Gauge field digitization in the Hamiltonian limit

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Motivation: Complex Action Problem (in very short)

Partition function as a path integral

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-S[\phi]} = \int \mathcal{D}\phi \, w[\phi]$$

If weights $w[\phi] \notin \mathbb{R}^+$ usual MCMC methods relying on importance sampling not applicable:

complex action problem

In principle, can be bypassed with the help of quantum computers

`[quant-ph/1811.03629]`

Digitizing gauge groups

In the NISQ era the main bottlenecks are the limited

- circuit depths
- **number of qubits**

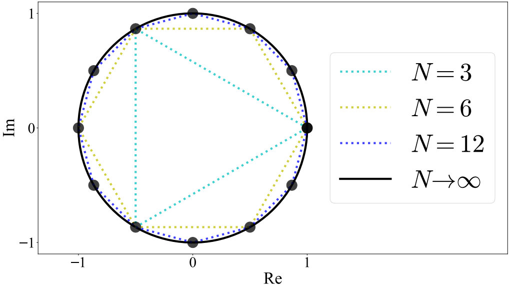
The Hilbert space for a gauge theory based on a continuous gauge group is infinite dimensional

$\Downarrow$

Shall be made discrete and finite via **digitization scheme**

e.g. U(1) discretized to Z(N)

$$g_\infty(\varphi \in \mathbb{R}) = e^{i\varphi} \quad \mapsto \quad g_N(n \in \mathbb{Z}^+) = e^{2\pi i n/N}$$



[hep-lat/1906.11213], [hep-lat/2201.09625]

The Hamiltonian limit

To derive the Kogut-Susskind Hamiltonian `[Phys.Rev.D 11.395]`:

1. temporal gauge $\sim$ **zero-temperature limit**
2. **anisotropic lattice** with action

$$S^{\mathrm{a}} = \sum_x \Big[-\beta_S \sum_{k<l} \mathrm{Re}\mathrm{Tr}\,\mathcal{U}_{kl}(x) - \beta_T \sum_k \mathrm{Re}\mathrm{Tr}\big[U_k(x+\hat{0})U_k^\dagger(x)\big]\Big]$$

3. taking the **Hamiltonian limit** as

$$\begin{cases} a_s = \text{fixed} & \sim \text{ discrete space} \\ a_t \to 0 & \sim \text{ continuous time} \end{cases}$$

   along trajectory set by Hamiltonian coupling g_{H}

4. **Kogut-Susskind Hamiltonian**

$$\mathcal{H}_{\mathrm{KS}} \cong \frac{g_{\mathrm{H}}^2}{2a_s} \sum_{\{ij\}} \ell_{ij}^2 + \frac{2}{g_{\mathrm{H}}^2 a_s} \sum_p \mathrm{Re}\mathrm{Tr}\,\mathcal{U}_p$$

   where ℓ_{ij} is a differential operator in group parameters acting on the link connecting sites i and $i+\hat{j}$; e.g in U(1) $\ell_{ij} \propto -i\mathrm{d}/\mathrm{d}\theta_{ij}$ acting on $e^{i\theta_{ij}}$

Scaling laws of gauge couplings in the Hamiltonian limit

In the Hamiltonian limit $\beta_S \to 0$ and $\beta_T \to \infty \ldots$ *How exactly?*

Hamiltonian from **transfer matrix**

$$\mathcal{Z} = \mathrm{Tr}\,\mathbf{T}^{N_t} \quad \text{from which} \quad \mathbf{T} = \mathbf{1} - a_t \mathcal{H} + \mathcal{O}(a_t^2)$$

To have non-trivial Hamiltonian in the continuous time limit, scaling laws are imposed

- **Temporal coupling**: continuous vs. discrete gauge groups

$$a_t \mathcal{H}_{\infty,\mathrm{kin}} \propto \Delta_\infty \qquad \Longrightarrow \qquad \beta_T \sim \frac{1}{a_t}$$

$$a_t \mathcal{H}_{N,\mathrm{kin}} \propto \Delta_N \qquad \Longrightarrow \qquad \beta_T \sim \log(1/a_t)$$

  with Δ_∞ a continuous and Δ_N a discrete Laplacian in group space
- **Spatial coupling**: $\beta_S \sim a_t$ (regardless)

