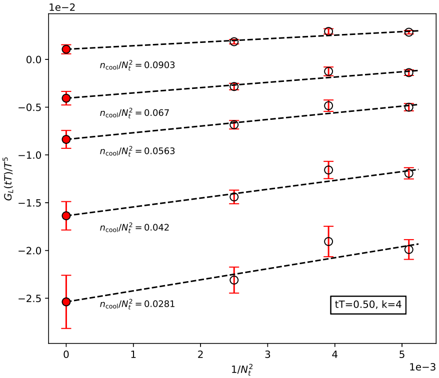
- cooling to dampen UV noise affecting the two-point corr. func. of $q(x)$
- continuum limit at fixed smoothing radius $r_s T \sim \frac{\sqrt{\frac{8}{3} n_{\mathrm{cool}}}}{N_t}$ + zero smoothing limit $r_s \to 0$

Continuum limit

We perform the continuum limit at fixed $\vec{p}/T$ with the following scaling

$$\frac{G_L^{\vec{p}}\left(tT, N_t, \frac{n_{\rm cool}}{N_t^2}\right)}{T^5} = \frac{G^{\vec{p}}\left(tT, \frac{n_{\rm cool}}{N_t^2}\right)}{T^5} + b^{\vec{p}}\left(tT, \frac{n_{\rm cool}}{N_t^2}\right)\frac{1}{N_t^2} + o\left(\frac{1}{N_t^2}\right)$$





$r_s \to 0$ limit

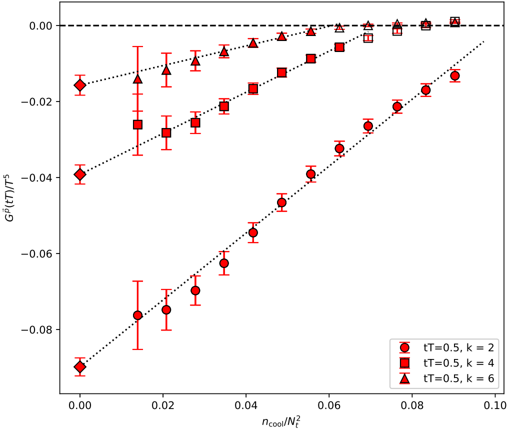
Zero smoothing limit according to the following linear behaviour

$$\frac{G^{\vec{p}}\left(tT, \frac{n_{\text{cool}}}{N_t^2}\right)}{T^5} = \frac{G^{\vec{p}}(tT)}{T^5} + c^{\vec{p}}(tT)\frac{n_{\text{cool}}}{N_t^2}$$

- lower bound fit range: $n_{\text{cool}}/N_t^2 \simeq 0.012$ $\to$ plateau of the topological susceptibility
- upper bound fit range:

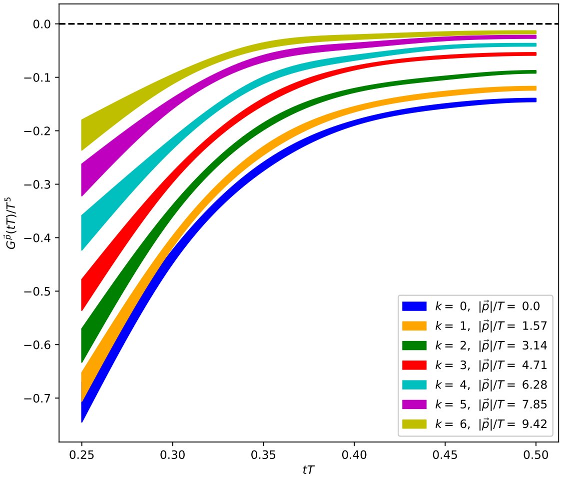
$$\frac{n_{\text{cool}}^{(\text{max})}}{N_t^2} \simeq \frac{3}{8}(tT)^2$$

