The imaginary-theta dependence of the SU(N) spectrum

Davide Vadacchino - University of Plymouth

(with C. Bonanno   C. Bonati   M. Papace)

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Introduction

The Yang-Mills Lagrangian density describing θ-vacua is

$$\tilde{S} = -\frac{1}{2g^2} \int \mathrm{d}^4x \, \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \imath\theta \int \mathrm{d}^4x \; q(x) \tag{1}$$

where

$$q(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\, F^{\mu\nu} F^{\rho\sigma} \tag{2}$$

The θ dependence in gauge theories has attracted a lot of interest especially at large-N.

- The $U(1)_A$ problem and the Witten-Veneziano formula.

  't Hooft 1976; Veneziano 1979; Witten 1979
- The strong-CP problem and the axion.

  Peccei and Quinn 1977; Weinberg 1978
- Lattice simulations at fixed Q.

  Aoki et al. 2007; Brower et al. 2003

Theta dependence of the spectrum

The greatest effort was devoted to studying the θ dependence of the free energy,

$$F(\theta, T) - F(0, T) = \frac{1}{2}\,\chi(T)\,\theta^2\,s(\theta, T) \tag{3}$$

where $\chi(T)$ is the topological susceptibility and $s(\theta, T) = 1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \ldots$

- In the chiral limit, perturbatively.

  Di Vecchia and Veneziano 1980; Grilli di Cortona et al. 2016
- At high-T, semiclassically.

  Gross, Pisarski, and Yaffe 1981; Schäfer and Shuryak 1998
- In D=2 for U(N) gauge theories, exactly.

  Bonati and Rossi 2019; Kovacs, Tomboulis, and Schram 1995

For the generic T regime, only lattice simulations. However, the θ term introduces a sign problem. Hence

- Either one works at $\theta = 0$,
- Or one works at imaginary $\theta = \imath\theta_I$.

Theta dependence of the spectrum

What are our expectations:

- At $\theta = 0$, ground state has quantum numbers 0^{++}.
- At $\theta \neq 0$, P explicitly broken, hence

$$m_G \to m(0^{++}), \quad \theta \to 0 \tag{4}$$

We will have, then

$$m_G(\theta) = m(0^{++}) \left\{1 + m_2\theta^2 + O(\theta^4)\right\}, \qquad \sigma(\theta) = \sigma \left\{1 + s_2\theta^2 + O(\theta^4)\right\} \tag{5}$$

our task: to estimate m_2 and s_2.

Two strategies are possible:

- Expand correlation functions in a power series in θ

Del Debbio, Manca, et al. 2006

$$G(t,\theta) = G^{(0)}(t) + \frac{1}{2}\theta^2 G^{(2)}(t) + \theta^4 + \cdots \tag{6}$$

  and simulate at $\theta = 0$.

- Simulate the theory at imaginary θ.

Numerical setup

We use a standard discretization

$$S_L(\theta_L) = S_W - \theta_L Q_{\text{clov}} \tag{7}$$

where

$$S_{\text{W}}[U_\mu] \equiv \beta \sum_x \sum_{\mu<\nu} \left(1 - \frac{1}{N} \Re \text{Tr} \mathcal{P}_{\mu\nu}\right), \qquad \beta \equiv \frac{2N}{g_0^2} \tag{8}$$

with

$$\mathcal{P}_{\mu\nu}(x) \equiv U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x), \qquad U_\mu(x) \equiv \exp\left(i \int_x^{x+\hat{\mu}} \text{d}\lambda^\mu \tau^A A_\mu^A(\lambda)\right), \tag{9}$$

and

$$Q_{\text{clov}}(x,\, t) = \frac{1}{2^9 \pi^2} \sum_x \sum_{\mu<\nu} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}\, \mathcal{C}_{\mu\nu}(t) \mathcal{C}_{\rho\sigma}(t) \tag{10}$$

is the clover operator built from $U_\mu(t)$.

Numerical Setup

The physical θ and θ_L are related as follows,

$$\theta = \imath Z_Q \theta_L \tag{11}$$

where $Q = Z_Q Q_{\text{clover}}$ and Z_Q is determined using cooling, from,

$$Z_Q = \frac{\langle Q Q_{\text{clov}} \rangle}{\langle Q^2 \rangle} \tag{12}$$

Panagopoulos and Vicari 2011

where

$$Q = \text{round}\left\{ \alpha Q^{(cool)}_{\text{clov}} \right\} \tag{13}$$

and α is chosen to minimize $\langle \left[\alpha Q^{(cool)}_{\text{clov}} - \text{round}(\alpha Q^{(cool)}_{\text{clov}}) \right]^2 \rangle$.

