Computing theta-dependent mass spectrum of the 2-flavor Schwinger model in the Hamiltonian formalism

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collaboration with
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JHEP11 (2023) 231 [2307.16655] and [2407.11391]

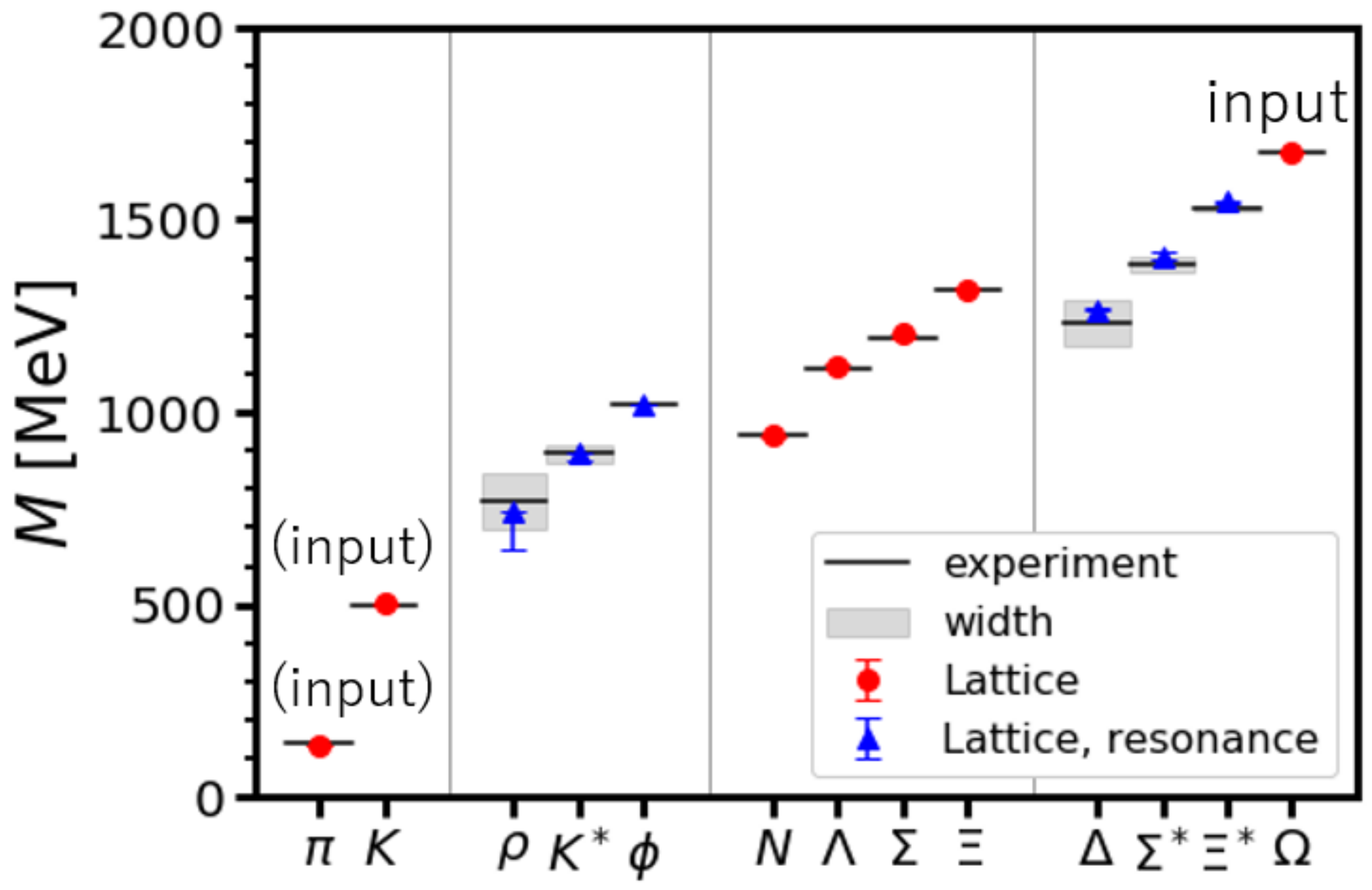
LATTICE 2024, 30 July 2024 @University of Liverpool

Mass spectra of gauge theories

motivation: numerically investigate low-energy spectra of confining gauge theories

😊 Lattice QCD can predict the hadron mass spectrum

😣 Theories with chemical potential or θ term are inaccessible due to the sign problem



[HAL QCD collab. (2024)]

Tensor network and quantum computing in the Hamiltonian formalism can be complementary approaches!

👍 free from the sign problem 👍 analyze excited states directly

aim of this work:
compute the hadron mass spectrum in the Hamiltonian formalism

"Mesons" in 2-flavor Schwinger model

Schwinger model = QED in 1+1d

- the simplest nontrivial gauge theory sharing some features with QCD

$$\mathscr{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f}\left[i\bar{\psi}_f\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_f - m\bar{\psi}_f\psi_f\right]$$

sign problem if $\theta \neq 0$

- quantum numbers:

  isospin J, parity P, G-parity $G = Ce^{i\pi J_y}$

- P and G are explicitly broken at $\theta \neq 0$,

  —> η becomes unstable

  due to $\eta \to \pi\,\pi$ decay and η - σ mixing

"mesons"

$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$: $J^{PG} = 1^{-+}$

$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right)$: $J^{PG} = 0^{--}$

$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$: $J^{PG} = 0^{++}$

Short summary

- JHEP11 (2023) 231 [2307.16655]:
  demonstrated three distinct methods to compute the mass spectrum at $\theta = 0$

  (1) correlation-function scheme

  (2) one-point-function scheme

  (3) dispersion-relation scheme

- [2407.11391]: improve and extend them to the case of $\theta \neq 0$

  (1)+(2) improved one-point-function scheme

  (3) dispersion-relation scheme

- θ -dependent spectra by these schemes are
  consistent with each other and with calculation in the bosonized model

Calculation strategy

[Kogut & Susskind (1975)]
[Dempsey et al. (2022)]

- setup: staggered fermion with open boundary

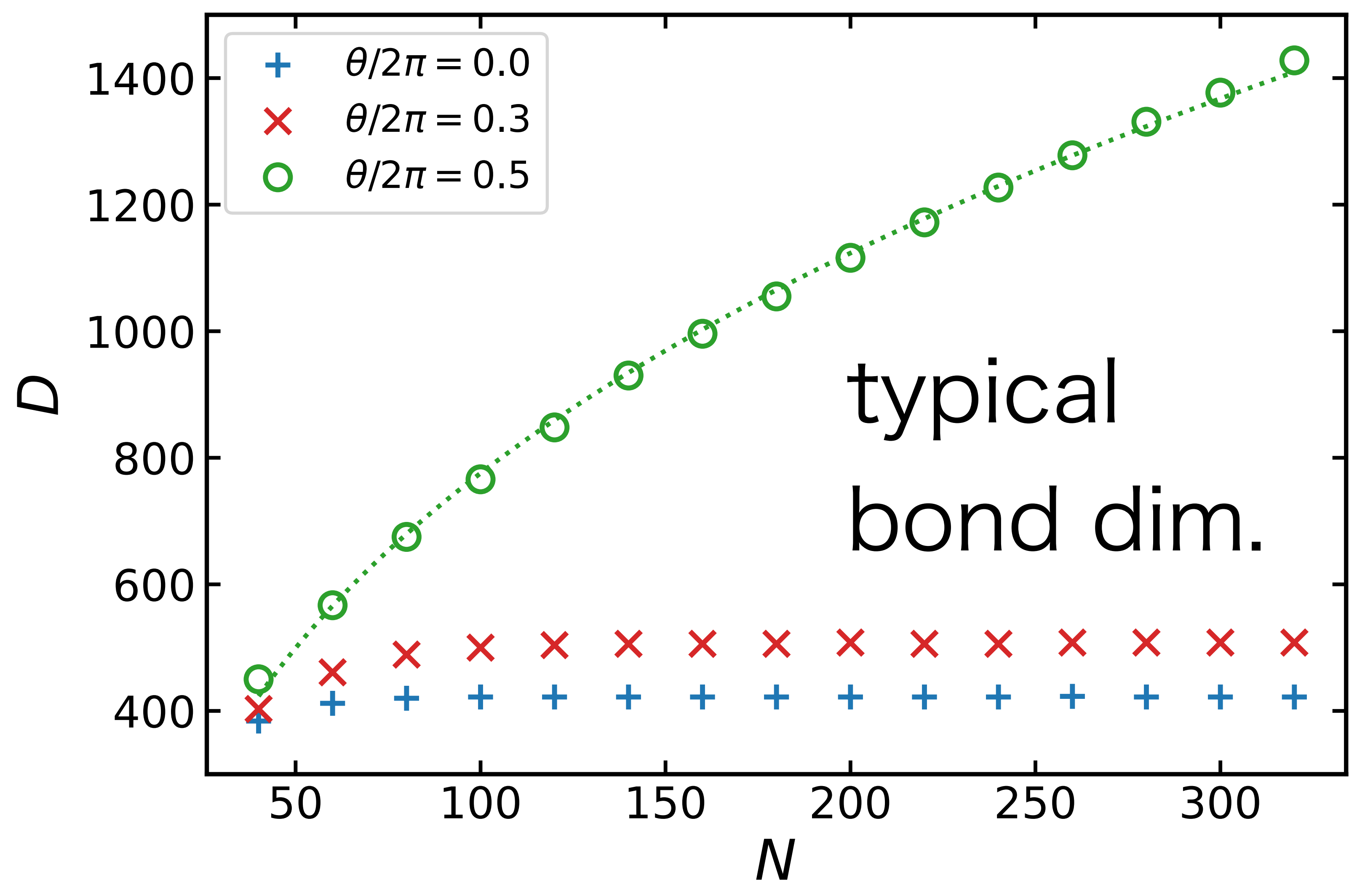
$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-2} \left(L_n + \frac{\theta}{2\pi} \right)^2 + \sum_{f=1}^{N_f} \left[\frac{-i}{2a} \sum_{n=0}^{N-2} \left(\chi_{f,n}^\dagger U_n \chi_{f,n+1} - \chi_{f,n+1}^\dagger U_n^\dagger \chi_{f,n} \right) + m_{\mathrm{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_{f,n}^\dagger \chi_{f,n} \right]$$

- rewrite to the spin Hamiltonian
  by Jordan-Wigner transformation
  after solving Gauss law and gauge fixing

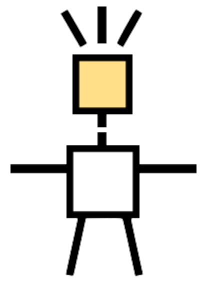
- obtain ground state $|\Psi_0\rangle$ and excited states $|\Psi_\ell\rangle$

  as MPS by DMRG with $H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}|$

  ℓ: level [Banuls et al. (2013)]



C++ library of ITensor is used
[Fishman et al. (2022)]



Simulation results

1. Operator mixing

2. Improved one-point-function scheme

3. Dispersion-relation scheme

Simulation results

1. **Operator mixing**
2. Improved one-point-function scheme
3. Dispersion-relation scheme

Operator mixing

<u>Resolve the θ-dependent operator mixing to define meson operators</u>

- diagonalize the correlation matrix

e.g.) isosinglet sector

$$\begin{pmatrix} \langle S(x)\,S(y)\rangle_c & \langle S(x)\,PS(y)\rangle_c \\ \langle PS(x)\,S(y)\rangle_c & \langle PS(x)\,PS(y)\rangle_c \end{pmatrix} = R(\delta)^{\mathrm{T}} \begin{pmatrix} \langle \sigma(x)\,\sigma(y)\rangle_c & 0 \\ 0 & \langle \eta(x)\,\eta(y)\rangle_c \end{pmatrix} R(\delta)$$

$$S(x) \leftrightarrow \bar{\psi}\psi(x)$$

$$PS(x) \leftrightarrow -i\bar{\psi}\gamma^5\psi(x)$$

$$R(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$$ $\cdots$ rotation matrix with the mixing angle δ

