Duality and entanglement in lattice gauge theories

Andrea Bulgarelli

Università degli Studi di Torino and Istituto Nazionale di Fisica Nucleare

Based on A. Bulgarelli and M. Panero *JHEP* 06 (2024) 041

The 41th International Symposium on Lattice Field Theory, 29th July 2024

Motivations

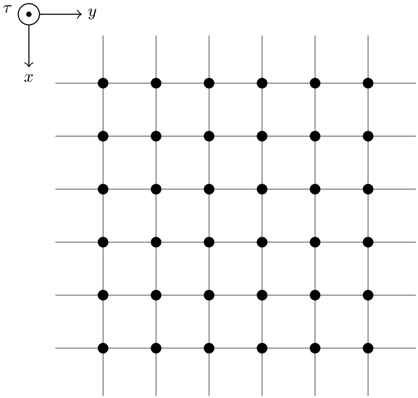
The study of entanglement in many-body systems has applications in different areas of physics:

- Quantum phases of matter and quantum phase transitions.
  - **Effective number** of degrees of freedom.
- High-energy physics.
  - **Confinement**.
- AdS/CFT and quantum gravity.
- Resource for quantum computing and quantum simulations.
- Quantum technologies.

Motivations

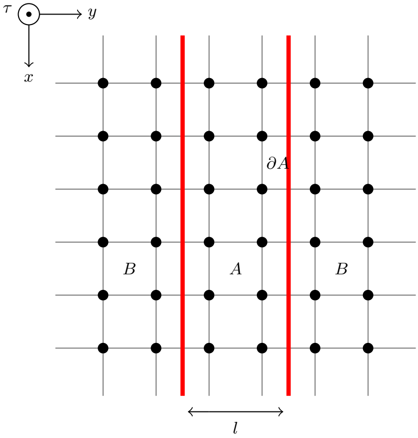
- Analytical calculations are possible only for simple, highly symmetric systems.
- In generic systems numerical simulations are needed. However typical techniques struggle to precisely estimate such highly non-local quantity. We faced this problem in [AB, Panero; 2304.03311].
- On top of that, in gauge theories, calculations of entanglement entropy are limited by ambiguities [Buividovich, Polikarpov; 0806.3376].

Entanglement in QFT



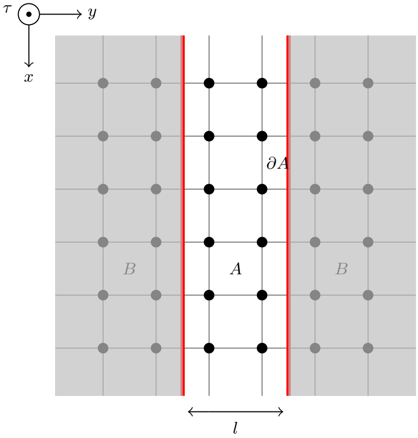
Entanglement in QFT

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



Entanglement in QFT

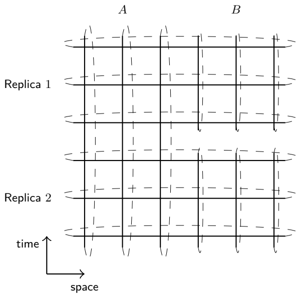
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n \qquad C_n = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_n}{\partial l}$$

Replica trick

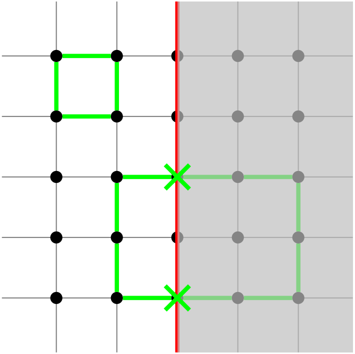
- A common way to calculate Rényi entropies is to exploit the replica trick [Calabrese, Cardy; hep-th/0405152].



$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z^n} \qquad C_n = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_n}{\partial l} \sim \frac{1}{a} \log \frac{Z_n(l+a)}{Z_n(l)}$$

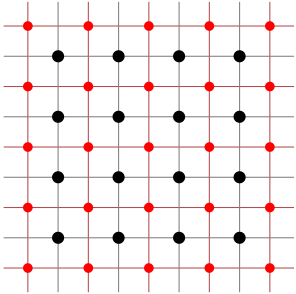
- This approach has been exploited in different works [Buividovich, Polikarpov; 0802.4247], [Itou et al.; 1512.01334], [Rabenstein et al.; 1812.04279], [Jokela et al.; 2304.08949].

$$\mathcal{H}_{\text{phys.}} \neq \mathcal{H}_{\text{phys.},A} \otimes \mathcal{H}_{\text{phys.},B}.$$



Kramers-Wannier duality

- Kramers-Wannier duality relates two different lattice models: $Z \propto Z_{\text{dual}}$
- In three spacetime dimensions it maps a **gauge theory** to a **spin model**.
- In particular: Ising $3D \leftrightarrow \mathbb{Z}_2$ gauge theory.