- smaller $n_{\text{cool}}^{(\text{max})}$ as k increases



$\vec{p}$ dependence of $\frac{G^{\vec{p}}(tT)}{T^5}$

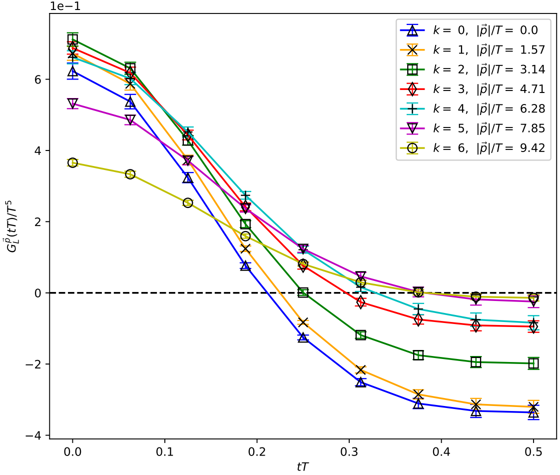
- suppression of the correlator for increasing values of $|\vec{p}|/T \to$ decay of $\Gamma_{\text{top}}(\vec{p})$
- significant suppression for $|\vec{p}|/T \simeq O(10)$
- inversion still to be performed to find the shape of $\Gamma_{\text{top}}(\vec{p})$



Preliminary results in $N_f = 2+1$ QCD

Lattice setup: tree-level Symanzik improved action for the gauge sector, rooted stout staggered discretization, physical point

- ensemble $N_s = 64$, $N_t = 16$, $a = 0.0536$ fm, $T = 230$ MeV
- $n_{\rm cool} = 12 \to n_{\rm cool}/N_t^2 \simeq 0.0469$
- similar suppression w.r.t. the quenched case



Summary

Conclusion

- computation of the spatial Fourier transform of the two-point correlation function of $q(x)$ with a double extrapolation procedure
- suppression of the correlator for increasing $\vec{p}$

Future outlooks

- inversion of the correlator in the quenched case to evaluate $\Gamma^{>}_{\mathrm{top}}(|\vec{p}|)$
- extend the computation to the QCD case at different temperatures to study how fermion dynamics affects the behaviour of $\Gamma^{>}_{\mathrm{top}}(|\vec{p}|)$

Backup slides

M. Hansen, A. Lupo, N. Tantalo [PRD 99, 094508 (2019)]

The goal is to invert the correlator:

$$G^{\vec{p}}(t) = -\int_0^\infty \frac{d\omega}{\pi}\rho(\omega,\vec{p})K_t(\omega) \quad \text{with} \quad K_t(\omega) \equiv \frac{\cosh[\omega/(2T) - \omega t]}{\sinh[\omega/2T]} \tag{1}$$

According to the Backus–Gilbert procedure, an estimation of $\rho(\omega)$ can be computed as a lin. combination of the correlator:

$$\bar{\rho}(\bar{\omega},\vec{p}) = -\pi \sum_{t=0}^{1/T} g_t(\bar{\omega}) G^{\vec{p}}(t) \tag{2}$$

and finally

$$\Gamma^{>}_{\rm top}(|\vec{p}|) = \left[\coth\left(\frac{\bar{\omega}}{2T}\right)\bar{\rho}(\bar{\omega},\vec{p})\right]_{\bar{\omega}=|\vec{p}|}$$

Combining Eqs. 1 and 2 ($\Delta(\omega, \bar{\omega})$ is the *resolution function*)

$$\bar{\rho}(\bar{\omega}) = \int_0^\infty d\omega \, \Delta(\omega, \bar{\omega}) \rho(\omega) \quad \text{with} \quad \Delta(\omega, \bar{\omega}) = \sum_{t=0}^{1/T} g_t(\bar{\omega}) K_t(\omega)$$

How to determine $g_t(\omega)$?

After choosing a *target function* $\delta(\omega, \bar{\omega})$, minimize

$$F[g_t] = (1 - \lambda) A_\alpha[g_t] + \frac{\lambda}{\mathcal{C}} B[g_t] \quad \text{with } \lambda \in [0, 1) \text{ free parameter}$$

where A_α is a measure of the distance between $\Delta(\omega, \bar{\omega})$ and $\delta(\omega, \bar{\omega})$ and $B[g_t]$ takes into account the uncertainties

$$A_\alpha[g_t] = \int_0^\infty d\omega \, [\Delta(\omega, \bar{\omega}) - \delta(\omega, \bar{\omega})]^2 e^{\alpha\omega} \qquad (\alpha < 2)$$

$$B[g_t] = \sum_{t,t'=0}^{1/T} \mathrm{Cov}_{t,t'} \, g_t g_{t'}$$