- define meson operators by $\begin{pmatrix} \sigma(x) \\ \eta(x) \end{pmatrix} := R(\delta) \begin{pmatrix} S(x) \\ PS(x) \end{pmatrix}$

Result of mixing angle

- **triplet sector**: $\delta_- \approx \theta/2$

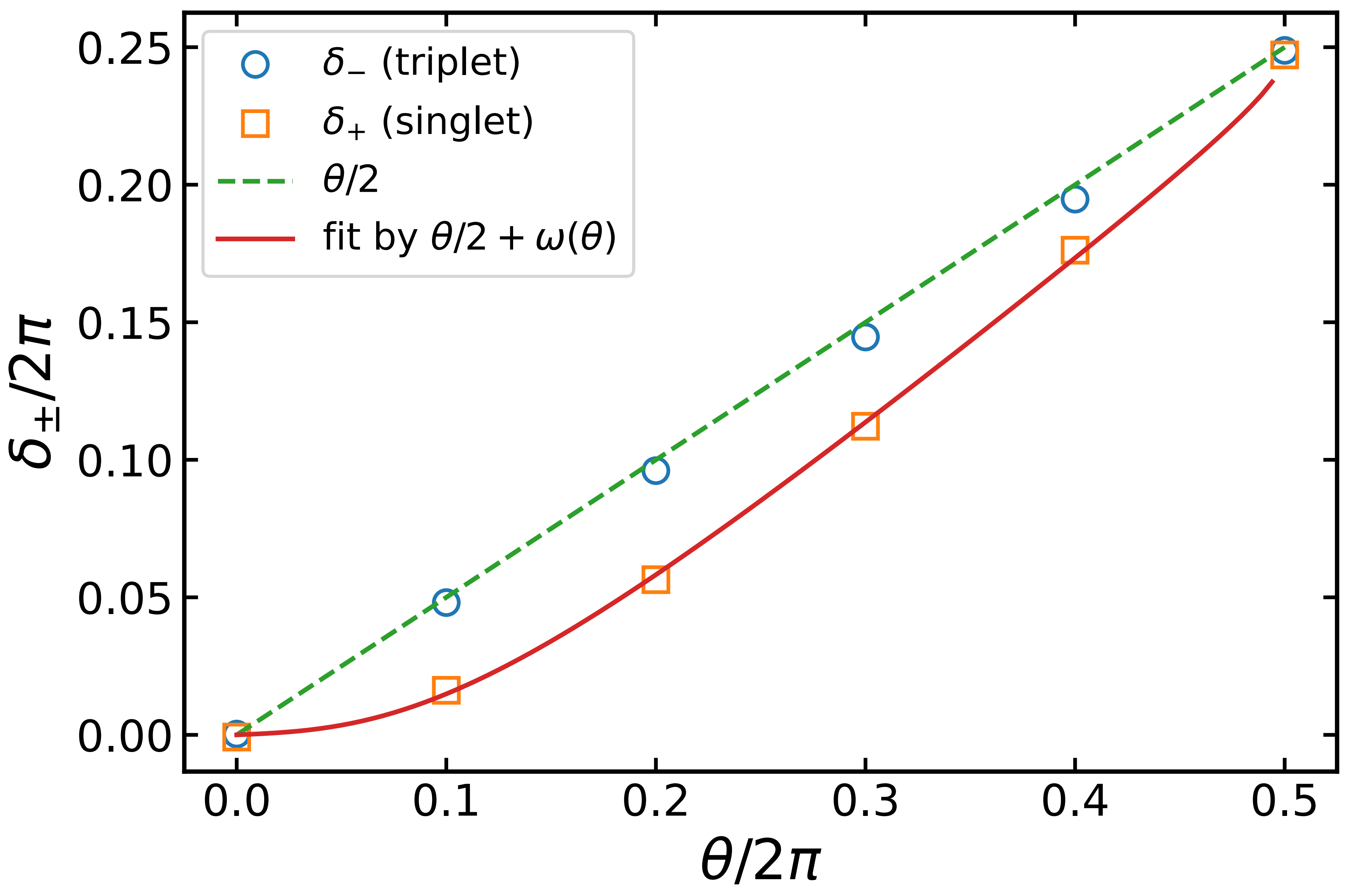
  trivial rotation $\exp\left[i(\theta/2)\gamma^5\right]$ since there is no mixing partner with π

- **singlet sector**: $\delta_+ \approx \theta/2 + \omega(\theta)$

  due to the nontrivial η - σ mixing

- The result of δ_+ can be fitted

  by the function obtained
  from the bosonized model



Simulation results

1. Operator mixing

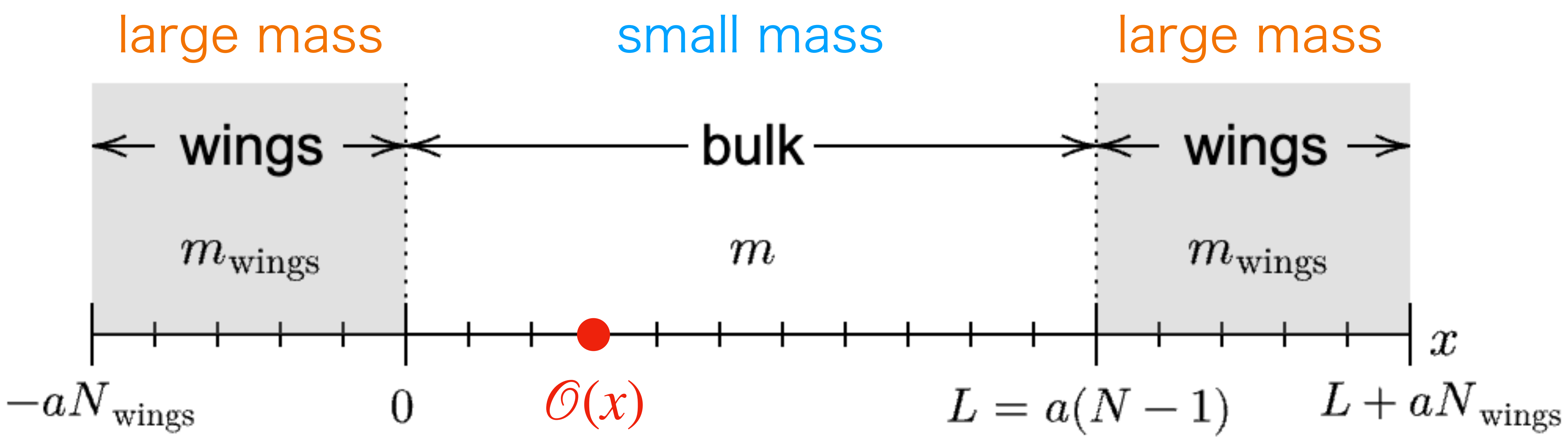
2. Improved one-point-function scheme

3. Dispersion-relation scheme

Improved one-point-function scheme

- We attach "the wings" to the lattice to control the boundary condition flexibly

  e.g.) Dirichlet b.c. $\cdots\ m_{\text{wings}} \gg m$



- Regarding the boundary as the source of mesons (~wall source), measure the bulk one-point function: $\langle \mathcal{O}(x) \rangle \sim \exp(-Mx)$

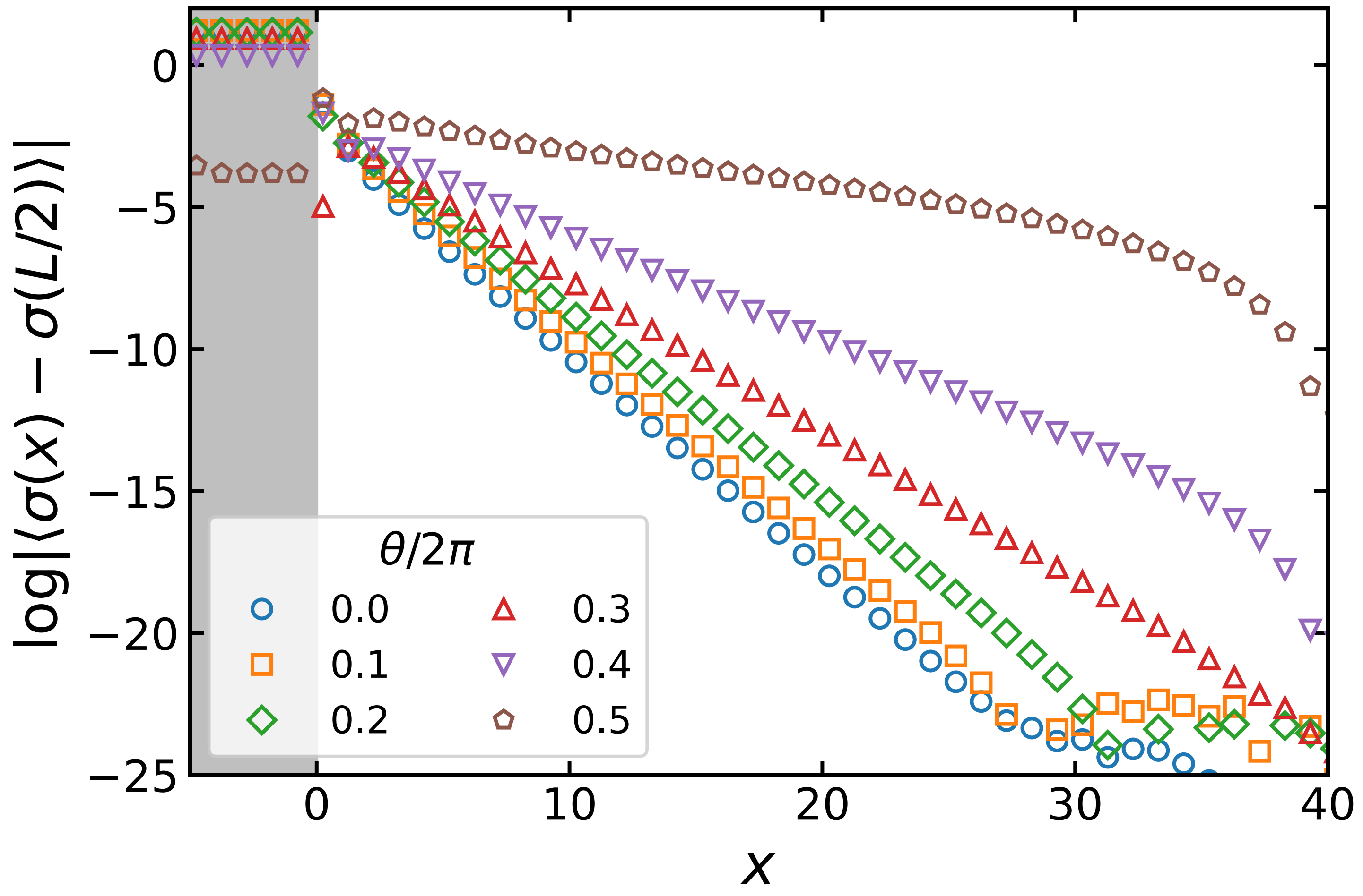
Result of sigma and eta mesons

- For the singlet mesons, we set the Dirichlet b.c. with $m_{\text{wings}} = m_0 \gg m$
- Assuming $\langle \sigma(x) \rangle \sim Ae^{-Mx} + Be^{-(M+\Delta M)x}$,