The Hamiltonians

U(1) Hamiltonian ($\beta_S \to 0$ s.t. $\beta_T = 4/\beta_S g_{\mathrm{H}}^4$):

$$a_s \mathcal{H}_\infty \cong -\frac{g_{\mathrm{H}}^2}{4} \sum_{\substack{i \in \Lambda_s \\ j=1,2,3}} \frac{\mathrm{d}^2}{\mathrm{d}\theta_{ij}^2} + \frac{2}{g_{\mathrm{H}}^2} \sum_p \mathrm{Re}\,\mathcal{U}_p$$

Z(N) Hamiltonian ($\beta_S \to 0$ s.t. $\beta_T = 1/(\cos(2\pi/N) - 1)\log(g_{\mathrm{H}}^4 \beta_S/4)$):

$$a_s \mathcal{H}_N \cong -\frac{g_{\mathrm{H}}^2}{4} \frac{N^2}{4\pi^2} \sum_{\substack{i \in \Lambda_s \\ j=1,2,3}} \log^2 \mathbf{P}_1 + \frac{2}{g_{\mathrm{H}}^2} \sum_p \mathrm{Re}\,\mathcal{U}_p$$

with $\mathbf{P}_1$ being a permutation matrix:

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

(it can be checked that $\log^2 \mathbf{P}_1$ is a discrete Laplacian up to a constant)

Anisotropic Euclidean simulations with discrete groups

Nuances of **working with discrete groups**:

- **freezing transition** occurs due to the discrete nature of the group
- which is of first order. . . **hysteresis**
- **parallel tempering** in $\beta_{S/T}$
- mixture of initial configurations

Nuances of **taking the Hamiltonian limit**:

- as $\beta_S \to 0$ and $\beta_T \to \infty$: overflow in $p_i = e^{x_i}/\mathcal{Z}$ and $\mathcal{Z} = \sum_i e^{x_i}$
- work with logarithms: "log-sum-exp trick"

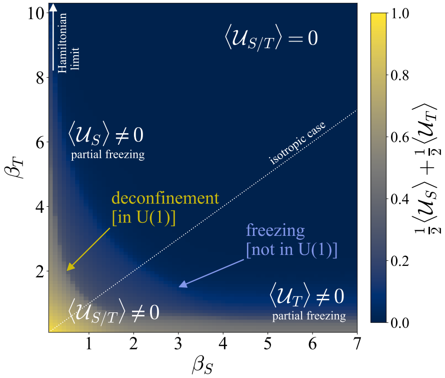
$$\log \mathcal{Z} = \log \left(e^{\tilde{x}} \sum_i e^{x_i - \tilde{x}} \right) = \tilde{x} + \log \left(\sum_i e^{x_i - \tilde{x}} \right) \qquad \text{with } \tilde{x} = \max\{x_i\}$$

Phase transitions in U(1) and Z(N) lattice gauge theories

Anisotropic scan in (β_S, β_T):

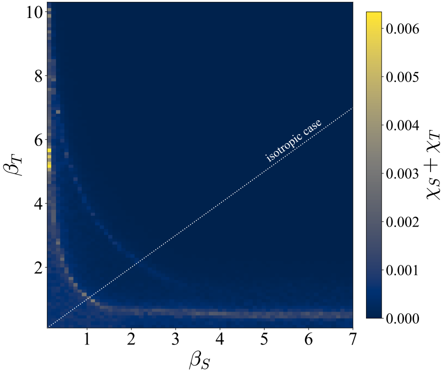
Average plaquettes:

