Quantum Computational Resources for lattice QCD in the strong-coupling limit

Based on arXiv:2406.18721, in collaboration with **Lucas Katschke**, Owe Philipsen and Wolfgang Unger.

Michael Fromm, Quantum Computing and Quantum Information, Lattice 2024, Liverpool

Lattice QCD in the strong-coupling limit …
Why bother ?

Limit of full Lattice Gauge QCD with staggered fermions

$$Z = \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \mathcal{D}U \, \mathrm{e}^{S_F + \beta S_G}$$

with **remnant staggered "chiral" symmetry** $\chi \to e^{i\epsilon(x)\theta}\chi, \bar{\chi} \to \bar{\chi} e^{i\epsilon(x)\theta}, \epsilon(x) = (-1)^{\sum_\mu x_\mu}$ (for one massless flavor)

besides **baryon number** $U_V(1)\; \chi \to e^{i\theta_B}\chi, \bar{\chi} \to \bar{\chi} e^{-i\theta_B}$

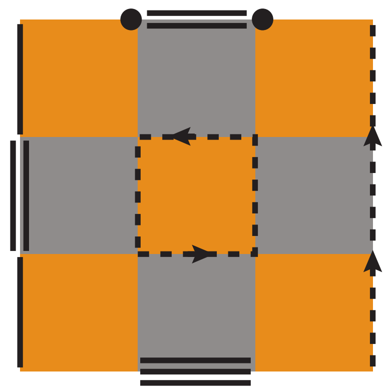
Accessible via

Analytic methods:

Hamiltonian Perturbation Theory, Effective Lagrangian (Kawamoto and Smit, 1981) -> **Spectrum**

Mean-field approximation (Miura, K. et al, 2017) -> **Phase diagram**

$$Z = \int \prod_x \left(\mathrm{d}\chi_x \mathrm{d}\bar{\chi}_x \mathrm{e}^{2am_q \bar{\chi}_x \chi_x} \prod_\mu z(x,\mu) \right)$$



Numerically: Algorithmic advantages for Euclidean Lattice Monte Carlo

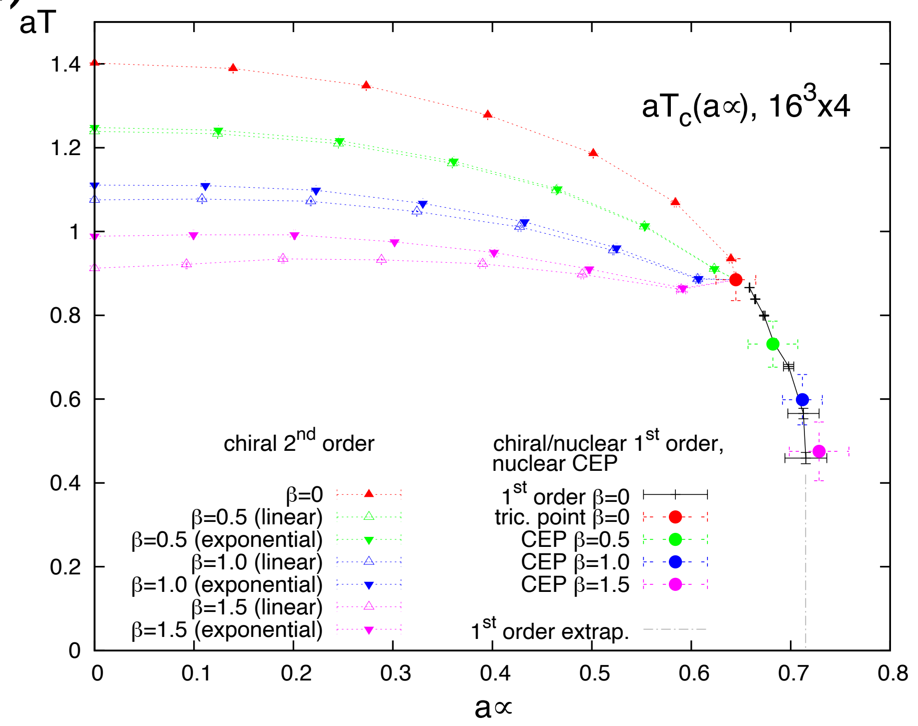
Sign problem milder (gauge dof integrated out first, then fermion integral)

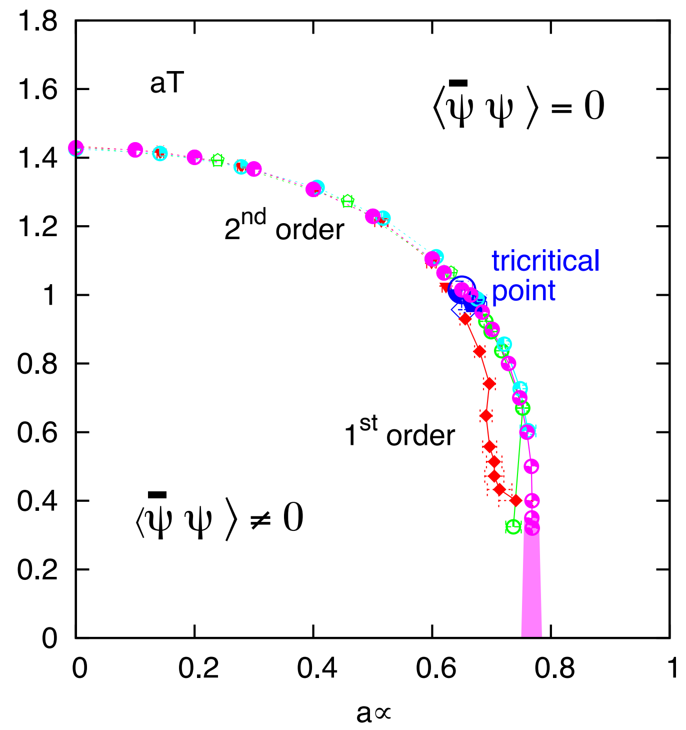
Discrete dof, different set of algos (e.g. worm algorithm, efficient)

Chiral limit of staggered fermions (with exact remnant chiral symmetry)

Continuous Euclidean Time (sign problem gone, static baryons)

Predictive also away from $\beta = 2N/g^2 = 0$ (e.g. via reweighting in β)

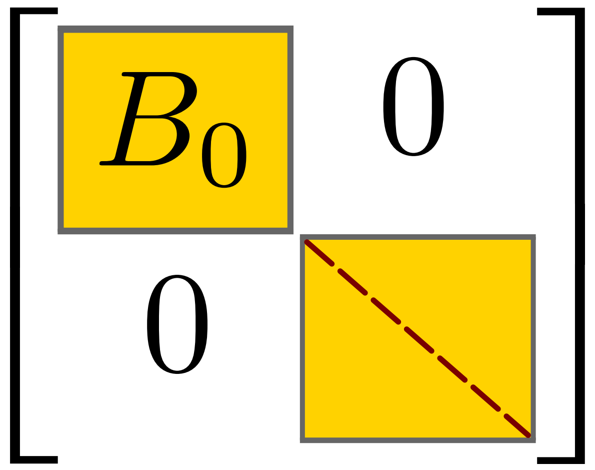




de Forcrand, Ph. et al., PRL 2014

Strong-coupling Hamiltonian from the Euclidean

Continuous Euclidean Time formulation gives rise to **Hamiltonian** (Unger et al., PoSLAT '21, '22, '23, → Wolfgang's talk)