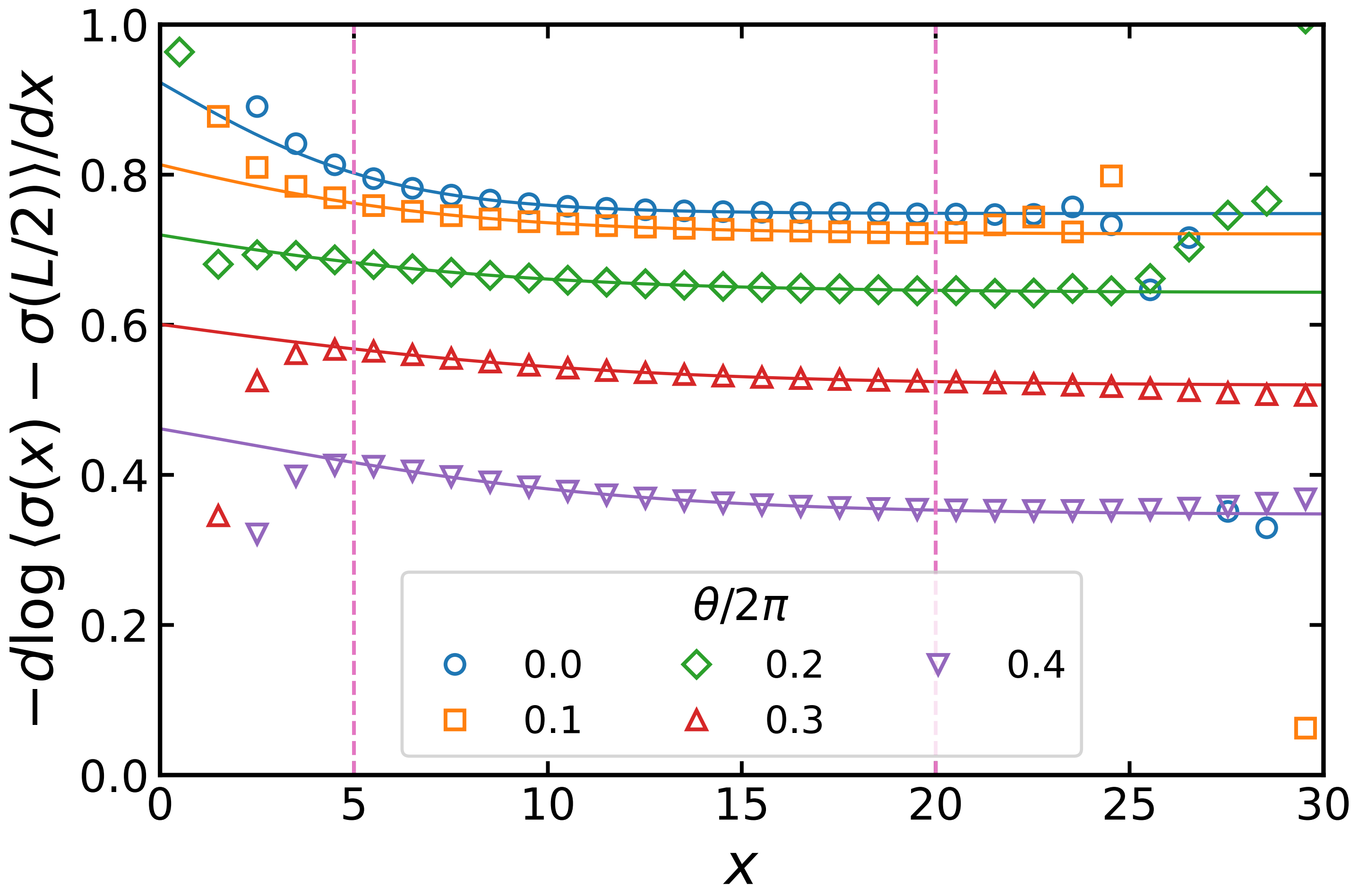
  we fit the effective mass by $M + \dfrac{\Delta M}{1 + Ce^{\Delta M x}}$ to obtain M

$$m = 0.1, \quad m_0 = 10$$

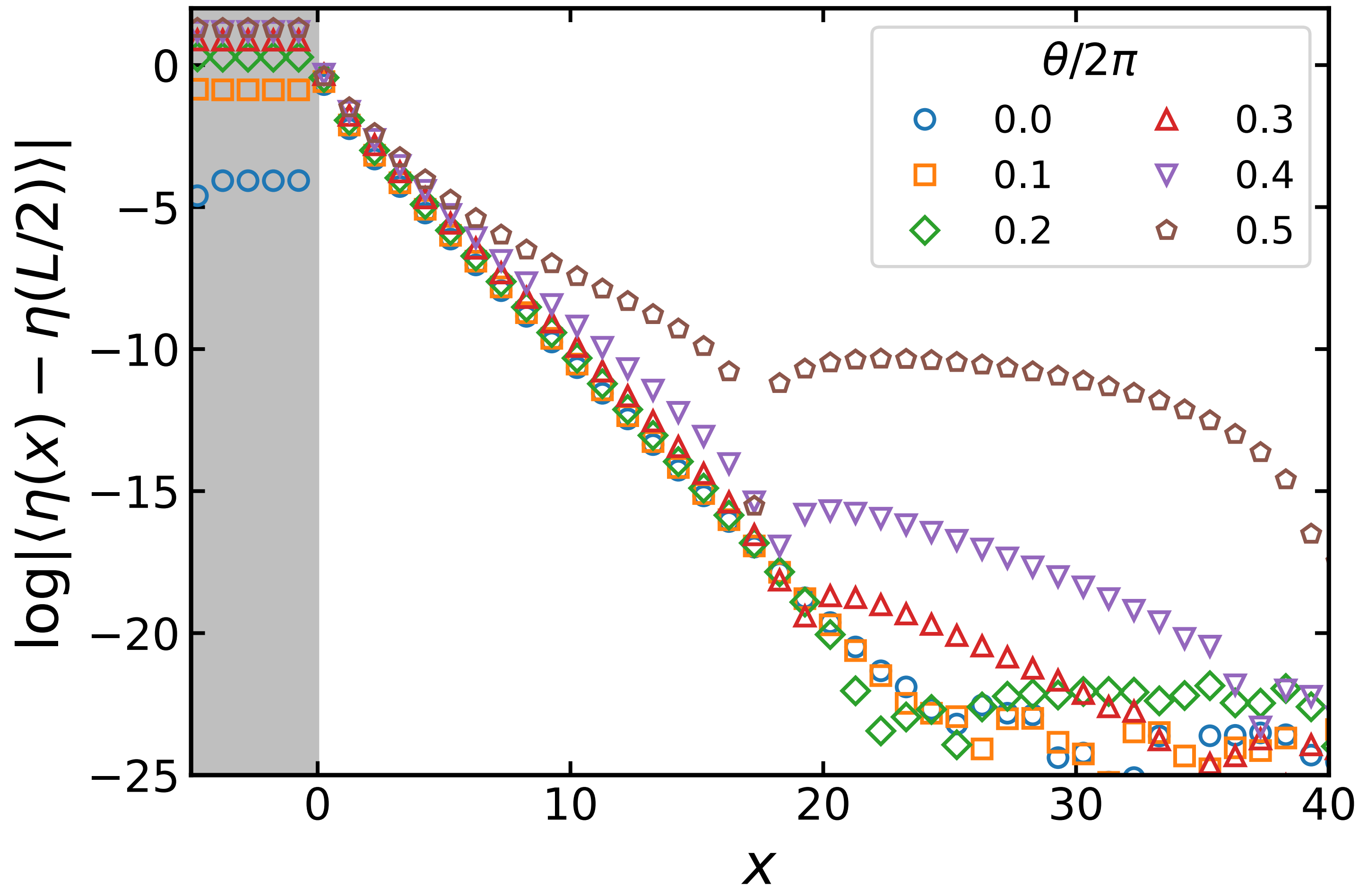
sigma meson:
stable at any θ



effective mass



eta meson:
unstable at $\theta \neq 0$



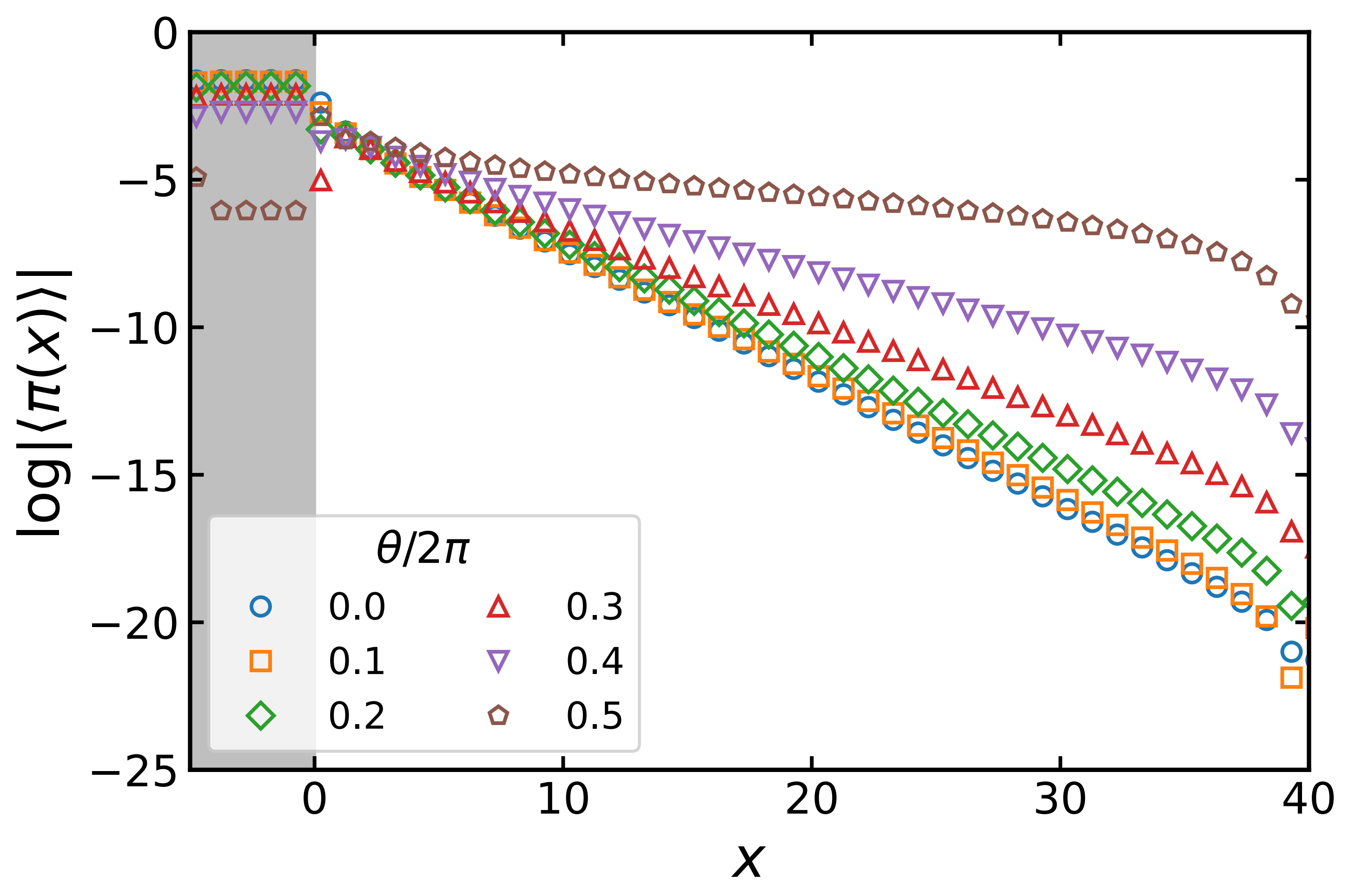
Result of pion

⚠️ $\langle \pi(x) \rangle = 0$ for the Dirichlet b.c. since such a boundary is isospin singlet

