SU(6) model revisited

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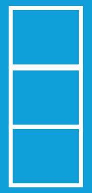
Why SU(6) model ?

$\mathcal{R} =$ 

(one of the chiral gauge theory)

・Weyl fermions follow the self-conjugate representation in this model.

c.f. Discussions on self-conjugate rep. in [Luscher1999, 2000]

⟹ This theory might be realized on the lattice with overlap fermion.

⟹ It may provide us a hint to realize all chiral gauge theory on the lattice.

・The vacuum structure is highly non-trivial, which is captured by 't Hooft anomaly matching condition. [Yamaguchi(2018)]

※Chiral sym. is spontaneously broken without the fermion bilinear condensate.

This is the main topic in this talk.

Our Work : Reanalysis of the SU(6) model

Motivation : To understand the non-trivial vacuum structure , using the ‘t Hooft anomaly matching.

Strategy :

- Analyzing all the ’t Hooft anomalies which are possible to arise.
- Constructing the IR effective theory by using ’t Hooft anomaly matching.

Our result : Under the assumption
that the order param. of SSB of chiral sym. is four-fermi operator,

mixed anomaly in this model is captured by only one scalar field in the IR region.

⇒ Evidence that
the vacuum structure consists of one scalar field w/ three-fold vacua.

Review of SU(6) model

Based on　[Yamaguchi(2018)]

Symmetry

- Discrete chiral symmetry $\mathbb{Z}_6^{(0)}$ (0-form symmetry, (conventional symmetry))

$$\psi \to e^{\frac{2\pi i}{6}} \psi$$

- Center symmetry $\mathbb{Z}_3^{(1)}$ (1-form symmetry)

$$W = e^{i \oint a} \to e^{\frac{2\pi i}{3}} W$$

Chiral anomaly

[Yamaguchi(2018)]

- Gauging **only** the 1-form center symmetry.

  i.e. Introduce the background gauge fields for $\mathbb{Z}_3^{(1)}$.

- Under the discrete chiral trsf. ; $\psi \to e^{\frac{2\pi i}{6}} \psi$

$$Z[B^{(2)}] \to Z[B^{(2)}] \underbrace{e^{2\pi i (-\frac{2}{3}) \int_{M_4} B^{(2)} \wedge B^{(2)}}}_{\text{Anomaly}}$$

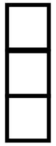
★ Confinement ⇒ **SSB of the chiral sym.** $\mathbb{Z}_{l=6} \to \mathbb{Z}_2,$

What is the order parameter?

[Yamaguchi(2018)]

Prohibition of bilinear condensate

$$< \psi\psi > = \epsilon^{\alpha\beta}\psi^{I}_{\alpha}\psi^{J}_{\beta} B_{IJ} = 0$$

SU(6) invariant bilinear form is anti-symmetric for 

Four-fermi operator can be a good candidate

$$< \psi\psi\psi\psi > \neq 0$$

Remaining questions

- What happens if we gauge $\mathbb{Z}_6^{(0)}$ as well?
  Are there additional anomalies in the theory?

- What is the order parameter of the SSB of $\mathbb{Z}_6^{(0)}$?
- What constitutes the IR theory in the gapped phase ?

Our goal

1. Obtain all the 't Hooft anomalies by fully gauging the entire symmetries.
2. Propose a consistent low energy theory.

Our Work

What we have done

1. Compute all the anomalies in UV region.
   - Anomalies from standard Stora-Zumino procedure
   - Anomalies from eta-invariant and Bordism group
2. Construct the IR effective theory.
   - Identify the topological term to reproduce eta-invariant.
   - Construct WZW-type action from the topological term (Not completed yet ...)