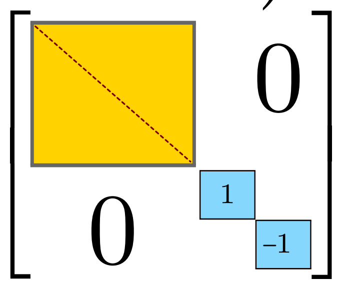


$$\mathcal{Z}_{CT}\left(aT, a\mu_B\right) = \mathrm{tr}_{\mathbb{H}}\left[e^{(-\hat{\mathcal{H}}+\hat{\mathcal{N}}a\mu_B)/aT}\right]$$

$N_f = 1$

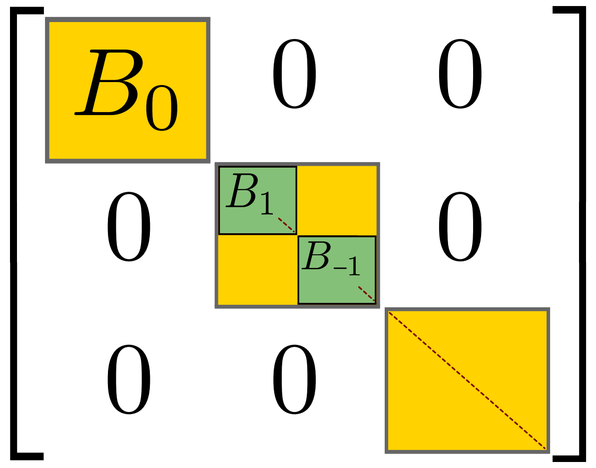
$$\hat{\mathcal{H}} = -\frac{1}{2}\sum_{\langle x,y\rangle}\left(\hat{J}_x^+\hat{J}_y^- + \hat{J}_x^-\hat{J}_y^+\right)$$ is that of $spin - \frac{N_c}{2}$ Heisenberg model, local Hilbert space has basis $|\mathfrak{h}_i\rangle \in \{0, \pi, 2\pi, 3\pi, B^+, B^-\}$

$$\hat{\mathcal{N}} = \sum_x \hat{\omega}_x$$



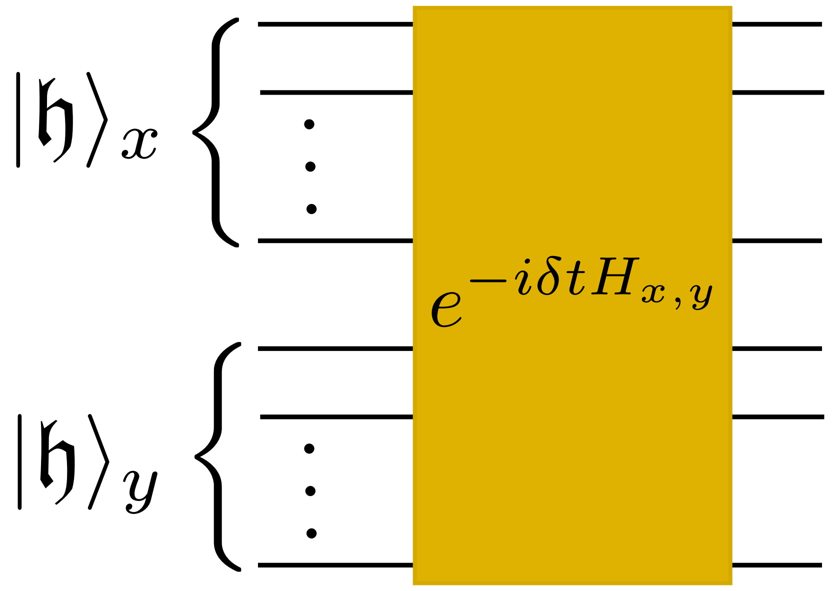
$N_f = 2$

$$\hat{\mathcal{H}} = -\frac{1}{2}\sum_{\langle x,y\rangle}\sum_{Q_i\in\{\pi^+,\pi^-,\pi_U,\pi_D\}} \hat{J}^+_{Q_i,x}\hat{J}^-_{Q_i,y} + \hat{J}^-_{Q_i,x}\hat{J}^+_{Q_i,y},$$ where $J^\pm_Q$ have a reducible representation with $d = 92$



- Things in common
  - No gauge redundancy, Grassmann constraint and Gauss' law respected implicitly
  - Discrete set of variables, Hilbert space finite-dimensional (if large), baryons static
  - Hamiltonian decomposes locally into baryonic "sectors" with baryon number $n_B \in \{-N_f, \ldots, N_f\}$, a structure **inherited** from the $J's$
  - block-diagonal structure <-> conserved quantities: e.g. local baryon number

Hamiltonian Evolution via Trotterization, Computational Basis: Qubits (I)

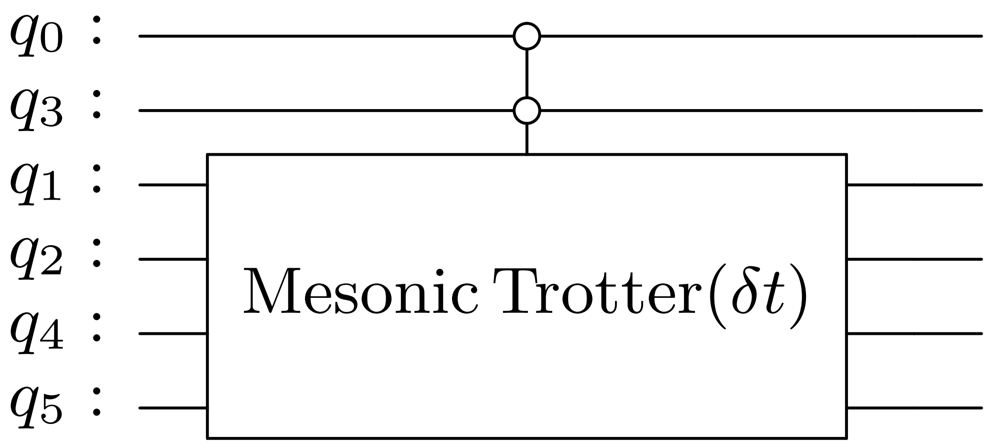


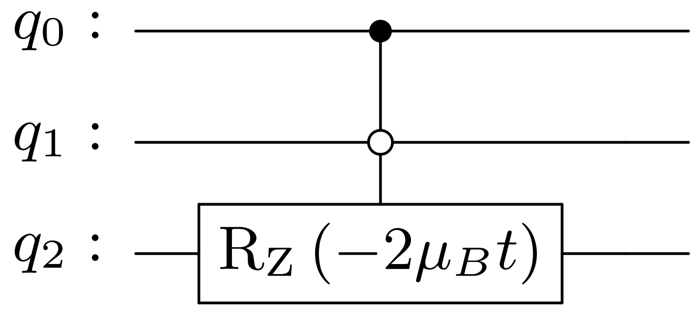
Use block-diagonal structure and reshape Hamiltonian locally