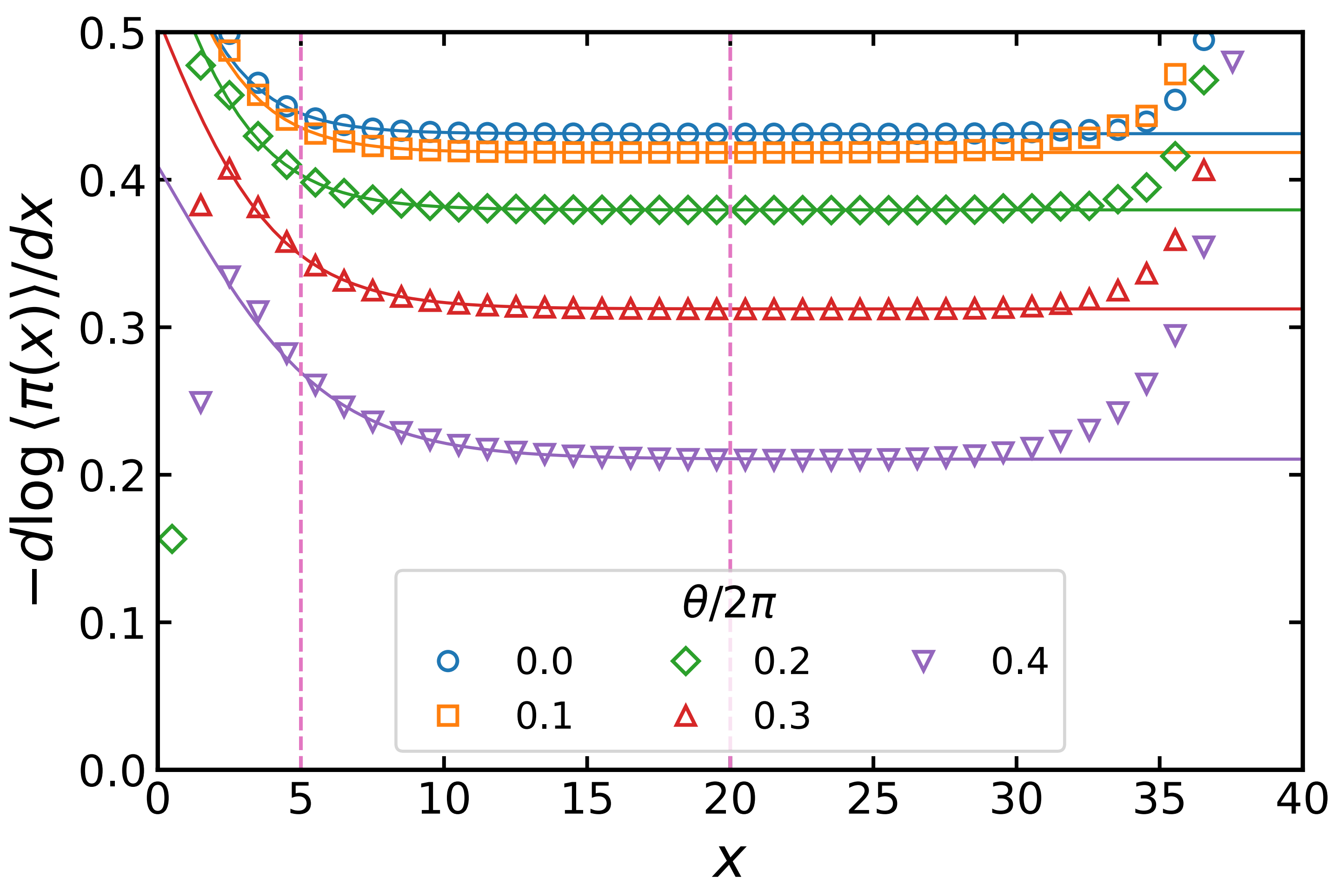
- We apply a flavor-asymmetric twist $m_{\text{wings}} = m_0 \exp(\pm i\Delta\gamma^5)$ in the wings to induce the isospin-breaking effect