0. Preliminary : 't Hooft Anomaly

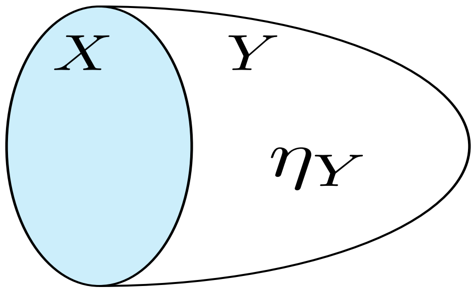
- Anomaly in 4-dim. mfd. X

$$Z[A] \xrightarrow{\text{gauge trsf.}} Z[A]\underbrace{e^{2\pi i\int_X \alpha}}_{\text{Anomaly}}$$

0. Preliminary : Anomaly (modern understanding)

[Witten '15] [Yonekura '16] [Witten-Yonekura '19]

- Extend 4-dim mfd. X to 5-mfd. Y
- Attach an appropriate invariant η_Y of Y .



$$\mathcal{Z}'[A] = \underbrace{|Z[A]|e^{-2\pi\eta_Y}}_{\text{Gauge invariant}}$$

← Eta-invariant

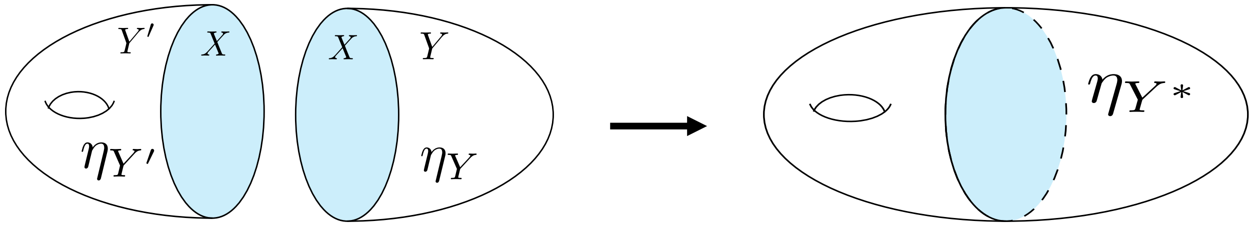
: Anomaly inflow

- Anomaly : Dependence on higher manifold

$$\frac{|Z[A]|e^{-2\pi\eta_Y}}{|Z[A]|e^{-2\pi\eta_{Y'}}} = \underline{e^{-2\pi\eta_{Y^*}}}$$

Anomaly

$Y^* = Y \cup Y'$



1. Compute all the anomalies in UV region

➢ Anomalies from Stora-Zumino procedure

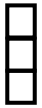
By introducing the background gauge field for $\mathbb{Z}_6$,

and using the Stora-Zumino procedure, [Stora, Zumino(1974)]

$\left[\mathbb{Z}_6^{(0)}\right] - \left[\mathbb{Z}_3^{(1)}\right]^2$ mixed anomaly, ⬅ Consistent w/ [Yamaguchi(2018)]

$$\frac{2\pi}{3!\,(2\pi)^3}\int(-3\cdot 6\cdot 6)\,A_6^{(1)}\wedge B_3^{(2)}\wedge B_3^{(2)} \;\in\; \mathbb{Z}_3$$

$\left[\mathbb{Z}_6^{(0)}\right]^3$ self-anomaly ⬅ Additional anomaly.

Dim. of $\mathcal{R}$ = 

$$\frac{2\pi}{3!\,(2\pi)^3}\cdot 20\int A_6^{(1)}\wedge dA_6^{(1)}\wedge dA_6^{(1)} \;\in\; \mathbb{Z}_9$$

1. Compute all the anomalies in UV region

➢ Anomaly from η-invariant and Bordism group

$\left[\mathbb{Z}_6^{(0)}\right]^3$ self-anomaly is a kind of non-perturbative anomalies.

⇒ Mathematically,
the η-invariant depends on the class of the 5-dimensional bordism group.

$$= \quad \Omega_5^{spin}(B\mathbb{Z}_6) \simeq \mathbb{Z}_9$$

[Chang-Tse Hsieh (2018)]

Above anomaly is consistent with our previous result by Stora-Zumino procedure.