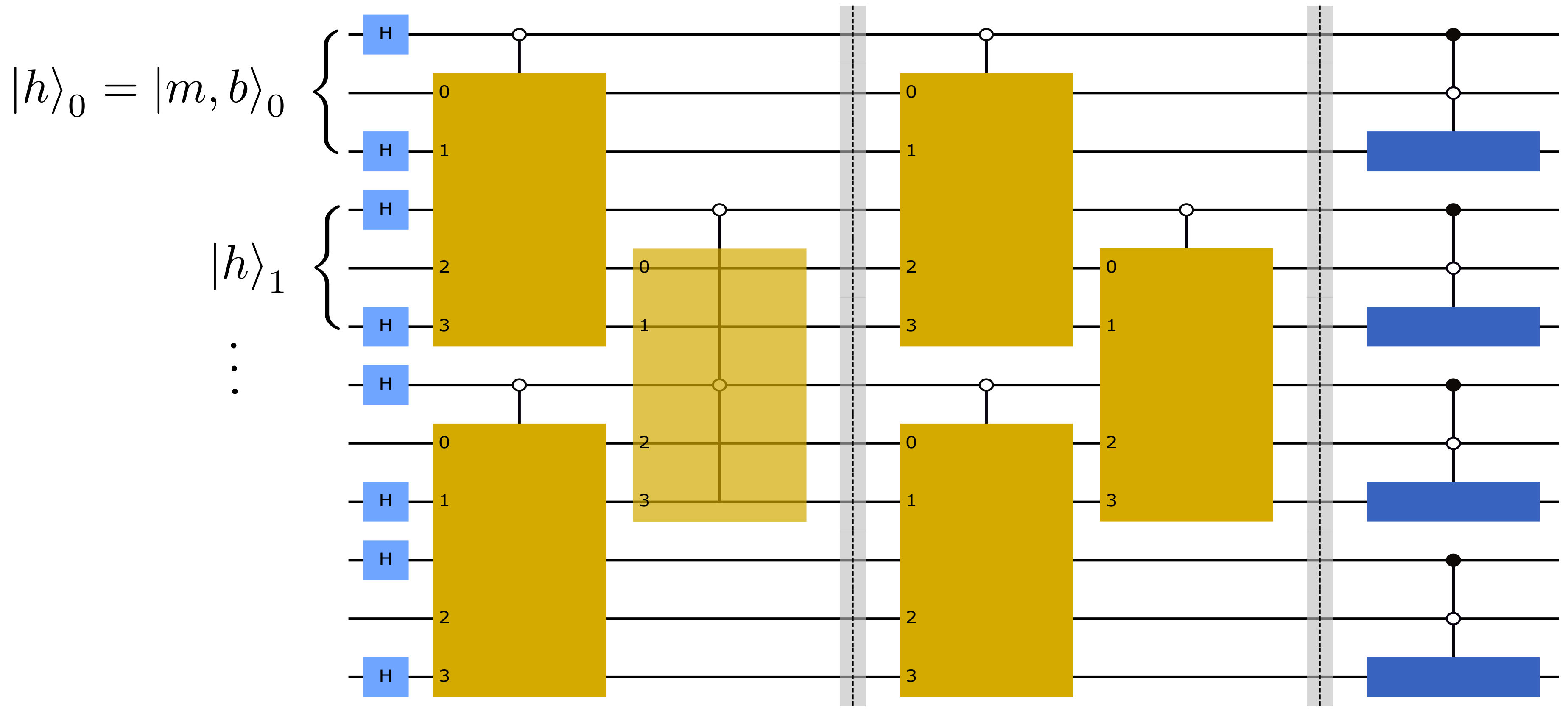
$$\hat{\mathscr{H}} = -\frac{1}{2}\sum_{\langle x,y\rangle} |0\rangle\langle 0|_x |0\rangle\langle 0|_y \left(\hat{J}^+_x\hat{J}^-_y + \hat{J}^-_x\hat{J}^+_y\right) + a\mu_B \sum_x |1\rangle\langle 1|_x \hat{\omega}_x$$

Observations:

- Can label conserved charge sectors by states of ancilla bits which act as control bit
- Static baryonic states have diagonal evolution
- Resources involved ? (→ 2308.03196)