$m = 0.1, \quad m_0 = 10$

pion

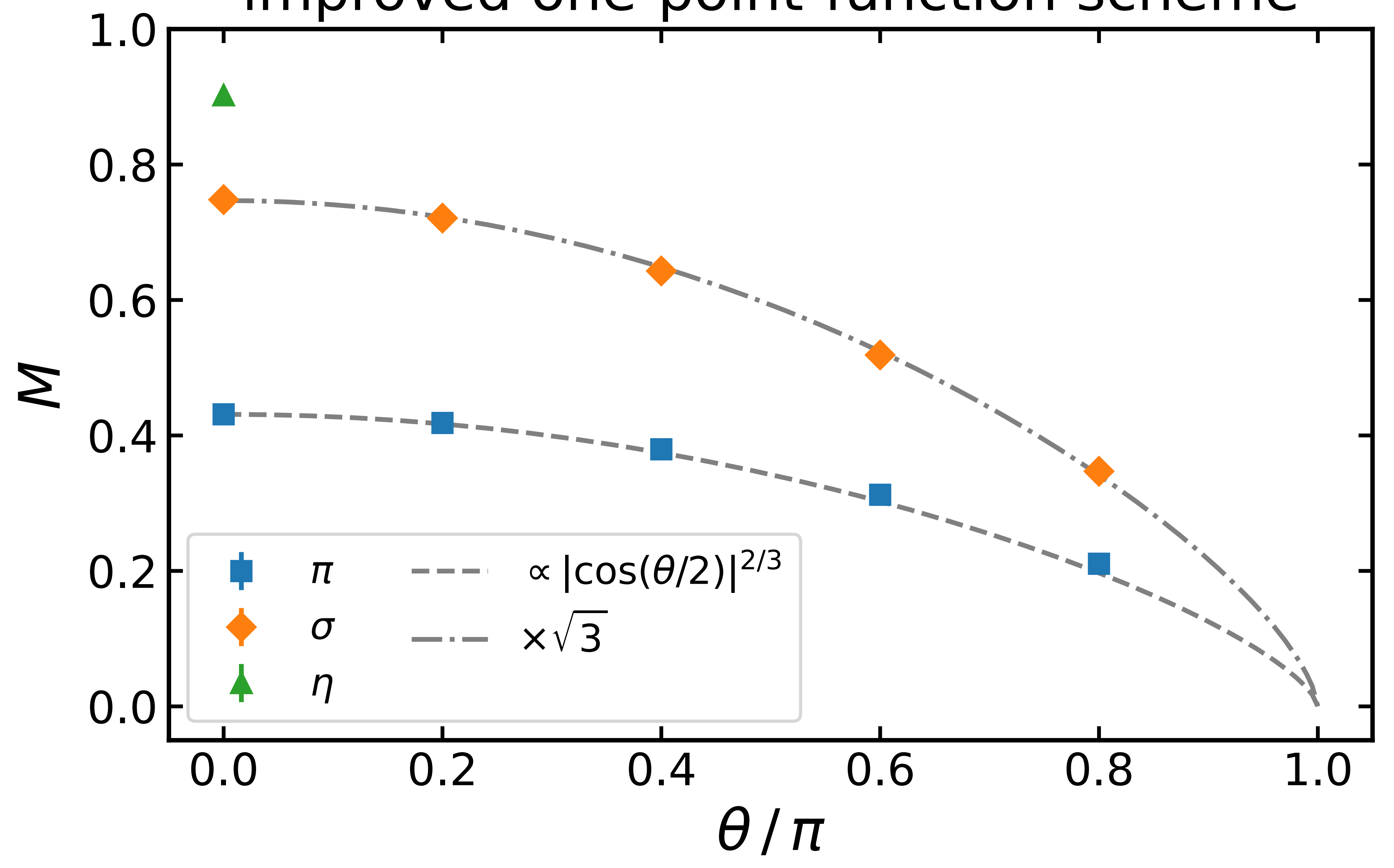


effective mass



summary

improved one-point-function scheme

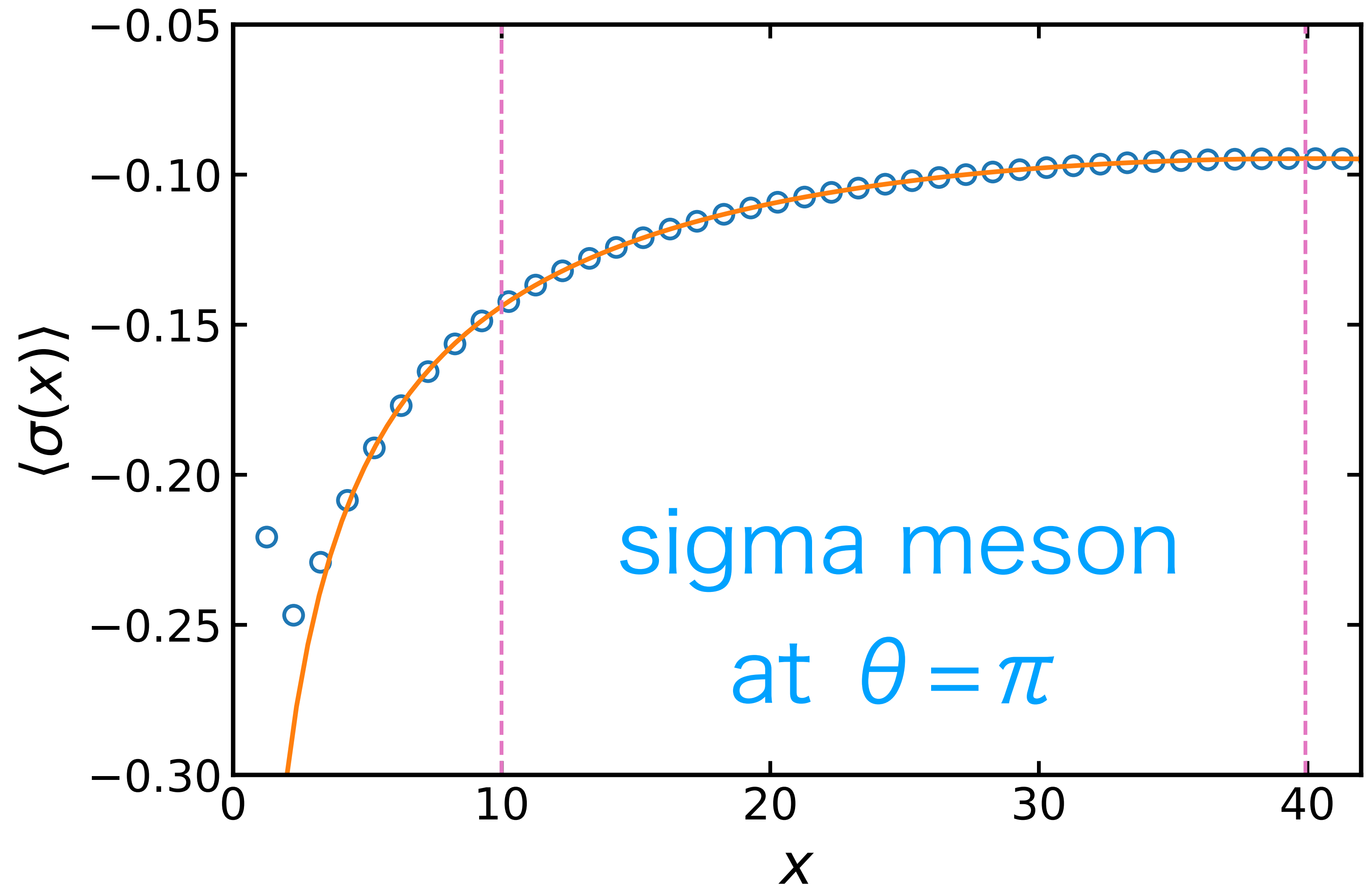


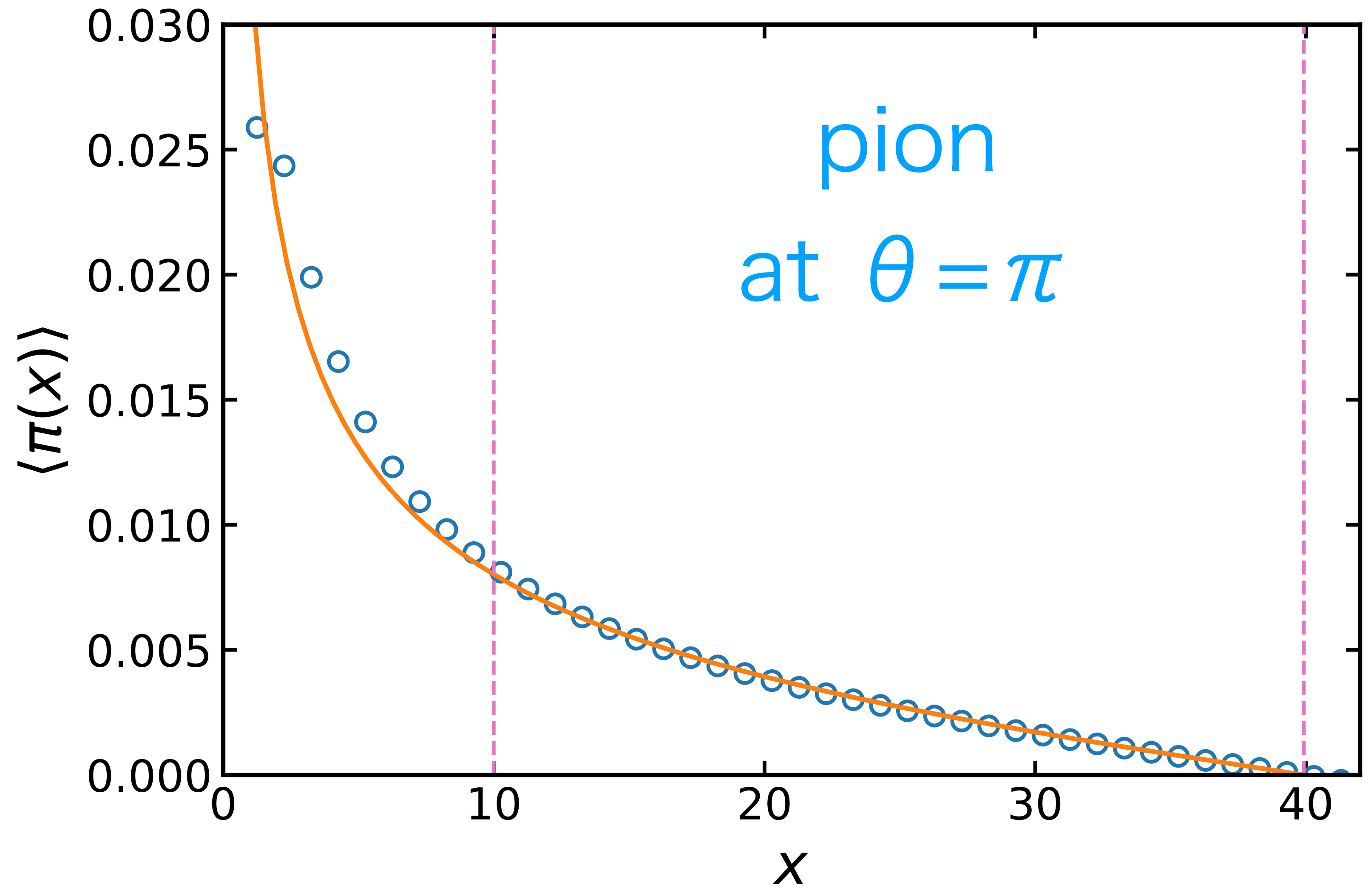
One-point functions at $\theta = \pi$

- the model is nearly conformal at $\theta = \pi$
  —> one-point functions are no longer exponential type

- We compare them with the calculation in the WZW model

$$\langle \sigma(x) \rangle \propto \frac{1}{\sqrt{\sin(\pi x/L)}} \text{ (Dirichlet b.c.)}, \quad \langle \pi(x) \rangle \propto \frac{\sin[\Delta(1-2x/L)]}{\sqrt{\sin(\pi x/L)}} \text{ (isospin-breaking b.c.)}$$





Simulation results

1. Operator mixing
2. Improved one-point-function scheme
3. **Dispersion-relation scheme**

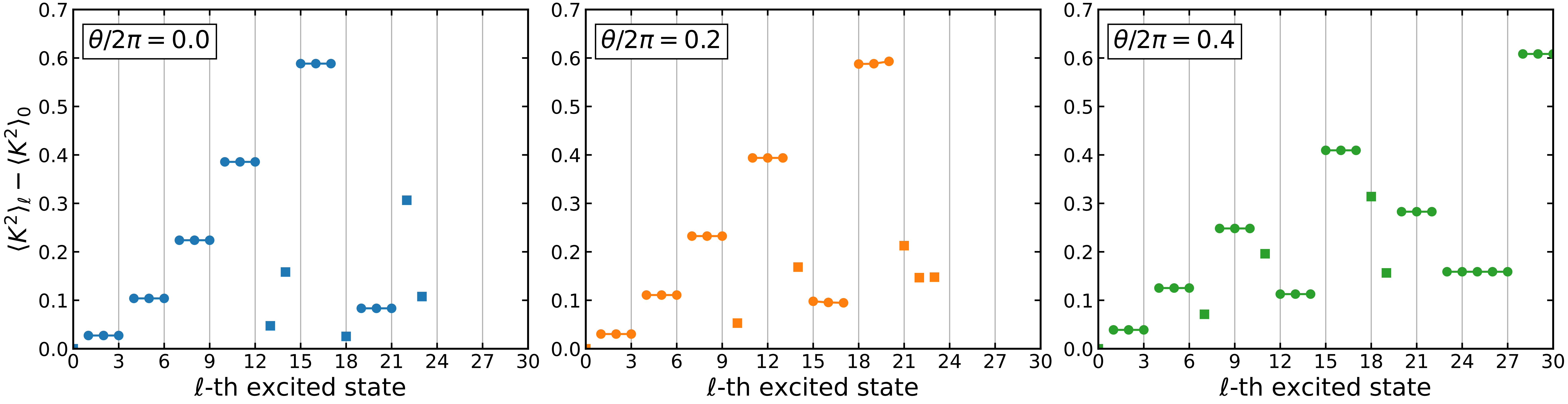
Dispersion-relation scheme

- measure the energy gap $\Delta E_\ell = E_\ell - E_0$ and momentum square $\Delta K_\ell^2 = \langle K^2 \rangle_\ell - \langle K^2 \rangle_0$
- identify the meson states by the isospin $(\mathbf{J}^2, J_z)$
- singlet projection to obtain σ efficiently: $H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| + W_J \mathbf{J}^2$