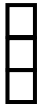
1. Compute all the anomalies in UV region

Thus, all the anomalies in SU(6) model are

- $\left[\mathbb{Z}_l^{(0)}\right] - \left[\mathbb{Z}_N^{(1)}\right]^2$ mixed anomaly

$$\frac{2\pi}{3!\,(2\pi)^3}\int(-3\cdot 6\cdot 6)\,A_6^{(1)}\wedge B_3^{(2)}\wedge B_3^{(2)} \;\in\; \mathbb{Z}_3$$

- $\left[\mathbb{Z}_{l=6}^{(0)}\right]^3$ self-anomaly

Dim. of $\mathcal{R} =$ 

$$\frac{2\pi}{3!\,(2\pi)^3}\cdot 20\int A_6^{(1)}\wedge dA_6^{(1)}\wedge dA_6^{(1)} \;\in\; \mathbb{Z}_9$$

2. Construct the IR effective theory

★ IR effective theory has to reproduce the UV anomaly.

('t Hooft anomaly matching condition)

We consider what degrees of free domes in IR region make UV anomalies.

- $\left[\mathbb{Z}_l^{(0)}\right] - \left[\mathbb{Z}_N^{(1)}\right]^2$ mixd anomaly

$$\sim \int A_6^{(1)} \wedge B_3^{(2)} \wedge B_3^{(2)} \in \mathbb{Z}_3$$

- $\left[\mathbb{Z}_{l=6}^{(0)}\right]^3$ self-anomaly

$$\sim \int A_6^{(1)} \wedge dA_6^{(1)} \wedge dA_6^{(1)} \in \mathbb{Z}_9$$

2. Construct the IR effective theory

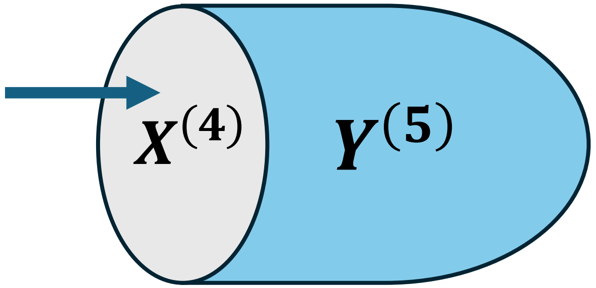
Idea : Wess-Zumino-Witten action

[Wess, Zumino(1971), Witten(1983)]

In the case of N_f-flavor massless QCD,

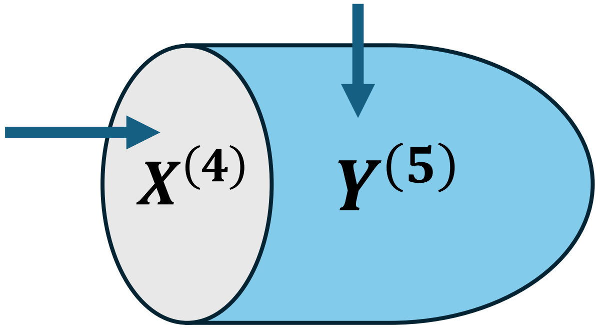
dressed gauge field: $A^U = A + d\pi$

Fermions: ψ
Gauge fields:A



UV theory

NG bosons: π



IR theory

The partition function w/ A^U is gauge invariant in the closed-mfd.

2. Construct the IR effective theory

Assumption : $< \psi\psi\psi\psi > \sim e^{2\pi i\,\phi} \neq 0$

$\exists$ scalar field $\phi \in \mathbb{Z}_3$ s.t. whose charge is $Q = 2$ under the $\mathbb{Z}_3$ trsf.

$\subset \mathbb{Z}_{l=6}$

$$\mathbb{Z}_3 : \phi \mapsto \phi - \frac{2\pi}{3}\Lambda, \qquad \oint d\Lambda \in 2\pi\mathbb{Z}$$

A part of the IR effective action is given by

$$\Gamma^{(5)}_{\mathrm{Mixed}} = \int \left(A^{(1)}_6 + d\phi\right) \wedge \left[\frac{2\pi}{3!\,(2\pi)^3}(-3lN\,)B^{(2)} \wedge B^{(2)}\right] + \int \frac{3}{2\pi}\phi \wedge db^{(3)}$$

dressed gauge field

(Gauge inv. in the closed mfd.)

★ Mixed anomaly is matched by just one scalar field.