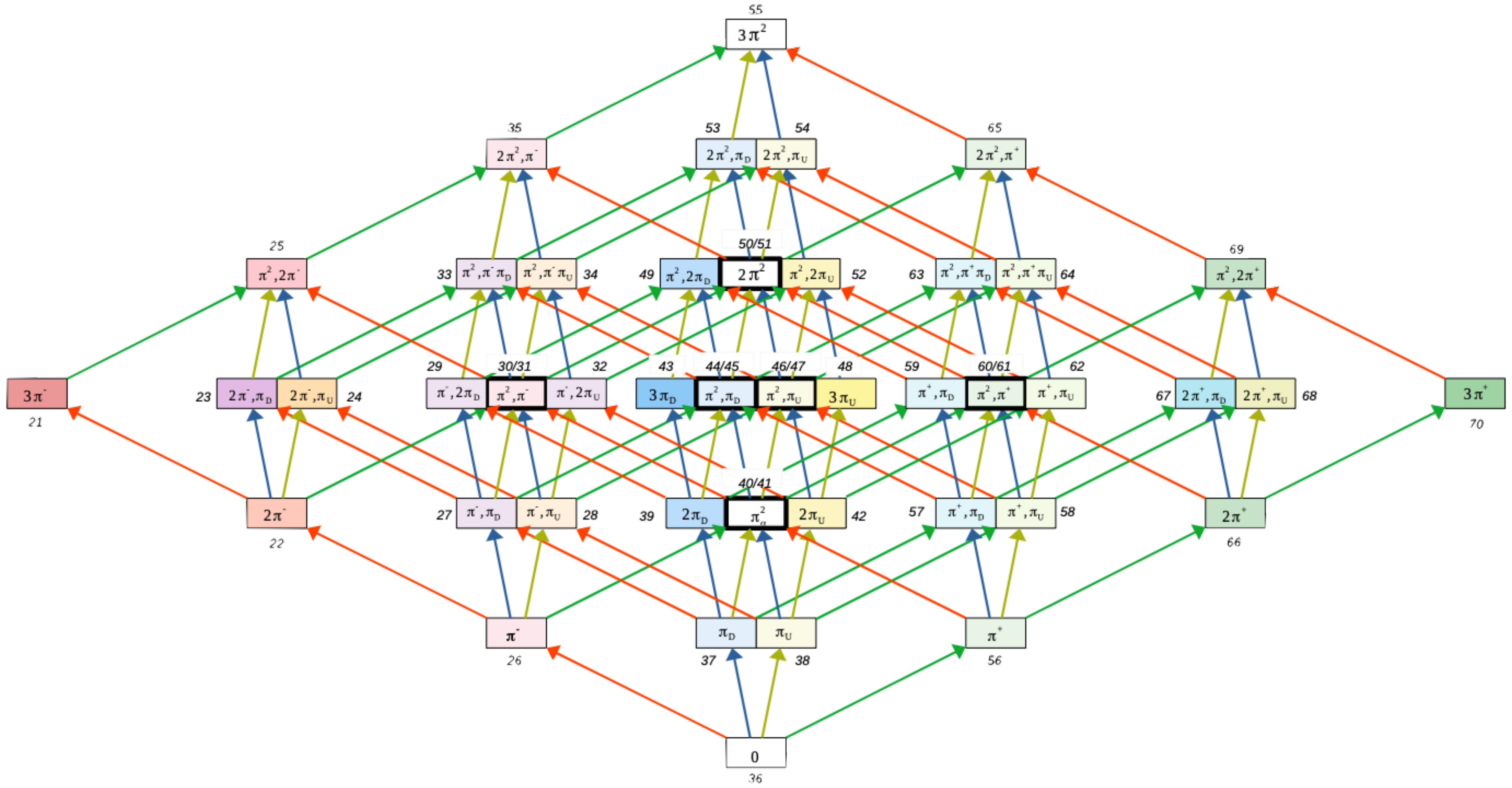


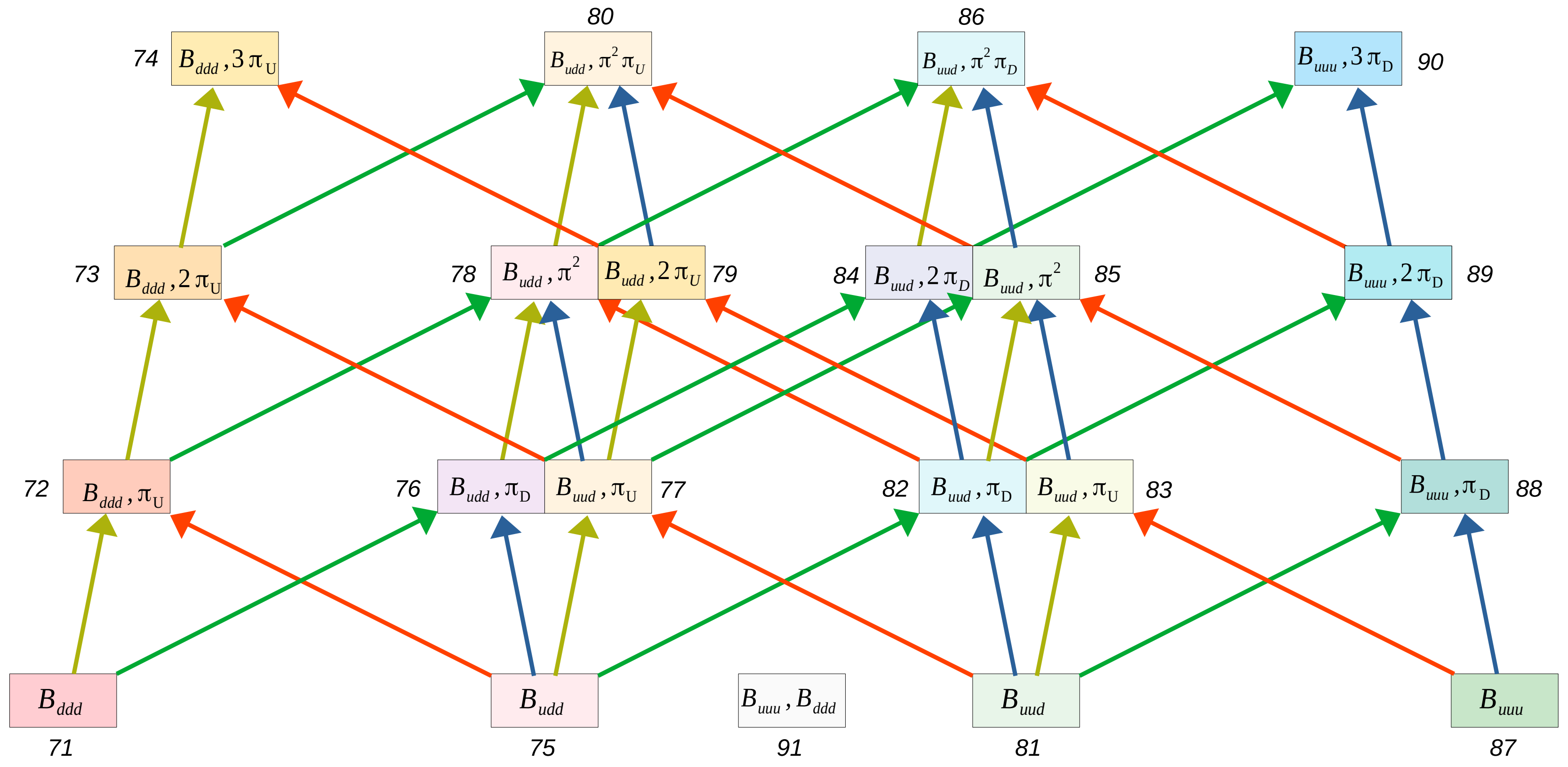
Hamiltonian Evolution via Trotterization, Computational Basis: Qubits (2)

- $N_f = 2$, now $n_{B_x} \in \{-2,\ldots,2\}$ Hamiltonian becomes

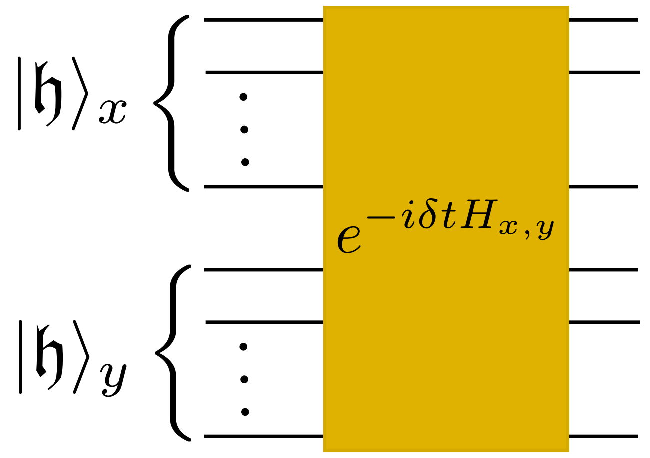
$$\hat{\mathscr{H}} = -\frac{1}{2}\sum_{\langle x,y\rangle}\sum_{i,j} |i\rangle\langle i|_x |j\rangle\langle j|_y \sum_{Q_k} \hat{J}^{(i)+}_{Q_k,x}\hat{J}^{(j)-}_{Q_k,y} + \hat{J}^{(i),-}_{Q_k,x}\hat{J}^{(j),+}_{Q_k,y}$$