momentum^2

triplets $\rightarrow \pi$

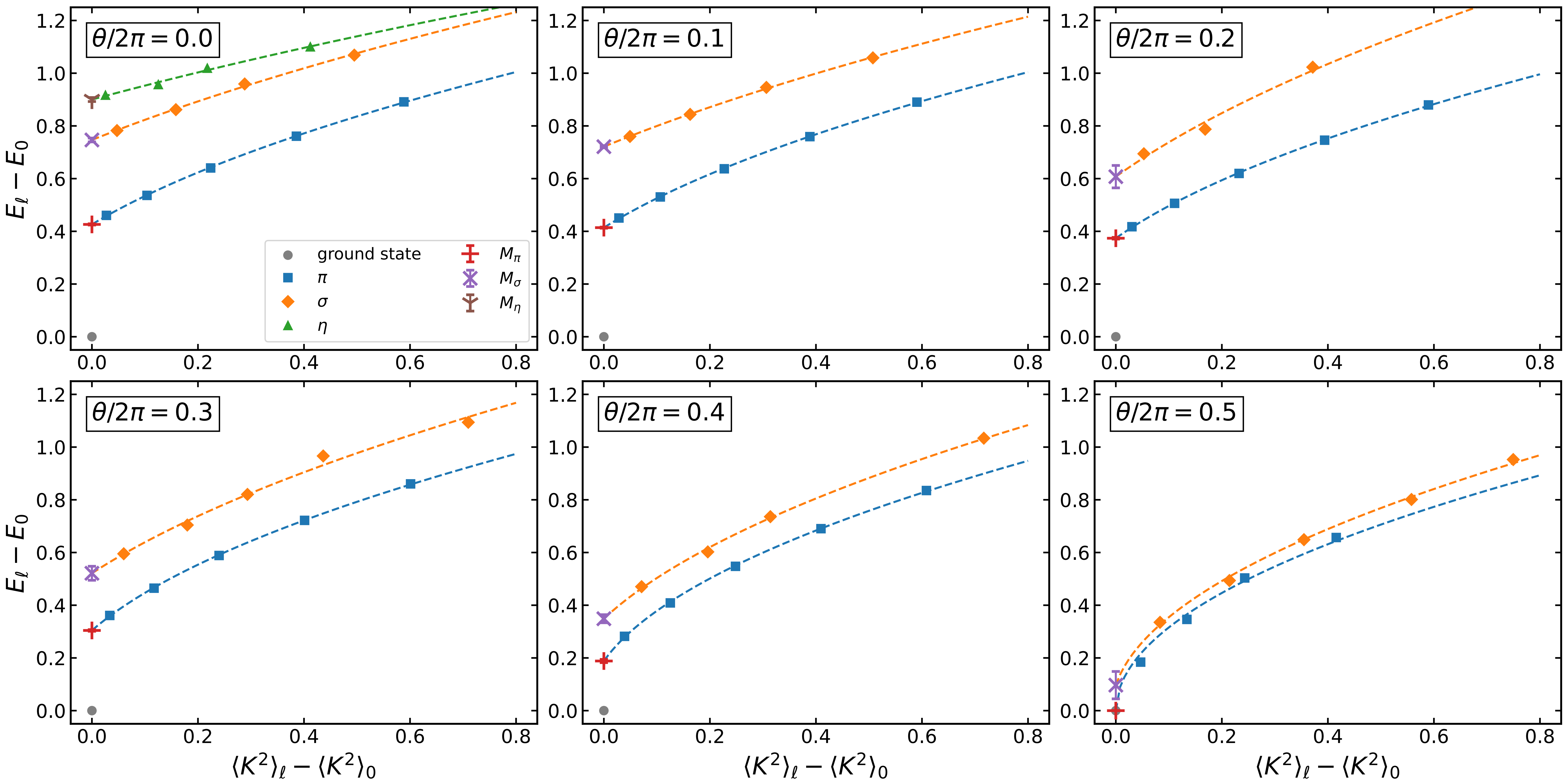
singlets $\rightarrow \sigma$



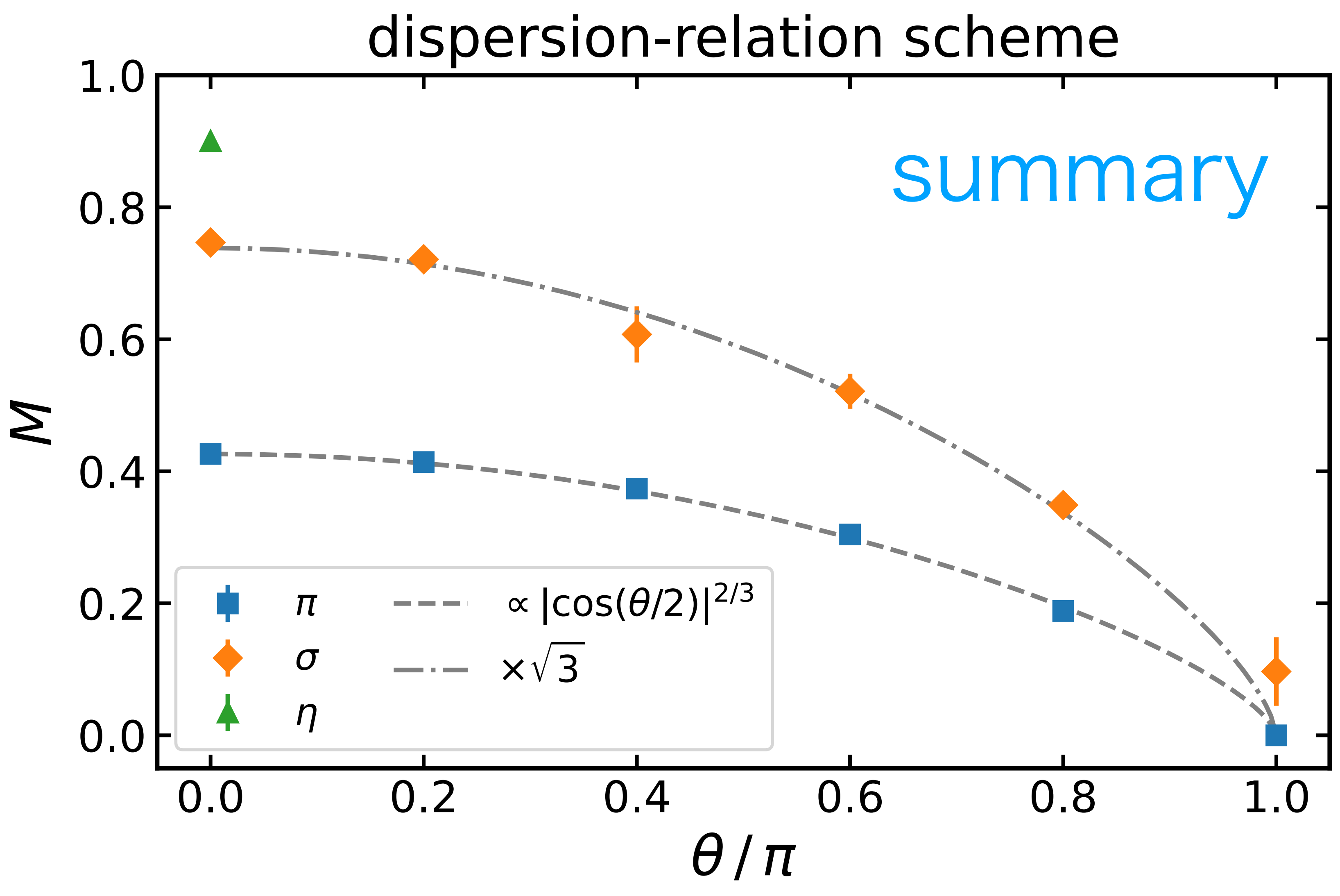
Result of dispersion relation

- plot ΔE_ℓ against ΔK_ℓ^2 and fit the data by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$ for each meson

energy vs momentum²



Around $\theta/2\pi = 0.2$, σ is contaminated by a remnant of η due to the mixing

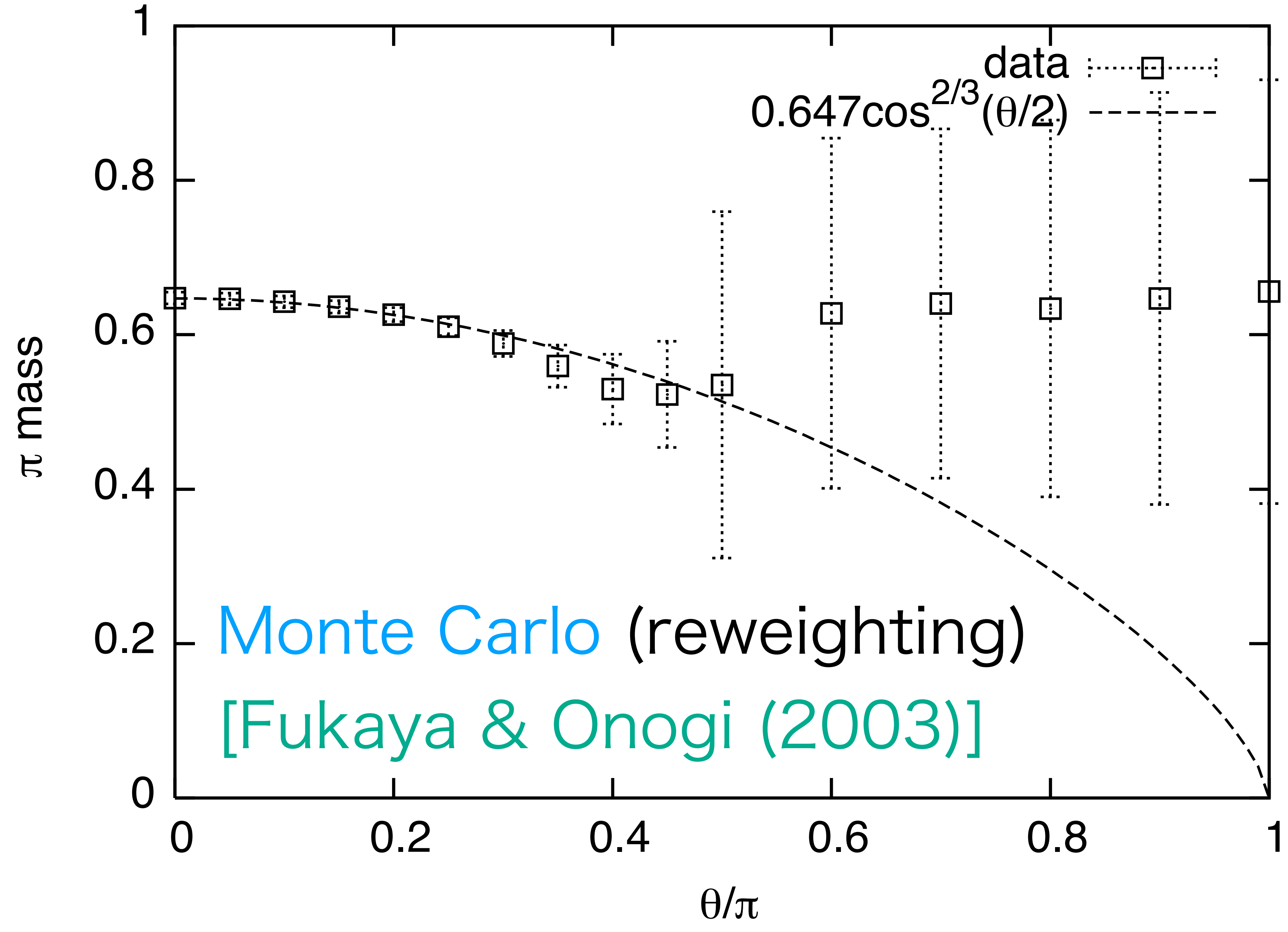


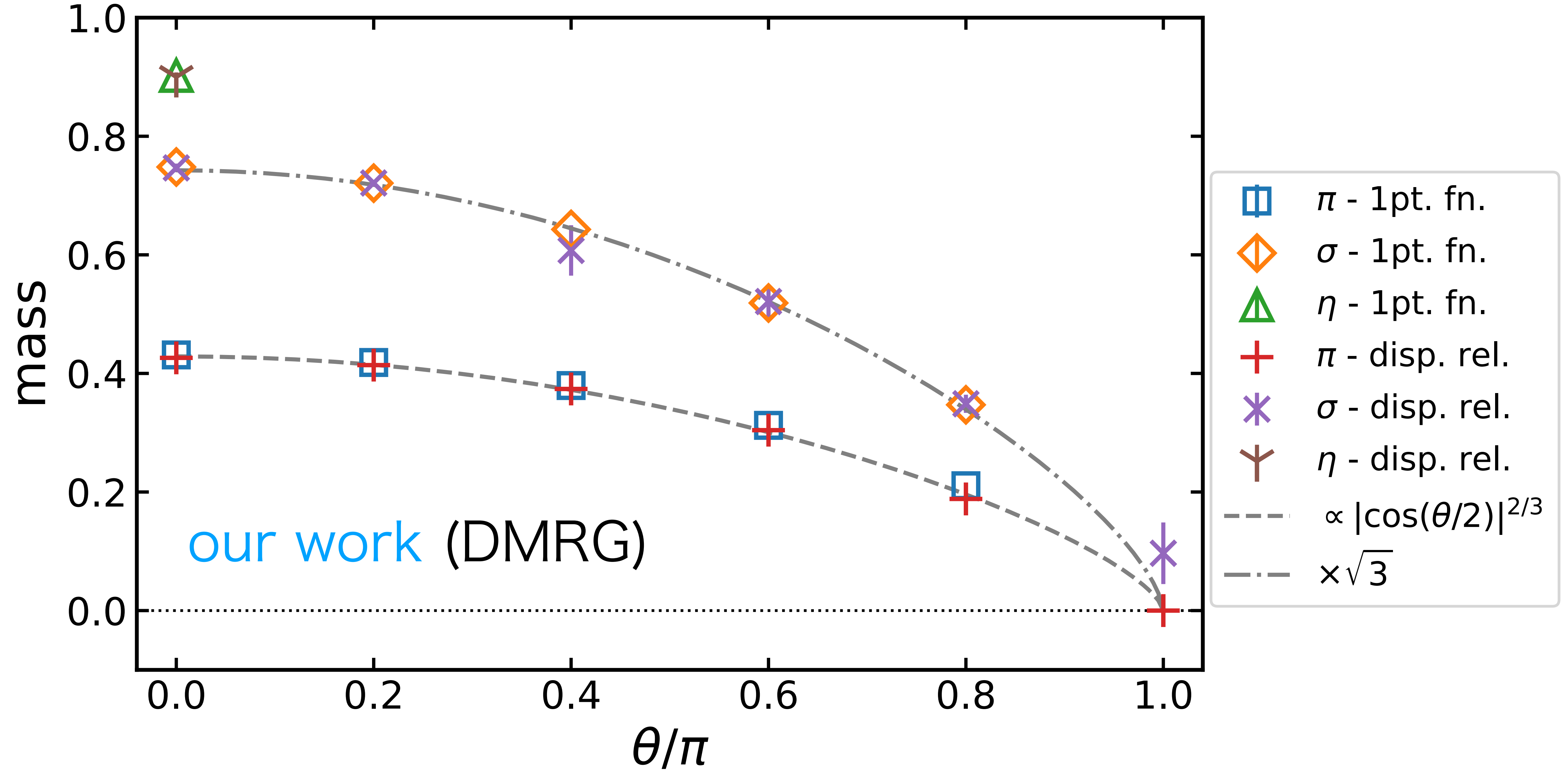
Summary

- The two schemes give consistent results and look promising
- consistent with predictions by the bosonization [Coleman (1976)] [Dashen et al. (1975)]

$$M_\pi(\theta) \propto |\cos(\theta/2)|^{2/3} \qquad M_\sigma(\theta)/M_\pi(\theta) = \sqrt{3}$$

The sign problem is circumvented!