2. Construct the IR effective theory

How about the $\left[\mathbb{Z}_6^{(0)}\right]^3$ self-anomaly ?? (Ongoing work)

・Topological term given by Stora-Zumino procedure is ill-defined mathematically.

$$\frac{2\pi}{3!\,(2\pi)^3}\,20\cdot A_6^{(1)}\wedge dA_6^{(1)}\wedge dA_6^{(1)} \to 2\pi\frac{1}{9}\,A_6\cup\beta_6 A_6\cup\beta_6 A_6 \not\in H^5(Y,2\pi\,\mathbb{Z}_6)$$

$\beta_6 : H^n(-,\mathbb{Z}_6)\to H^{n+1}(-,\mathbb{Z}_6)$

・The well-defined topological term is

[Wan, Wang (2019)]

(written by co-chain form)

$$\eta_Y[Y]=\beta_9\,(\beta_3 A_3\cup\beta_3 A_3)$$

Where

$A_3\in\mathbb{Z}_3\subset\mathbb{Z}_6$

Bockstein homomorphism : $\beta_9 : H^n(-,\mathbb{Z}_3)\to H^{n+1}(-,\mathbb{Z}_9)$, $\beta_3 : H^n(-,\mathbb{Z}_3)\to H^{n+1}(-,\mathbb{Z}_3)$

2. Construct the IR effective theory

How about the $\left[\mathbb{Z}_{\mathbf{6}}^{(\mathbf{0})}\right]^{\mathbf{3}}$ self-anomaly ?? (Ongoing work)

・The well-defined topological term is

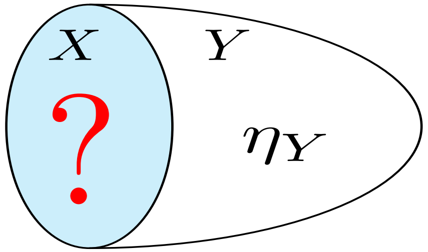
$$\eta_Y[Y] = \beta_9\,(\beta_3 A_3 \cup \beta_3 A_3)$$

[Wan, Wang (2019)]

(written by co-chain form)

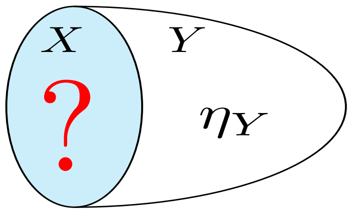
・What compensates the $\left[\mathbb{Z}_{\mathbf{6}}^{(\mathbf{0})}\right]^{\mathbf{3}}$ topological term in 4-dim. ?

Work in progress



2. Construct the IR effective theory

Discussion : What compensates the $\left[\mathbb{Z}_{l=6}^{(0)}\right]^3$ topological term in 4-dim. ?



- $\beta_9\,(\beta_3 A_3 \cup \beta_3 A_3)$ might be similar with CS term s.t. $\sim A_3\, dA_3 dA_3$.

Indeed,

- $A_3\, dA_3 dA_3$ is used to treat $\left[\mathbb{Z}_l^{(0)}\right]^3$ anomaly in SUSY QCD etc…

[Delmastro, Gomis, Hsin, Komargodski (2023)]

- By using $A_3\, dA_3 dA_3$, we can describe the anomalies from Domain-Wall.

[Kaidi, Nardoni, Zafrir, Zheng (2023)]

⇒ Self-anomaly might be also matched by the scalar field $\phi \in \mathbb{Z}_3$

Summary

・All the anomalies in the $SU(6)$ model are

$$\left[\mathbb{Z}_l^{(0)}\right] - \left[\mathbb{Z}_N^{(1)}\right]^2 \text{ mixd anomaly} \quad \text{and} \quad \left[\mathbb{Z}_{l=6}^{(0)}\right]^3 \text{ self-anomaly.}$$

・The mixed anomaly can be matched using only one scalar field, $\phi \in C^0(-, \mathbb{Z}_3)$.

・The correct topological term corresponding to the self-anomaly is $\eta_Y[Y] = \beta_9\left(\beta_3 A_3 \cup \beta_3 A_3\right)$.