- Compared to $N_f = 1$, the …
  - $J^{(i)\pm}_{Q_k,x}$ are high dimensional ($d_{max} = 50$, for $n_B = 0$)
    - Higher qubit count for storage
    - High gate depth already for single Trotter step
  - Time evolution of baryonic sectors mix $|i\rangle\langle i|_x \otimes |j\rangle\langle j|_y$





Hamiltonian Evolution via Trotterization, Computational Basis: Qubits (2)



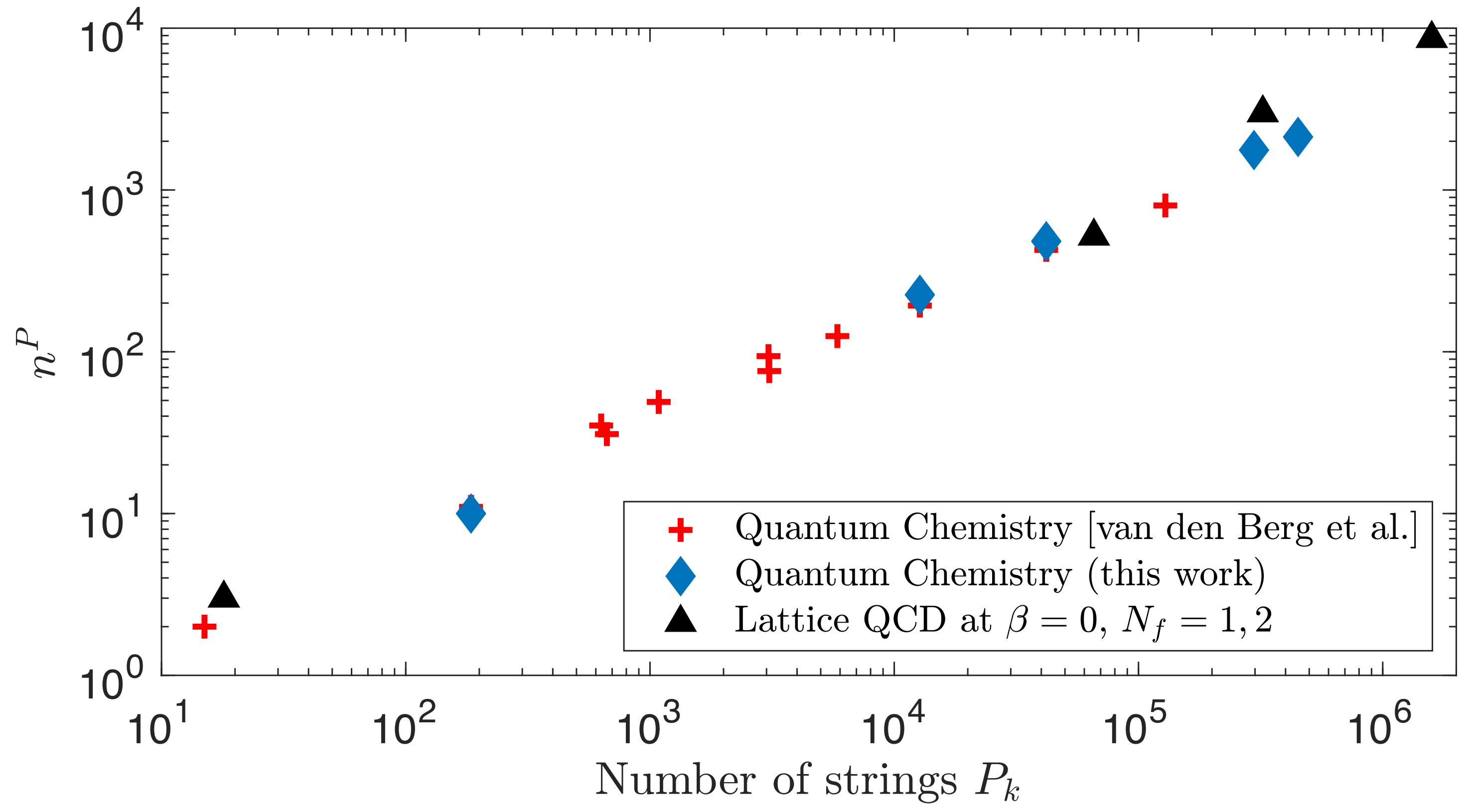
$$= \prod_{ij} e^{i\frac{\delta t}{2}|i\rangle\langle i|_x |j\rangle\langle j|_y \sum_{Q_k} \hat{J}^{(i)+}_{Q_k,x}\hat{J}^{(j)-}_{Q_k,y} + \hat{J}^{(i),-}_{Q_k,x}\hat{J}^{(j),+}_{Q_k,y}}$$

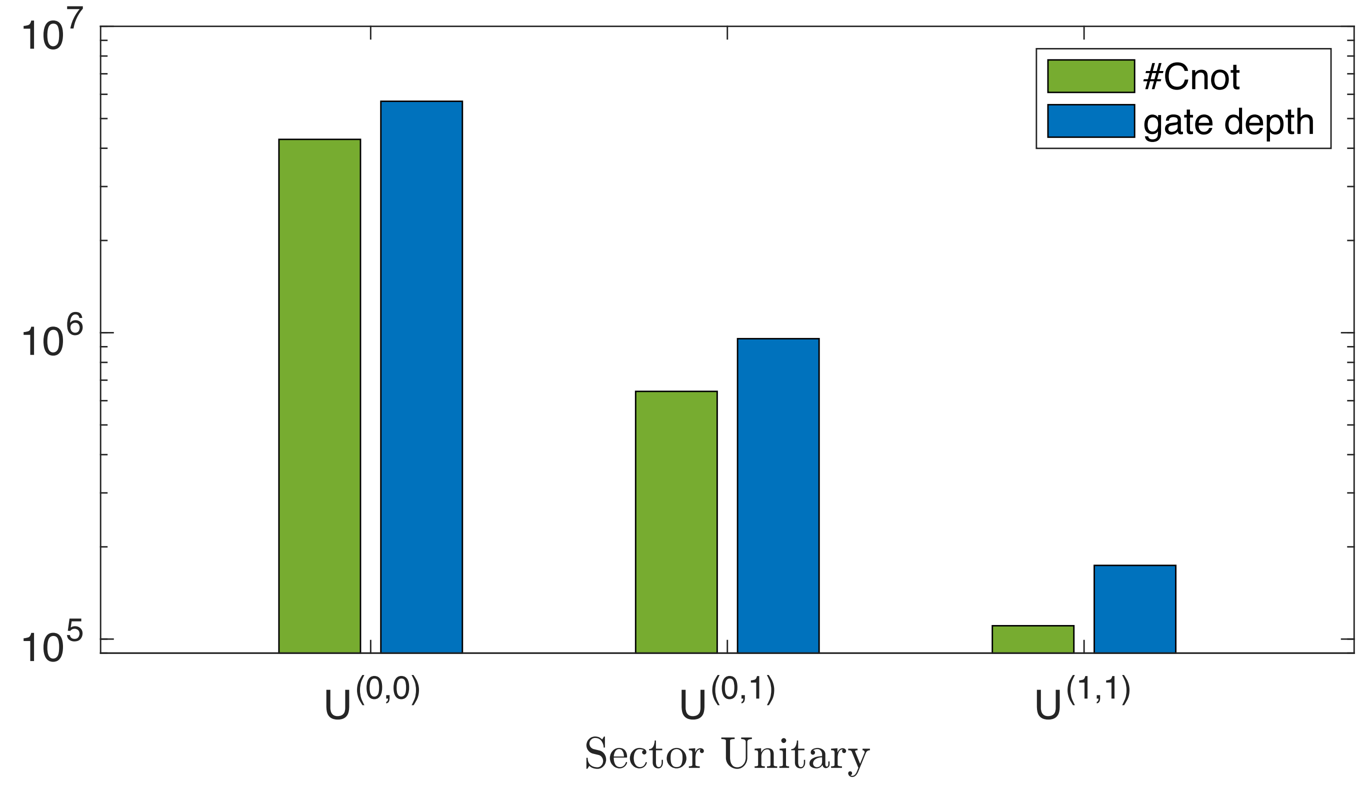
$$\equiv U^{c,(0,0)}U^{c,(1,0)}U^{c,(0,1)}U^{c,(1,1)}$$