- It was shown that the universal information encoded in the entanglement entropy is **exactly mapped** [Casini, Huerta; 1406.2991], [Moitra, Soni, Trivedi; 1811.06986], in particular

$$C_n^{\text{gauge}}(\beta^\star, l) = C_n^{\text{spin}}(\beta, l).$$

Kramers-Wannier duality

For gauge theories that admit a dual spin model, it is possible to:

Explicitly derive the **geometry** of the replica space for the gauge theory.

Simulate the spin model to **compute** the entropic c-function of the gauge theory.

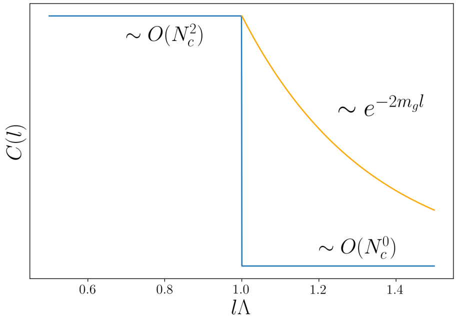
Monte Carlo simulations: setup

- In this work we performed a Monte Carlo study of the entropic c-function in the $(2+1)$-**dimensional** $\mathbb{Z}_2$ **gauge theory** exploiting duality.

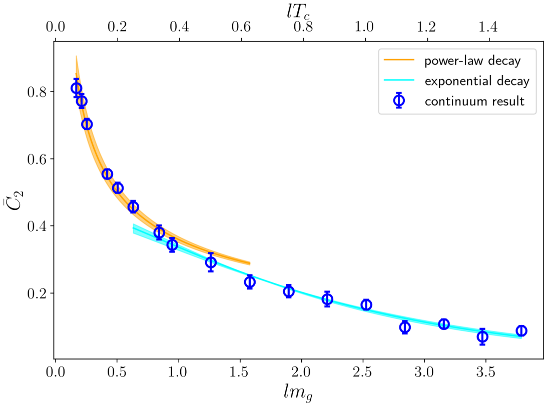
- In order to compute the derivative of the second Rényi entropy we used a highly efficient **non-equilibrium algorithm** introduced in [Alba; 1609.02157], [AB, Panero; 2304.03311].
  - See talks by E. Cellini, A. Nada, Algorithms parallel, Wednesday.

- Efficient algorithms and GPU parallelization allowed us to perform **thermodynamic and continuum** extrapolations of our results.

- In confining theories that admit a dual holographic description it was shown [Klebanov et al.; 0709.2140] that the entropic c-function has a sharp (first-order) transition.
- The scale of this transition is set by a scale which is of the same order of the hadronic scale Λ_{QCD}, the critical temperature T_c or the mass of the lightest glueball m_{g}.



Our result



$$f(lm_g; B, c) = \frac{B}{(lm_g)^c} \qquad B = 0.360(9) \quad c = 0.48(2)$$

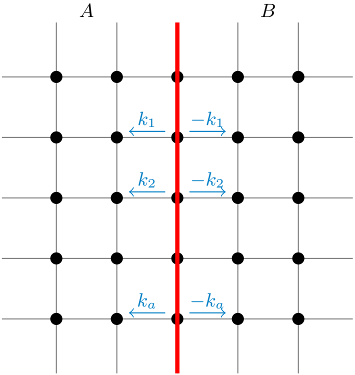
$$f(lm_g; A, \alpha) = A lm_g \int \mathrm{d}t \exp\left(-2\sqrt{1+t^2}\alpha lm_g\right) \qquad A = 0.33(3) \quad \alpha = 0.360(19)$$

Conclusions

- In this work we performed the first thermodynamic and continuum study of the entropic c-function of a confining gauge theory.
- Our results can be directly generalized to other Abelian gauge theories, such as the $U(1)$ **gauge theory**.
- It would be important to understand if the geometries obtained through the duality can be extended to **non-Abelian gauge theories**.
  - See talks by T. Rindlisbacher later in this parallel and by R. Amorosso, Confinement parallel, Wednesday.
- Further developments on the algorithmic side are also necessary to study more complicated theories: **Stochastic Normalizing Flows** for entanglement entropy.