Future prospect

・$\eta_Y[Y]$ could give mathematically rigorous treatment of the CS term.

・This anomaly might be also captured by $\phi \in C^0(-, \mathbb{Z}_3)$.

・The vacuum structure consists of one scalar field w/ three-fold vacua.

Understanding such vacuum structures will give a theoretical guide for future lattice simulations.

Backup Slides

Introduction : Chiral gauge theory

Chiral gauge theory : The theory in which the gauge interactions of fermions are asymmetric b/w right- and left-handed particles.

Strong interaction → ~~Perturbation~~

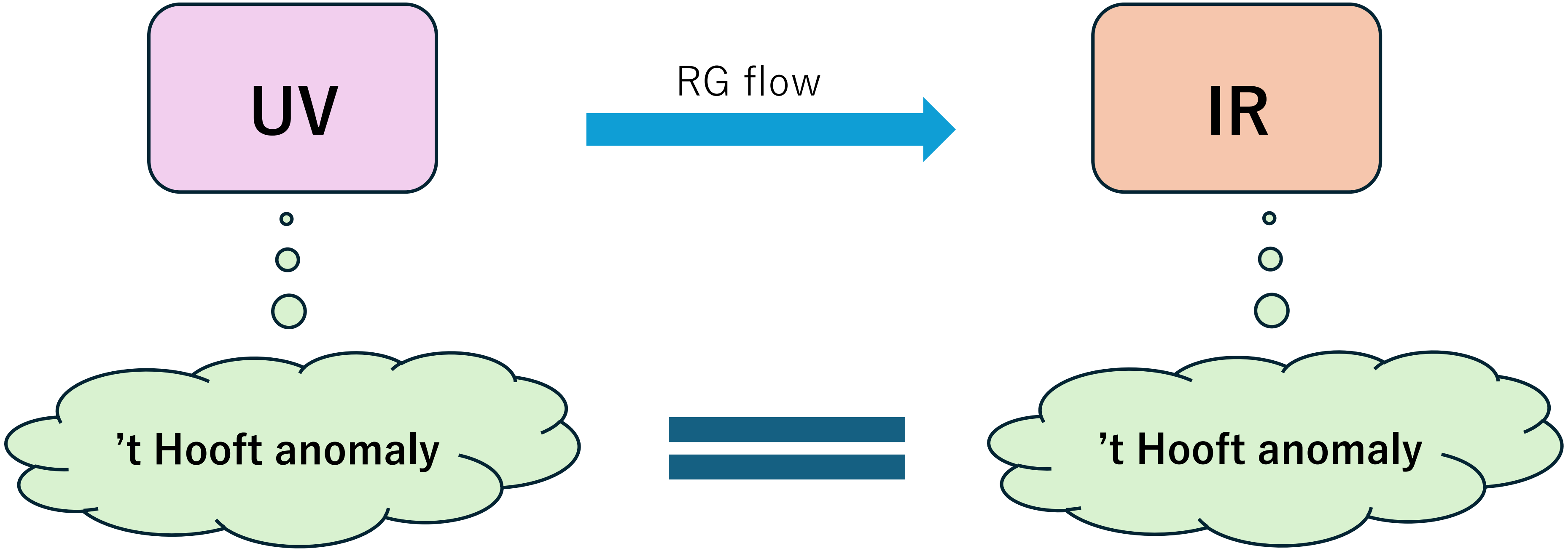
Nielsen-Ninomiya theorem [Nielsen, Ninomiya(1981)] → ~~Lattice simulation~~

The details of the theory is still largely unknown (especially in the vacuum structure).

Introduction : ’t Hooft anomaly matching condition

Symmetry can constrain the dynamics of chiral gauge theory as well.

‘t Hooft anomaly matching condition ['t Hooft (1980)]



Introduction : Generalized symmetry

(Higher form symmetry)

Generalized symmetry (p-form symmetry) : [Gaiotto, Kapustin, Seiberg, Willett(2014)]

・Acts on p-dim. charged operators such as lines.

・Sym. generators are co-dimension-(p+1) topological operators.