Future prospect

- Extension to 2+1 dimensions, where the gauge field is dynamical
- Application to the model with chemical potential:
  How the spectrum changes in the high-density region?
- Analyses using the wave functions of the excited states:
  scattering problem, entanglement property, etc.

Thank you for listening.

Discussion

(1)correlation-function scheme

👍 generic method applicable to any case / off-diagonal elements

😥 sensitive to the bond dimension of MPS —> 😊 quantum computation?

(2)one-point-function scheme

👍 NOT sensitive to the bond dimension / easy to compute

😥 only the lowest state of the same quantum number as the boundary

(3)dispersion-relation scheme

👍 obtain various states heuristically / directly see wave functions

😥 how to generate excited states efficiently?

Hamiltonian formalism

- Hamiltonian is written only by fermionic operators

$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-2} \left[\sum_{f=1}^{N_f} \sum_{k=0}^{n} \chi_{f,k}^{\dagger} \chi_{f,k} + \frac{N_f}{2} \left(\frac{(-1)^n - 1}{2} - n \right) + \frac{\theta}{2\pi} \right]^2 + \sum_{f=1}^{N_f} \left[\frac{-i}{2a} \sum_{n=0}^{N-2} \left(\chi_{f,n}^{\dagger} \chi_{f,n+1} - \chi_{f,n+1}^{\dagger} \chi_{f,n} \right) + m_{\mathrm{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_{f,n}^{\dagger} \chi_{f,n} \right]$$

- Jordan-Wigner transformation: fermion operator —> spin operator

$$\chi_{1,n} = \sigma_{1,n}^{-} \prod_{j=0}^{n-1} (-\sigma_{2,j}^{z} \sigma_{1,j}^{z}) \qquad \chi_{2,n} = \sigma_{2,n}^{-} (-i\sigma_{1,n}^{z}) \prod_{j=0}^{n-1} (-\sigma_{2,j}^{z} \sigma_{1,j}^{z})$$

$$\sigma_{f,n}^{\pm} = \frac{1}{2} (\sigma_{f,n}^{x} \pm i\sigma_{f,n}^{y}) \qquad \left[\sigma_{f,n}^{a}, \sigma_{f',n'}^{b} \right] = 2i\, \delta_{ff'}\, \delta_{nn'}\, \epsilon^{abc}\, \sigma_{f,n}^{c}$$

- useful to apply the tensor network method or quantum computation

Hamiltonian formalism

- spin Hamiltonian: $H = H_{\text{gauge}} + H_{\text{kin}} + H_{\text{mass}}$

$$H_{\text{gauge}} = \frac{g^2 a}{8} \sum_{n=0}^{N-2} \left[\sum_{f=1}^{N_f} \sum_{k=0}^{n} \sigma^z_{f,k} + N_f \frac{(-1)^n + 1}{2} + \frac{\theta}{\pi} \right]^2$$

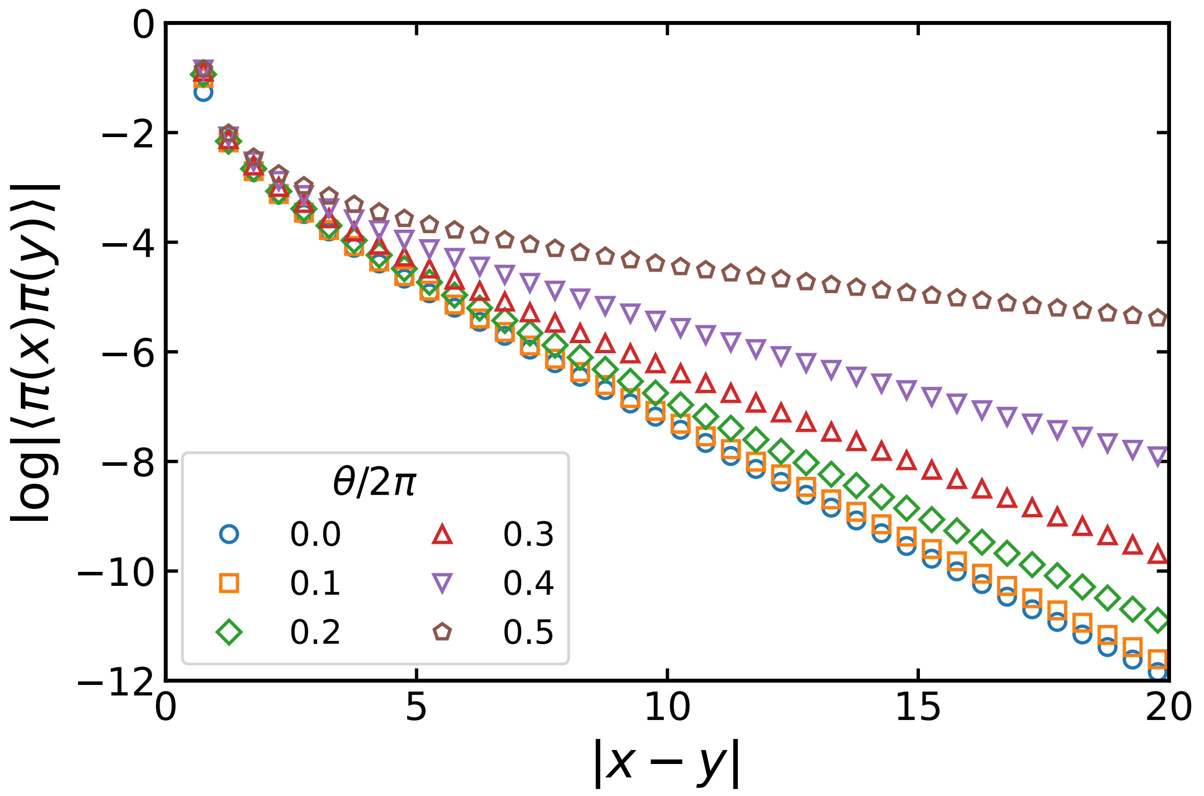
$$H_{\text{kin}} = \frac{-i}{2a} \sum_{n=0}^{N-2} \left(\sigma^+_{1,n} \sigma^z_{2,n} \sigma^-_{1,n+1} - \sigma^-_{1,n} \sigma^z_{2,n} \sigma^+_{1,n+1} + \sigma^+_{2,n} \sigma^z_{1,n+1} \sigma^-_{2,n+1} - \sigma^-_{2,n} \sigma^z_{1,n+1} \sigma^+_{2,n+1} \right)$$

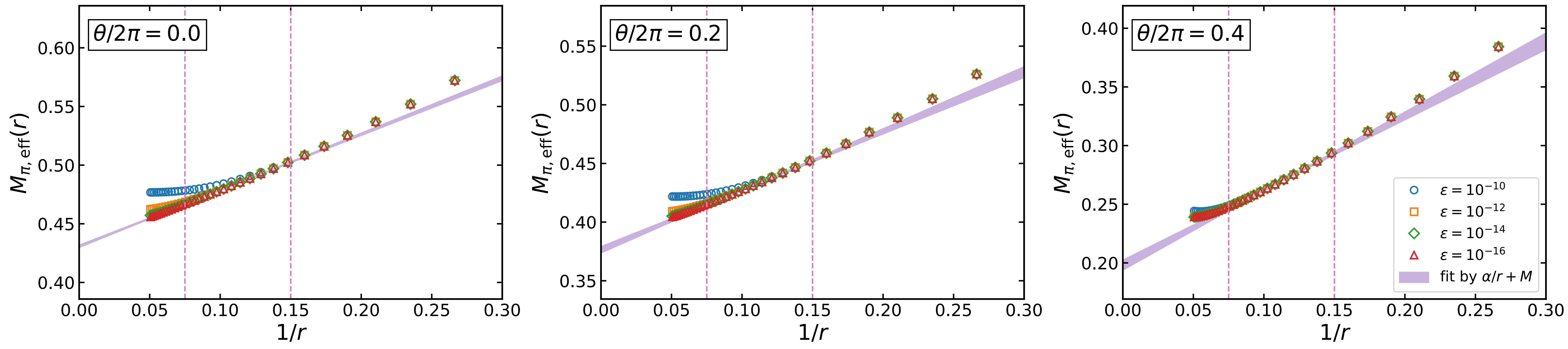
$$H_{\text{mass}} = \frac{m_{\text{lat}}}{2} \sum_{f=1}^{N_f} \sum_{n=0}^{N-1} (-1)^n \sigma^z_{f,n} + \frac{m_{\text{lat}}}{2} N_f \frac{1 - (-1)^N}{2}$$

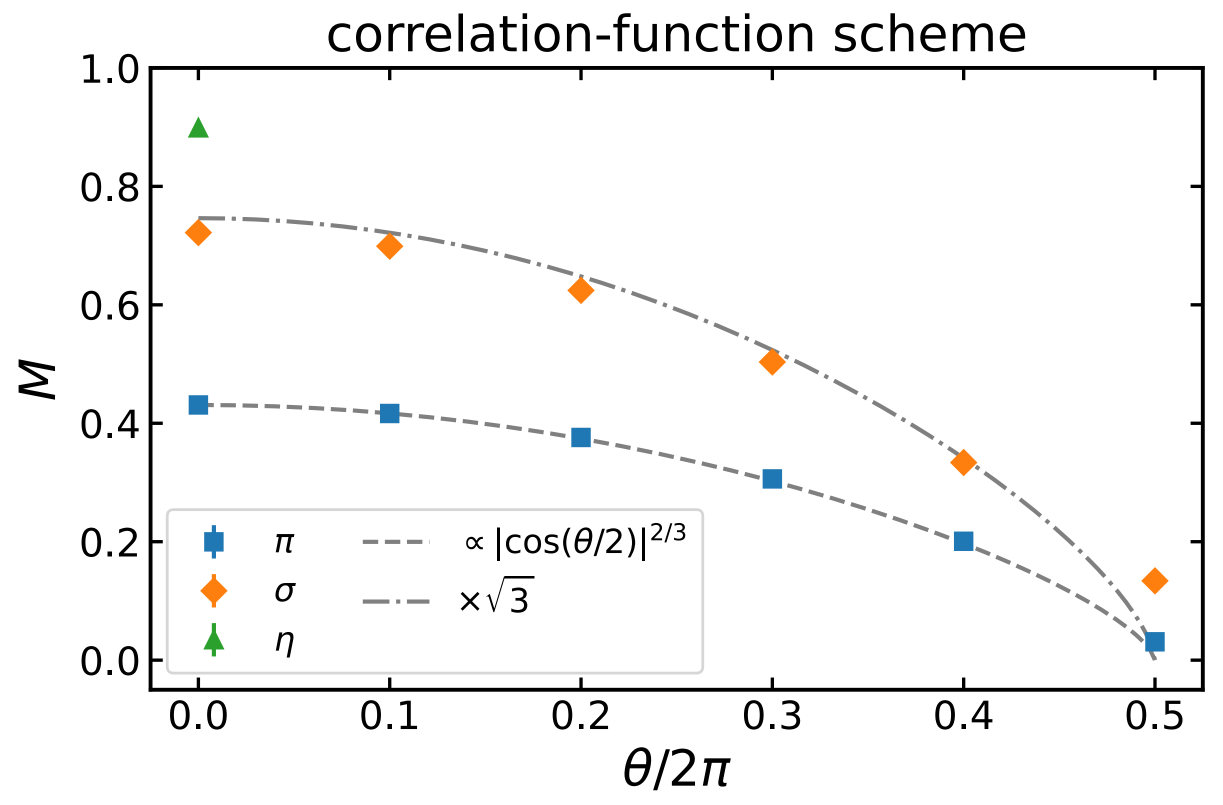
- We compute eigenstates of this Hamiltonian by the tensor network method