$Z(N=7), |\Lambda| = N_s^3 \times N_t = 4^3 \times 32$

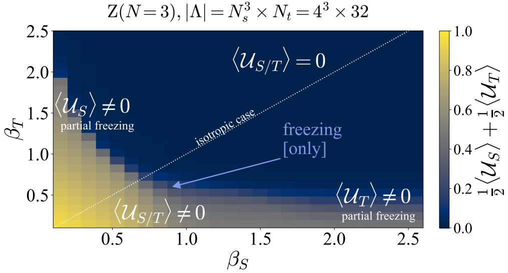


Plaquette susceptibilities:

$Z(N=7), |\Lambda| = N_s^3 \times N_t = 4^3 \times 32$

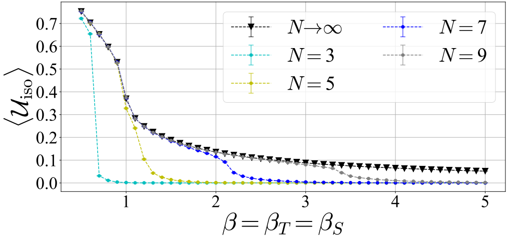


Phase transitions in U(1) and Z(N) lattice gauge theories



For $N \lesssim 5$ freezing washes away deconfinement

Freezing goes away as $N \to \infty$

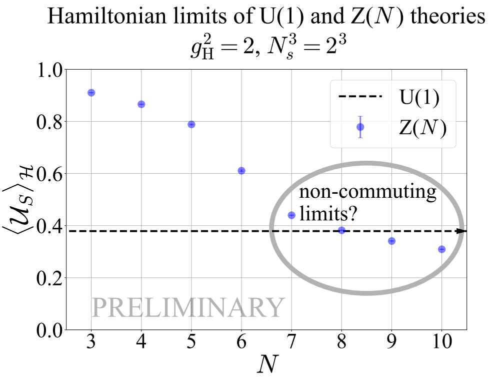


Hamiltonian limit from Euclidean simulations

Do the Hamiltonian limit and the $N \to \infty$ limit commute?

For fixed N_s **2(+1) numerical limits** are to be computed for U(1) or Z(N):

1. $N_t \to \infty$,
2. $\beta_S \to 0$ and $\beta_T \to \infty$ (with appropriate scaling law)
3. $N \to \infty$ for Z(N).



Summary and Outlook

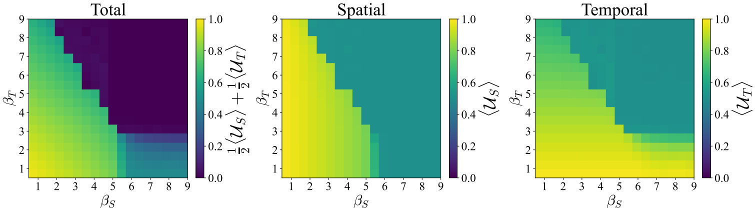
- **Freezing transition** due to discrete nature of the group Z(N) absent in U(1)
- Studying the **freezing transition with anisotropic lattices** for the first time
- Scans in the (β_S, β_T) coupling space reveals **rich phase diagram** for Z(N) gauge theories
- **Deconfinement transition** also in Z($\gtrsim 5$) and U(1) theories, otherwise flushed away by freezing
- **Partial freezing transitions** if β_S small (large) and β_T large (small)
- ***Remaining question***: what happens with freezing transition in the Hamiltonian limit as $N \to \infty$, i.e., the discretization is taken to be finer?

Summary and Outlook

Future prospects: extending analysis to SU(2) and SU(3)

- Largest non-abelian subgroup of SU(3) is S(1080)
  (studied in detail in `[hep-lat/1906.11213]` with isotropic lattices)

$N_t \times N_s^3 = 40 \times 2^3$



  also shows partial freezing

- $SU(3) \cong S(1080) \quad \Leftrightarrow \quad U(1) \cong Z(3)$ $\sim$ need something finer!
- for SU(2) finer digitizations (not based on subgroups) were introduced: T. Hartung et.al. `hep-lat/2201.09625`: *Digitising SU(2) gauge fields and the freezing transition*

Thank you!

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