- Just like the one-flavor case, time evolution factorizes into controlled unitaries

Diagonalization of $\sum_{Q_k} \hat{J}^{(i)+}_{Q_k,x}\hat{J}^{(j)-}_{Q_k,y} + \hat{J}^{(i),-}_{Q_k,x}\hat{J}^{(j),+}_{Q_k,y} = \sum_{l}^{N_{ij}} c_l P_l$

- N_{ij} can be $\mathcal{O}(10^6)$, depending on baryonic sectors (i,j) at (x,y)
- Ressources ? Techniques from quantum chemistry …
  - Partition Pauli strings P_l into commuting sets (n_P such sets)
  - Simultaneous diagonalization reduces gate depth





Change Computational Basis: Qudits (I)

Intuitively computation with d–level system promises less complexity…

$d \leq 7$ demonstrated with trapped ions (Ringerbauer, M. et al, Nat.Phys. 2022)

$d_{N_f=1} = 6$, local Hilbert space can be stored

Computationally it's the Heisenberg model

$$e^{-i\delta t H_{x,y}} \approx e^{-i\delta t J_x^1 \otimes J_y^1} e^{-i\delta t J_x^2 \otimes J_y^2} \approx \prod_{i=1}^{9} U_{\mathscr{X}\mathscr{X}_i} U_{\mathscr{Y}\mathscr{Y}_i}$$

where

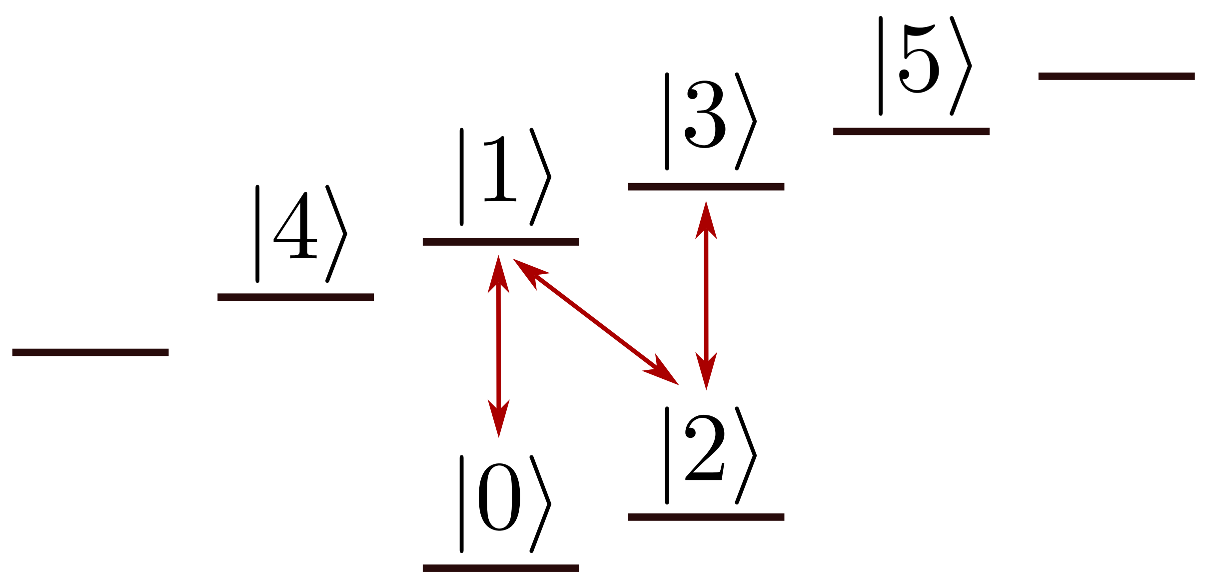
$U_{\mathscr{X}\mathscr{X}} = e^{-i\alpha \mathscr{X}^x_{kl} \otimes \mathscr{X}^y_{mn}}$, $U_{\mathscr{Y}\mathscr{Y}} = e^{-i\beta \mathscr{Y}^x_{kl} \otimes \mathscr{Y}^y_{mn}}$ with d–dim. Pauli's $\mathscr{X}, \mathscr{Y}$