Correlation-function sheme

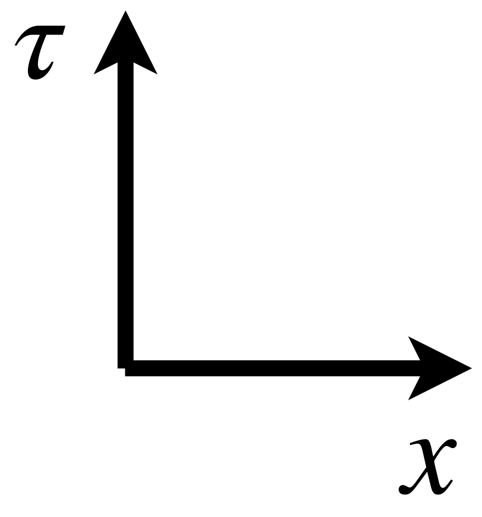
- spatial 2-point correlation function: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle \sim \frac{1}{r^\alpha} e^{-Mr}$ $\quad r = |x - y|$
- effective mass: $M_{\pi,\mathrm{eff}}(r) = -\frac{d}{dr} \log C_\pi(r) \sim \frac{\alpha}{r} + M$
- 1/r behavior is observed only when the bond dim. is large
- mass is given by $r \to \infty$ extrapolation



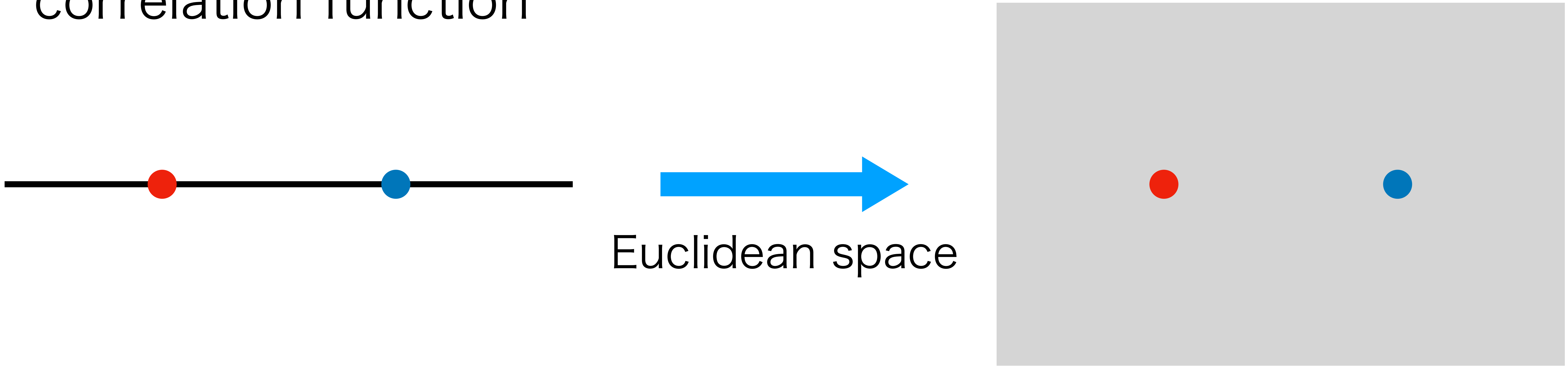




Momentum projection

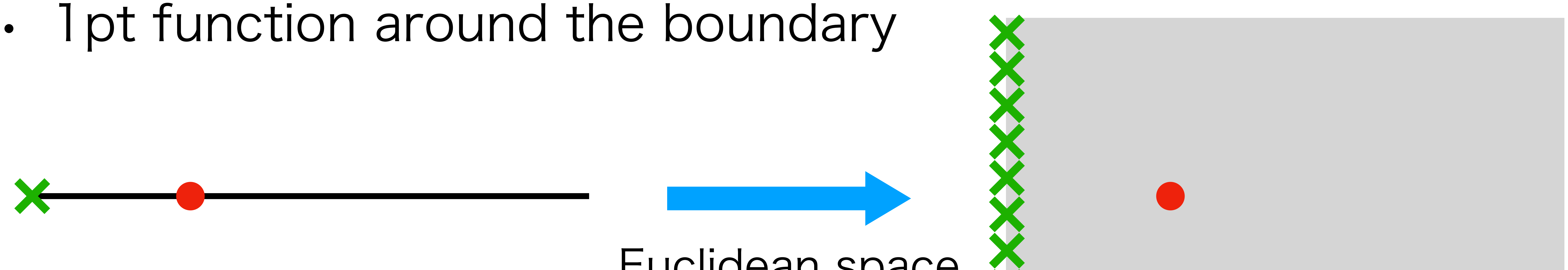


- correlation function



$$\langle 0 | \mathcal{O}(x,\tau)\mathcal{O}(y,\tau) | 0 \rangle$$

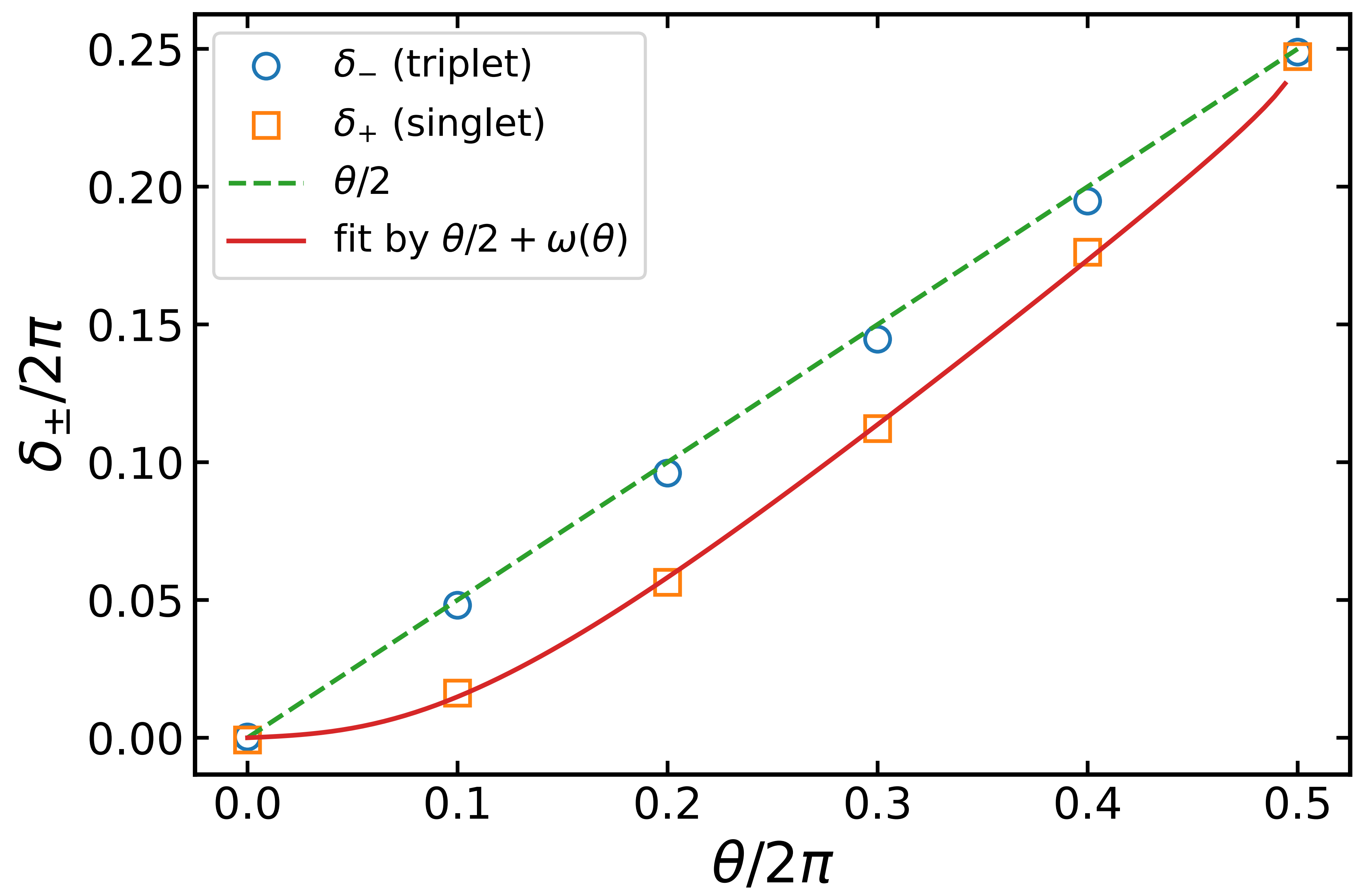
- 1pt function around the boundary



$$\langle \mathrm{bdry} | \mathcal{O}(x,\tau) | 0 \rangle \sim \int d\tau \langle 0 | \mathcal{O}(0,\tau)\mathcal{O}(x,\tau) | 0 \rangle$$

Mixing angle

singlet sector: $\delta_+ \approx \theta/2 + \omega(\theta)$



- In the bosonized model, $\omega(\theta)$ is given by the argument of R which diagonalizes the mass matrix

$$\mathcal{M} \propto \begin{pmatrix} 1 & A\sin(\theta/2)\,|\cos(\theta/2)\,|^{1/3} \\ A\sin(\theta/2)\,|\cos(\theta/2)\,|^{1/3} & B\,|\cos(\theta/2)\,|^{4/3} \end{pmatrix} = R(\omega(\theta))^{\mathrm{T}}\Lambda\, R(\omega(\theta))$$

$$R(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

fitting result

$A = -0.23(2),\ B = 0.76(4)$

Isospin quantum numbers

- isospin operators: conserved charge of SU(2) isospin symmetry

$$J_a = \frac{1}{2}\int dx \sum_{f,f'} \psi_f^\dagger (\sigma^a)_{f,f'} \psi_{f'} \qquad a \in \{x, y, z\}$$

- lattice version

$$J_z = \frac{1}{2}\sum_{n=0}^{N-1}\left(\chi_{1,n}^\dagger \chi_{1,n} - \chi_{2,n}^\dagger \chi_{2,n}\right), \quad J_+ = \sum_{n=0}^{N-1} \chi_{1,n}^\dagger \chi_{2,n} = (J_-)^\dagger, \quad \mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_+J_-) + J_z^2$$

- They exactly commute with the lattice Hamiltonian.

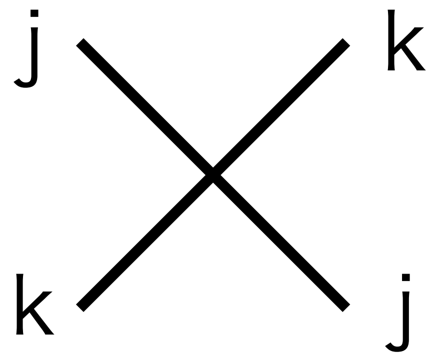
$$[H, J_z] = [H, J_\pm] = [H, \mathbf{J}^2] = 0$$

Charge conjugation

- charge conjugation: exchange particles/anti-particles
  = exchange even/odd sites and flip each spin
  = 1-site translation and σ^x operator

$$C := \prod_{f=1}^{N_f} \left(\prod_{n=0}^{N-1} \sigma^x_{f,n} \right) \left(\prod_{n=0}^{N-2} (\mathrm{SWAP})_{f;N-2-n,N-1-n} \right)$$

$$(\mathrm{SWAP})_{f;j,k} = \frac{1}{2} \left(\mathbf{1}_{f,j} \mathbf{1}_{f,k} + \sum_a \sigma^a_{f,j} \sigma^a_{f,k} \right) \longrightarrow$$



$[H, C] \neq 0$ due to the boundary

- G-parity: $G = C \exp(i\pi J_y)$ acting on the whole multiplet

1-site translation