Backup slides

- Consider Abelian gauge theories on the lattice.
- In a basis that diagonalizes the electric field operators we have a well defined notion of electric flux on every link.
- Given a bipartition of the lattice we can define the net electric flux through the boundary $k = \sum_a k_a$.
- $\mathcal{H} = \bigoplus_k \mathcal{H}_A^{(k)} \otimes \mathcal{H}_B^{(-k)}$



Operator algebra, centers and entropy

- It was shown in [Buividovich, Polikarpov; 0806.3376], [Casini, Huerta, Rosabal; 1312.1183] that the entanglement entropy of an Abelian gauge theory admits the following decomposition

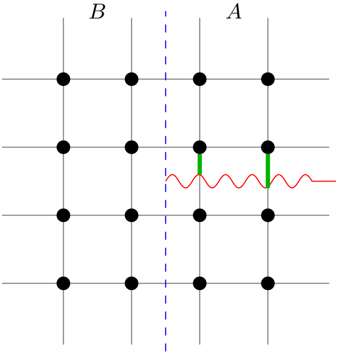
$$S_A = -\sum_k p^{(k)} \log p^{(k)} - \sum_k p^{(k)} S(\rho_A^{(k)}).$$

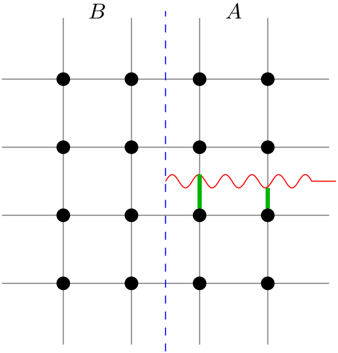
- The first term depends on the classical probability distribution of the flux through the boundary.
- There are arguments suggesting that in the continuum limit the universal part of S_A takes contributions only from the second, distillable term [Casini, Huerta; 1406.2991], [Moitra, Soni, Trivedi; 1811.06986].
- Still, the non-factorization of $\mathcal{H}$ makes the definition of a replica geometry less clear.

Entanglement entropy and duality

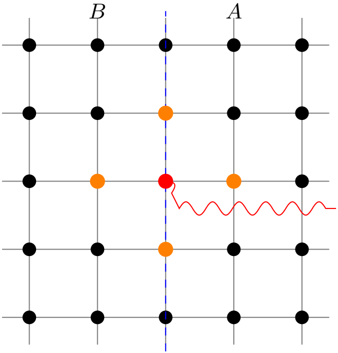
	Spin model (e.g., $D=3$ Ising model)		Dual model (e.g., $D=3$ $\mathbb{Z}_2$ gauge theory)
Lagrangian	Z_n	**this work** $\longrightarrow$	$Z_n^\star$
	replica trick + discretization $\uparrow$		$\uparrow$ [Lin, Radičević; 1808.05939]; **this work**
Hamiltonian	C_n with $\mathcal{H}_A \otimes \mathcal{H}_B$	$\longrightarrow$ [Casini, Huerta; 1406.2991]; [Radičević; 1605.09396]; [Moitra, Soni, Trivedi; 1811.06986]	$C_n^\star$ superselection sectors

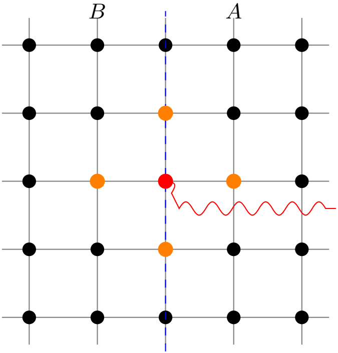
$\mathbb{Z}_N$ spin model in $2D$

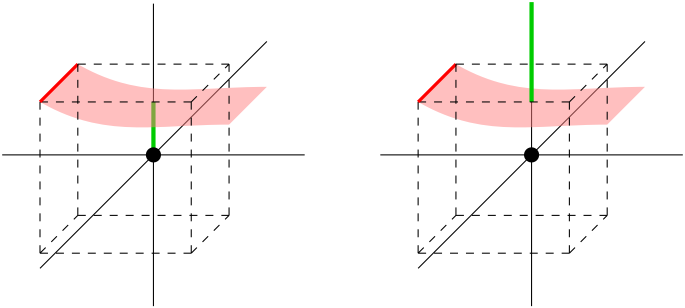




$\mathbb{Z}_N$ spin model in $2D$



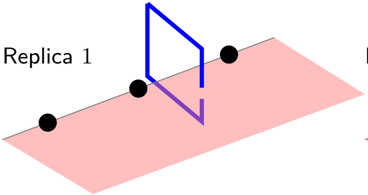




- In a system of n replicas gauge fields on the boundary belong to $4n$ plaquettes.
- Notice that this geometry constraints gauge transformations along the entangling surface to be the same on all replicas.

Abelian gauge theories in $3D$: central-plaquette geometry

- Geometrically, an other possibility is to locate the entangling surface along the links of the spin model.
- In this geometry spins along the entangling surface have an enlarged number of nearest neighbors.