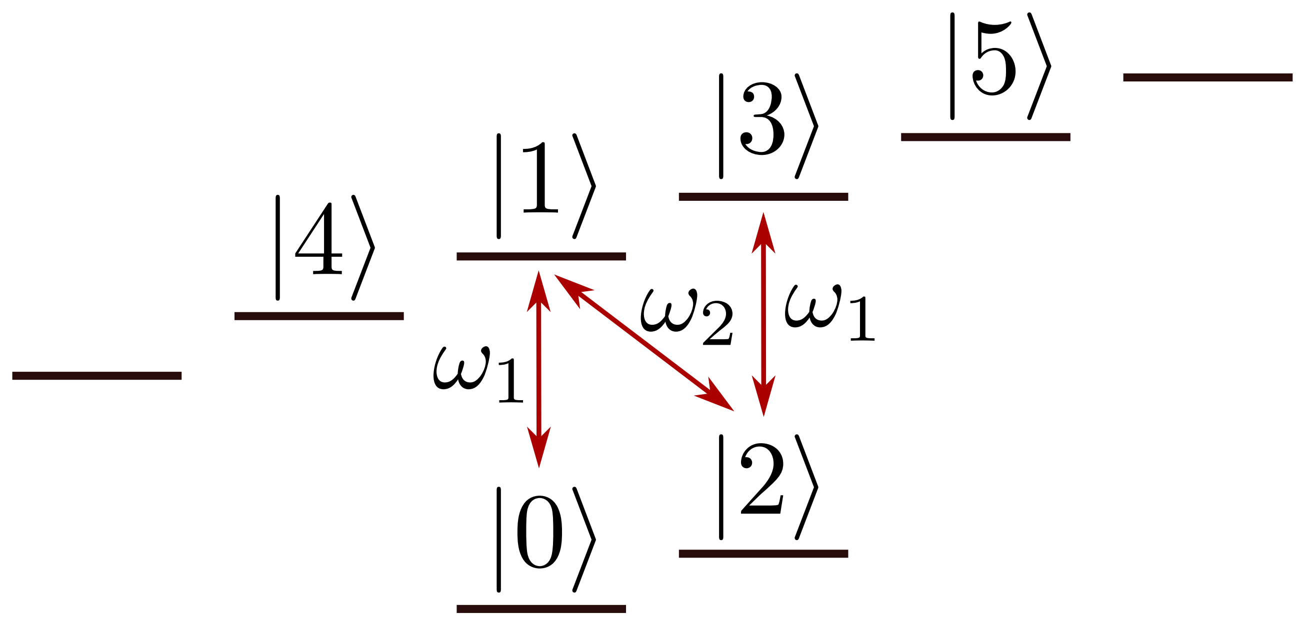
Each U **already** elementary operation (Mølmer, K. and Sørensen, A., PRL 1999)

Can even do better (Low, P.J. et al., PRR 2020)…

Generalized MS–gates: 2–qudit entangling gate with multiple transitions simultaneously

$$e^{-i\delta t H_{x,y}} \approx \tilde{U}_{\mathscr{X}\mathscr{X}} \tilde{U}_{\mathscr{Y}\mathscr{Y}}$$



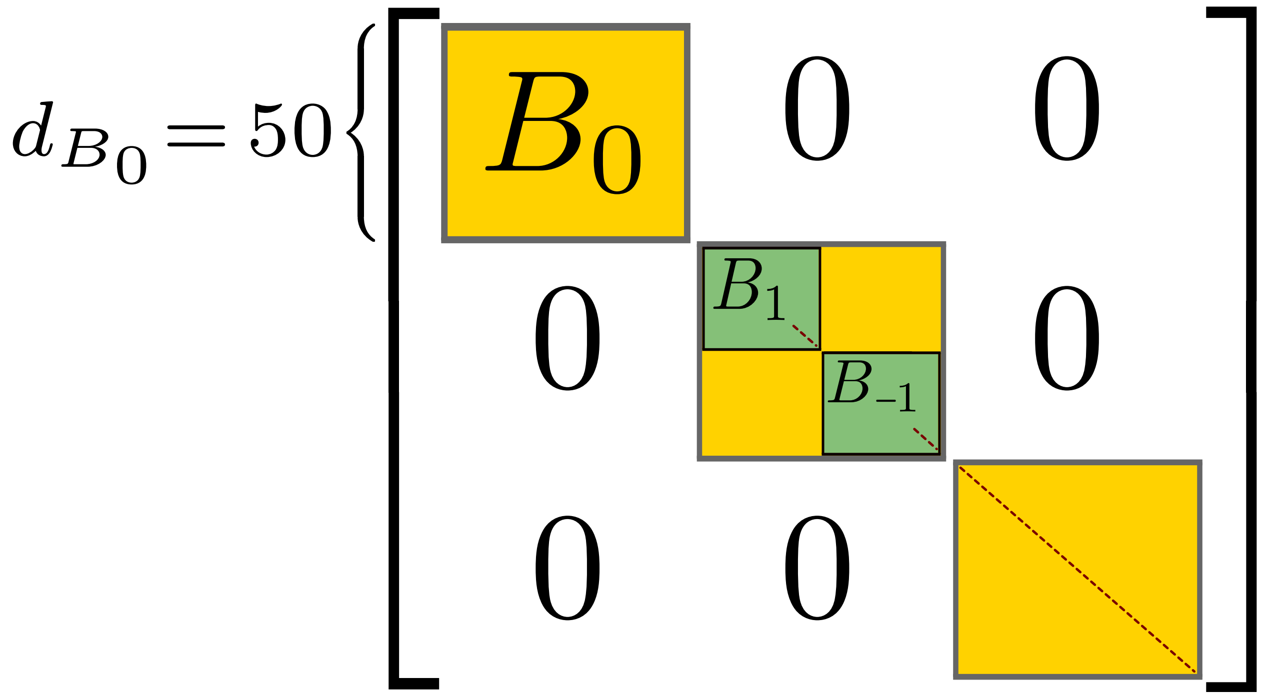


Change Computational Basis: Qudits (2)

$$d_{N_f=2} = 92$$

rewriting with ancillary qudits …

$$\hat{\mathcal{H}} = -\frac{1}{2}\sum_{\langle x,y\rangle}\sum_{i,j} \underbrace{|i\rangle\langle i|_x}_{d_a=3} |j\rangle\langle j|_y \sum_{Q_k} \underbrace{\hat{J}^{(i)+}_{Q_k,x}}_{d_{B_0}} \hat{J}^{(j)-}_{Q_k,y} + \hat{J}^{(i),-}_{Q_k,x}\hat{J}^{(j),+}_{Q_k,y}$$



