Deflation and polynomial preconditioning in the application of the overlap operator at nonzero chemical potential

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Gustavo Ramirez-Hidalgo
(Bergische Universität Wuppertal)
&
Jacob Finkenrath
(CERN)

FOR 5269

BERGISCHE UNIVERSITÄT WUPPERTAL

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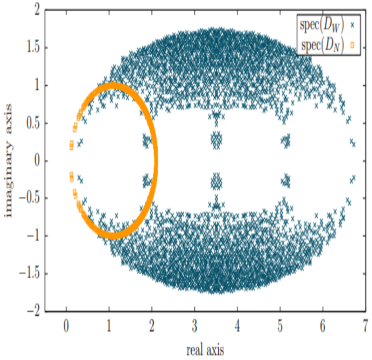

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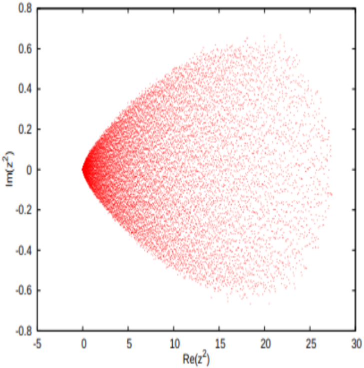
- on a 6^4 lattice ($\mu = 0.3, \beta_g = 5.1$), spectrum of $Q_\mu^2(m_0^{ker})$:



Preconditioning at the solver level

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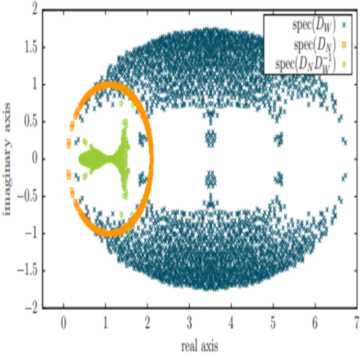
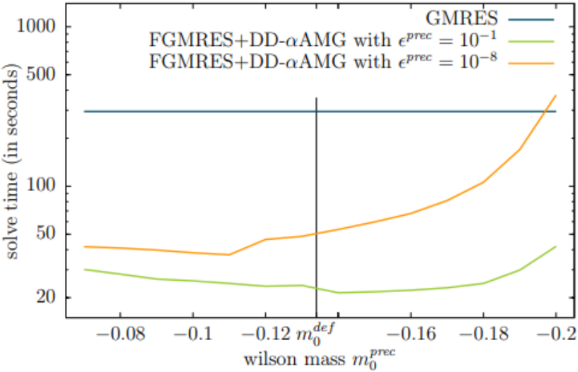


Preconditioning at the solver level

Impact of preconditioning, at the level of D_N, via $\mathrm{AMG}(D_w(m_0^{prec}), \epsilon)$ (no chemical potential, 32×32^3, smeared, work by Brannick, Frommer, Kahl, Leder, Rottmann, Strebel - 2014):



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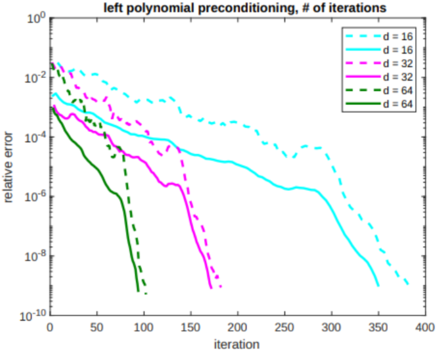
In the LQCD context, this means:

$$(Q_\mu^2)^{-1/2}b = q(Q_\mu^2)(Q_\mu^2(q(Q_\mu^2))^2)^{-1/2}b \text{ with } q(Q_\mu^2) \approx (Q_\mu^2)^{-1/2}$$

(note how Arnoldi is done with Q_μ^2).

Preconditioning at the sign-function level

With a chemical potential $\mu = 0.3$, 64×32^3, non-smeared, d is the degree of the polynomial, the **dashed** lines are a cheap approximation of the error. Tolerances are 10^{-5} for the table and 10^{-9} for the figure.



d	iterations	mvms	inner products	time 64 nodes (in s)	time 256 nodes (in s)
1	1 600	3 200	1 279 200	127.8	105.8
8	296	8 910	42 510	25.7	9.8
16	140	8 742	9 991	12.3	7.8
32	72	9 198	3 125	11.6	7.4
64	33	8 636	2 578	10.6	5.5

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Analyzing the interplay of polynomial preconditioning with deflation, in the context of the sign function, is **ongoing work**.

“Modern” overlap solver

while not converged **do**
    for $i = 1 : m_{out}$ **do**
        …
        for $j = 1 : m_{in}$ **do**
            …
            $v_{in}^{(1)} \leftarrow \mathrm{AMG}\left(D_w(m_0^{prec}, \epsilon)\right) w_{in}^{(1)}$ (hp)
            …
            $v_{in}^{(2)} \leftarrow \left(Q_\mu^2 \left(q(Q_\mu^2)\right)^2\right)^{-1/2} \left(w_{in}^{(2)} - R_m L_m^H w_{in}^{(2)}\right)$ (sp/hp)
            …
        end for
        …
        $v_{out} \leftarrow \left(Q_\mu^2 \left(q(Q_\mu^2)\right)^2\right)^{-1/2} \left(w_{out} - R_m L_m^H w_{out}\right)$ (dp)
        …
    end for
end while

Thank you!