Del Debbio, Panagopoulos, and Vicari 2002

Hence, in terms of θ_L,

$$m_G(\theta_L) = m(0^{++}) \left\{1 - m_2 Z_Q^2 \theta_L^2 + O(\theta_L^4)\right\}, \quad \sigma(\theta_L) = \sigma \left\{1 - s_2 Z_Q^2 \theta_L^2 + O(\theta_L^4)\right\}$$

Calculation of the spectrum

Variational method

We consider the operators

$$\mathcal{O}(t) = \sum_i v_i \mathcal{O}_i(t) \ , \quad \mathcal{O}_i(t) = \sum_{\vec{x}} \mathrm{Tr}\, U_{\mathcal{C}i}(\vec{x},\, t) \tag{14}$$

for a set of lattice paths $\mathcal{C}_i$, and in a specific space-time symmetry channel,

- contractible paths correspond to glueball states
- non-contractible paths correspond to torelons

We determine v_i from the GEVP,

$$C_{ij}(t) v_i = \lambda(t,\, t_0) C_{ij}(t_0) v_j \ , \quad C_{ij}(t) = \frac{1}{aL} \sum_{t'} \langle \mathcal{O}_i(t - t') \mathcal{O}_j(t') \rangle \tag{15}$$

and then define the optimal correlator as $C_G(t) = C_{ij}(t) v_i v_j$,

Calculation of the spectrum

To determine m_G, we look for a plateau in

$$am_G^{\text{eff}}(t) = -\log\left\{\frac{C_G(t+a)}{C_G(t)}\right\} \tag{16}$$

and then fit the functional form

$$C_G(t) = A_G\left[e^{-m_G t} + e^{m_G(aL-t)}\right] \tag{17}$$

on the corresponding interval, using A_G and m_G as fitting parameters.

The string tension is determined from the mass of the torelon as follows,

$$a^2\sigma = \frac{am_{\text{tor}}}{L} + \frac{\pi}{3L^2} \tag{18}$$

Numerical Setup

Ensembles of configurations of pure gauge SU(N) theories were collected in the following settings:

- at $N = 3$, using the standard HB+OR combination at 5 values of β, L such that $\sqrt{\sigma}L \geq 3.5$, approx. $O(60k)$ configurations
- at $N = 6$, using Parallel Tempering on Boundary Conditions (PTBC) at 2 values of β, approx. $O(5k)$. Bonanno, Bonati, and D'Elia 2021; Bonanno, Clemente, et al. 2024; Hasenbusch 2017

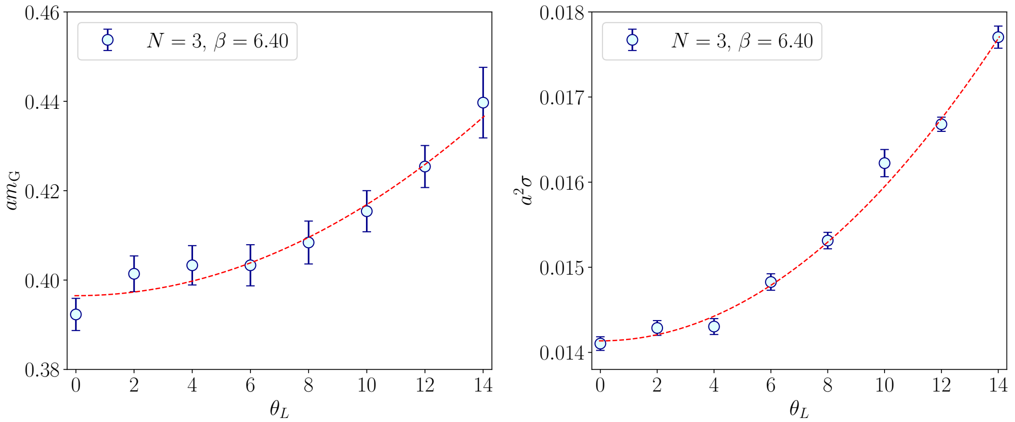
N	L/a	β	N_{conf}
3	16 – 30	5.95 – 6.40	$60k$
6	14, 16	25.056, 24.452	$5k$

- The variational basis consisted of ~ 100 blocked-smeared operators.
- After obtaining $\sigma(\theta_L)$ and $m_G(\theta_L)$, we fit

$$f(\theta_L) = A_1 \left(1 + A_2 \theta_L^2\right) \tag{19}$$

with A_1 and A_2 fitting parameters and f either σ or m_G.