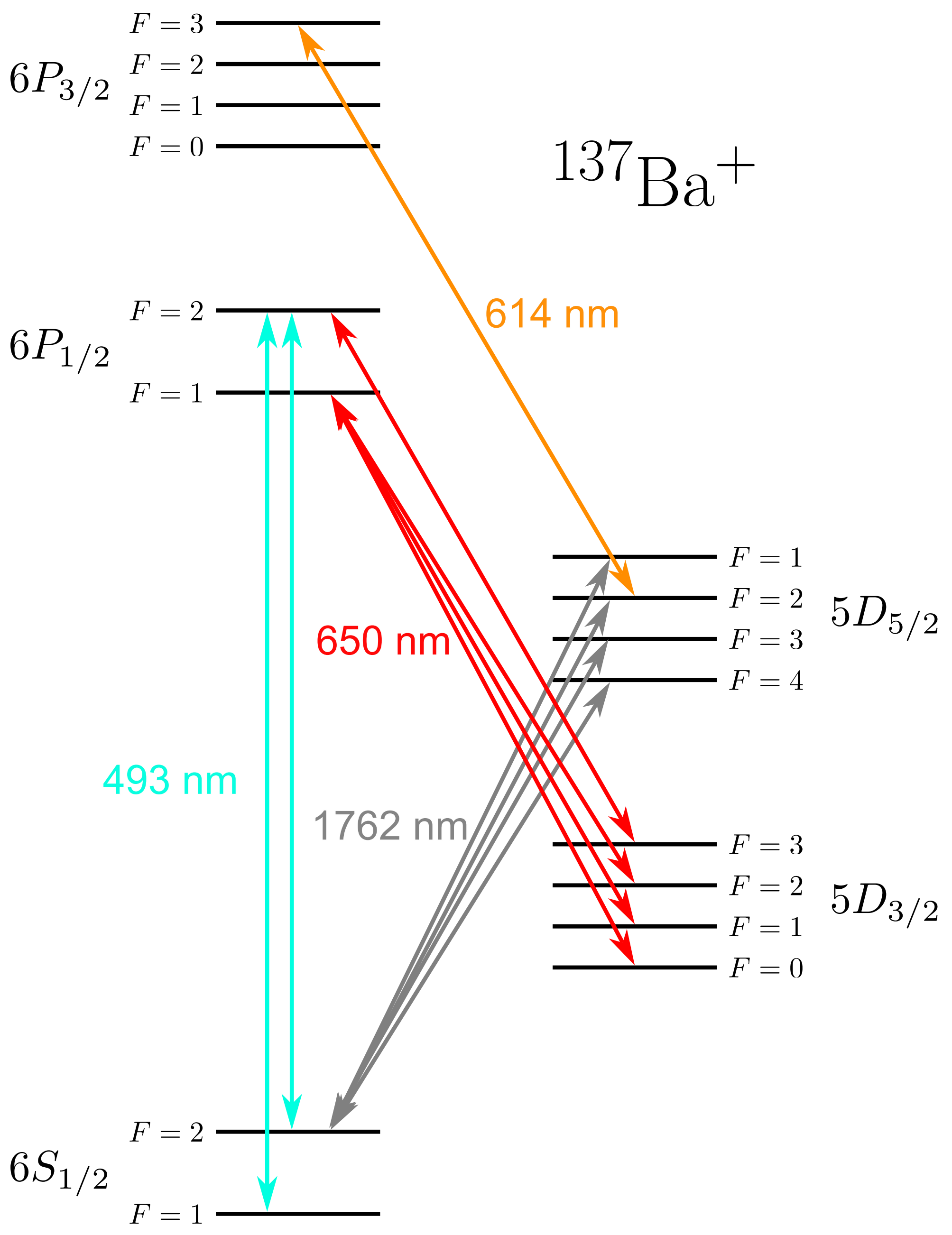
Mixed qudit environment :

Qutrit controlling the baryonic sector

Qudit with large d controlling the state

$d = 50$, feasible ?

^{137}Ba^{+} trapped ion with up to $d = 25$ (Low, P.J. et al., arXiv:2306.03340)



Change Computational Basis: Qumodes

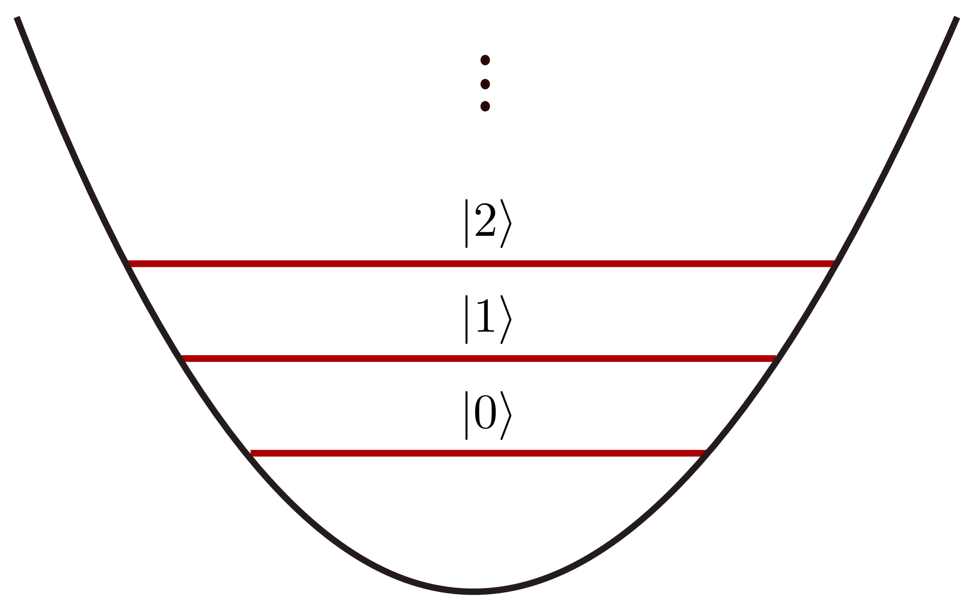
Digitization with qumodes

- Local state $|\mathfrak{h}_x\rangle$ encoded in vibrational modes / photon occupation number $|n\rangle_x$, $\hat{n}_x = a_x^\dagger a_x$, $\left(a, a^\dagger\right) \leftrightarrow \left(\hat{x}, \hat{p}\right)$, tailored to encode continuous variables (see e.g. Jha, R.G., PRA 2024 $O(3)$ model)

- Discrete model ? $N_f = 1$– model encodes information in angular momentum states → Jordan–Schwinger–map (Schwinger 1952)

$$\hat{J}_x^+ = \hat{a}_{1,x}^\dagger \hat{a}_{2,x}, \ \hat{J}_x^- = \hat{a}_{2,x}^\dagger \hat{a}_{1,x}$$

i.e. two qumodes per site x with local **constraint** $n_1 + n_2 = 2j = N_c$



- Challenge: Gate set $\{e^{is\hat{x}_k}, e^{is\hat{a}_k^\dagger \hat{a}_k}, e^{is\hat{x}_k^2}, e^{is\hat{x}_k^3}, e^{is\hat{x}_k\hat{x}_j}\} \equiv \{Z(s), R(s), P(s), V(s), CZ(s)\}$

- Hamiltonian involves quartic interaction $\sim \hat{x}_{1,x}\hat{x}_{2,x}\hat{x}_{1,y}\hat{x}_{2,y}$

- Non-Gaussian quartic gate $Q(s) = e^{is\hat{x}_k^4}$ needs to be decomposed → expensive

Summary and Conclusion

Table III. Storage requirement for lattice volume N

Information Carrier	$N_f = 1$	$N_f = 2$
Qubit	$3N$	$(3+6)N^*$
Qudit ($d > 2$)	N	$N + N^{**}$
Qumode	$2N + N^{**}$	$> 2N$

* We used unary encoding of the three ancilla states representing the baryon sectors $|B|$.
** With mixed architecture, i.e. either mixed-qudit or qubit-qumode.

Table IV. Entangling gate count for one mesonic nn-Trotter step

Information Carrier	$N_f = 1$	$N_f = 2$
Qubit	$\mathcal{O}(10)$	$\mathcal{O}(10^6)$
Qudit ($d > 2$)	2	$\mathcal{O}(10^2)$
Qumode	$\mathcal{O}(10^2) - \mathcal{O}(10^3)^*$	–

* Depending on the availability of the quartic gate.

Thank you!