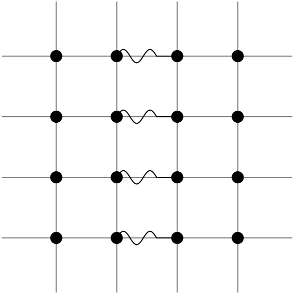
- This "central-plaquette" geometry was introduced in [Chen et al., 1503.01766].

Boundary conditions

- In general the topology of the lattice manifold has an effect on the duality transformation.

- For example, in the $2D$ Ising model on a torus

$$Z(\beta) \propto Z^\star_{pp}(\beta^\star) + Z^\star_{pa}(\beta^\star) + Z^\star_{ap}(\beta^\star) + Z^\star_{aa}(\beta^\star).$$



Jarzynski's theorem

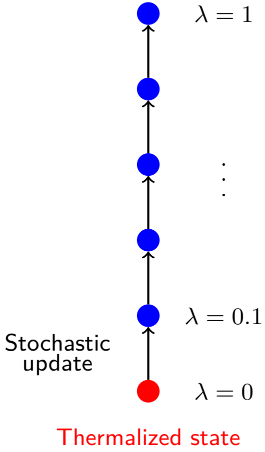
- In a Monte Carlo simulation the off-equilibrium evolution is discretized in a set of steps $\{\lambda_k\}$.
- At each step of the evolution the work performed on the system to change parameters λ from λ_k to λ_{k+1} is

$$\delta W_k = S_{\lambda_{k+1}}[\phi_k] - S_{\lambda_k}[\phi_k]$$

- Jarzynski's theorem then states that

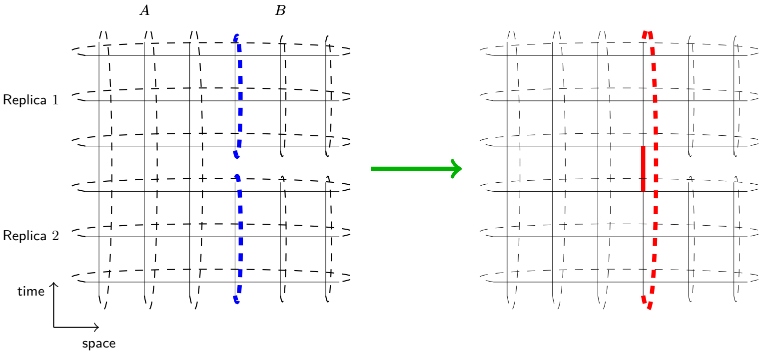
$$\left\langle \exp\left(-\sum_k \delta W_k\right)\right\rangle = \frac{Z_{\lambda=1}}{Z_{\lambda=0}}$$

where the average is over the ensamble of out-of-equilibrium trajectories.

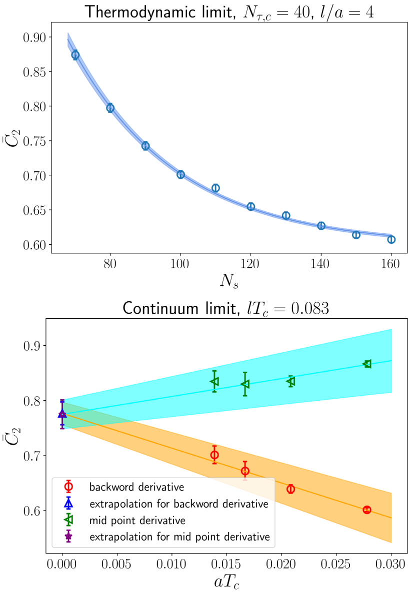


Non-equilibrium algorithm

$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n}\frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$



Thermodynamic and continuum limit



Theoretical predictions

- At large length scales ($m_{\mathrm{g}} l \gg 1$), the $\mathbb{Z}_2$ gauge theory can be modeled as a gas of weakly interacting bosons.

- In [Klebanov et. al.; 0709.2140] it was conjectured that in that regime the entropic c-function can be approximated by

$$C_n^{(2+1)}(m_g l) \sim l \int \mathrm{d}\mathbf{k} C_n^{(1+1)}(\sqrt{m_g^2 + \mathbf{k}^2} l),$$

$$C_n^{(1+1)}(m_g l) \underset{m_g l \gg 1}{\sim} \exp(-2\alpha m_g l), \qquad \alpha = 1.$$

Theoretical predictions

- In opposite regime $m_{\mathrm{g}} l \ll 1$ it is less clear how to model the behavior of the entropic c-function.
- In a recent work [Florio; 2312.05298] it was found that the logarithmic negativity in the Schwinger model shows a polynomial decay at short length scales.
- Motivated by this observation we fitted our data at small lengths with a polynomial function

$$f(m_{\mathrm{g}} l; B, c) = B(m_{\mathrm{g}} l)^{-c}.$$