At $N = 3$ and fixed β



We fit,

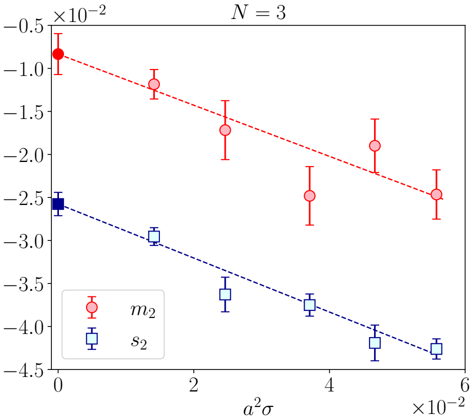
$$f(\theta_L) = A_1 \left(1 + A_2 \theta_L^2\right) \tag{20}$$

and obtain

$$m_2(\beta = 6.40) = -0.0118(16)\ , \quad s_2(\beta = 6.40) = -0.0295(10)\ . \tag{21}$$

At $N = 3$, continuum extrapolations

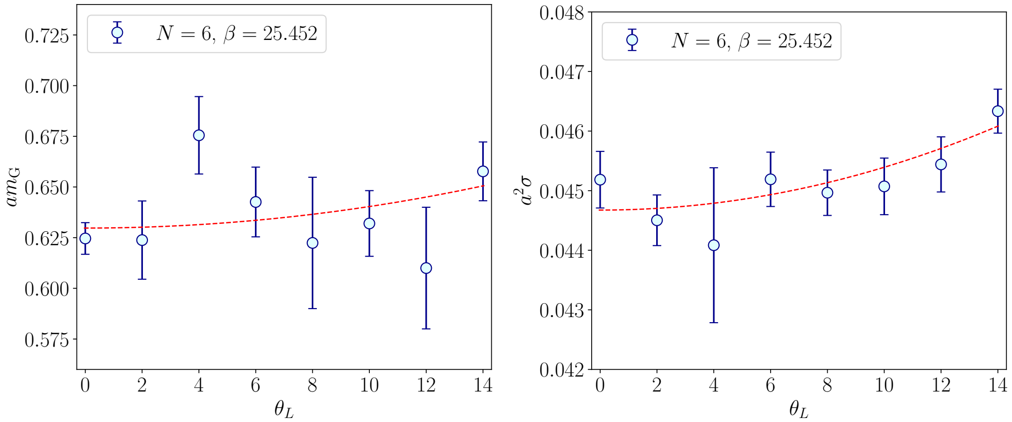
Assuming $O(a^2)$ scaling corrections,



we obtain

$$m_2 = -0.0083(23) \ , \quad s_2 = -0.0258(14) \ . \tag{22}$$

At $N = 6$, and fixed β



We fit,

$$f(\theta_L) = A_1\left(1 + A_2\theta_L^2\right) \tag{23}$$

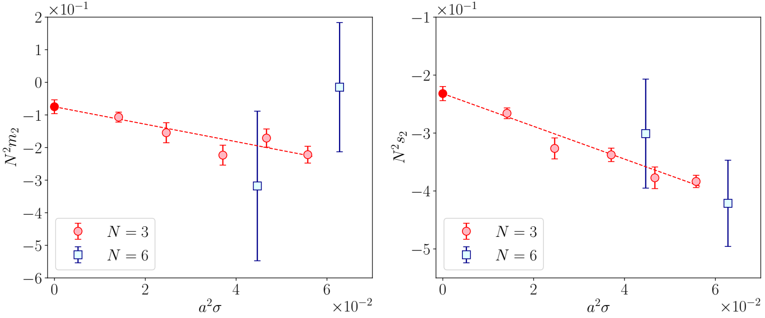
and obtain

$$m_2(\beta = 25.452) = -0.0088(64)\ , \quad s_2(\beta = 25.452) = -0.0084(26)\ . \tag{24}$$

The large-N scaling

We cannot test directly expected scaling. Assuming it is valid for $N \geq 3$,

$$s_2 \simeq \frac{\bar{s}_2}{N^2} + O(N^{-4}), \qquad m_2 \simeq \frac{\bar{m}_2}{N^2} + O(N^{-4}) \tag{25}$$



We obtain, from the $N = 3$ data,

$$\bar{s}_2 \simeq -0.23(1), \quad \bar{m}_2 \simeq -0.075(20) \tag{26}$$

the $N = 6$ data seem consistent with this scaling.

Conclusions

- The leading order $O(\theta^2)$ dependence on θ of the ground state mass and the string tension was studied, at imaginary θ. At $N = 3$,

$$m_2(N=3) \simeq -0.0083(23), \quad s_2(N=3) \simeq -0.0258(14) \tag{27}$$

- We could not estimate these coefficients at $N = 6$, but could check that the data are not in contrast with the expected scaling with

$$s_2 N^2 \simeq \bar{s}_2 \simeq -0.23(1), \quad m_2 N^2 \simeq \bar{m}_2 \simeq -0.075(20) \tag{28}$$

- This method is effective in reducing uncertainties compared to the Taylor's expansion method.
- The relative error in simulating at frozen topology can be estimated at $\sim 0.08\%$ at $V \simeq (1.5fm)^4$.

Future directions:

- Investigate $N > 3$ to better determine $\bar{s}_2$ and $\bar{m}_2$.
- Study the excited spectrum.

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