't Hooft anomaly matching, including generalized symmetries, provides richer insights into chiral gauge theory.

[Gaiotto, Kapustin, Komargoski, Seiberg (2017), Konishi (2018), Yamaguchi (2018) etc.]

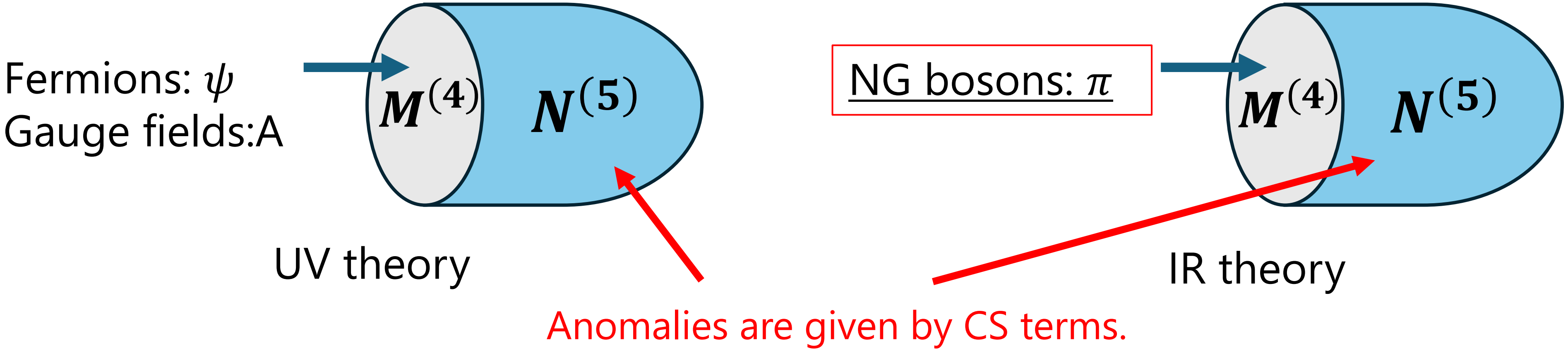
IR effective theory

5-dim. mfd. Σ, 4-dim. mfd. M, s.t. $\partial\Sigma = M$

★ Wess-Zumino-Witten action

[Wess, Zumino(1971), Witten(1983)]

- anomaly inflow : Anomaly in **N** = Anomaly in $\mathbf{M} = \boldsymbol{\partial}\mathbf{N}$
- 't Hooft anomaly matching condition : **UV** anomaly = **IR** anomaly



$$S^{(5)}_{\text{gauge-inv.}}(\mathrm{N}^{(5)}, A^U) = S_{WZW}(M^{(4)}, \pi) + S_{\text{inflow}}(\mathrm{N}^{(5)}, A)$$

※Assuming the confinement phase.

Analogy from BF Theory

$\Gamma^{(5)}_{\text{Mixed}}$ can deformed as

$$\Gamma^{(5)}_{\text{Mixed}} = \frac{3}{2\pi}\int_{\text{N}^{(5)}} (A_q + d\phi)\wedge(db^{(3)} + D^{(4)}) + \int_{N^5} (A_q + d\phi)\wedge\left(\frac{\dim R}{48}\right) p_1\left(M^{(4)}\right)$$

$$= \frac{3}{2\pi}\int_{\text{N}^{(5)}} d\phi\wedge\left(db^{(3)} + D^{(4)}\right) - A_q\wedge b^{(3)} + \frac{3}{2\pi}\int_{N^{(5)}} A_q\wedge D^{(4)} + \int_{N^5} (A_q + d\phi)\wedge\left(\frac{\dim R}{48}\right) p_1\left(M^{(4)}\right)$$

BF action with gauge interaction

Anomaly inflow term

Where $D^{(4)} = -\frac{2\pi}{3}\left(\frac{q}{2\pi}B_q^{(2)}\wedge\frac{q}{2\pi}B_q^{(2)}\right) \in H^4(-, 2\pi\mathbb{Z}_3)$

Analogy from BF Theory

Gauge transformation laws :

$$\mathbb{Z}_q^{(0)}: \quad \phi \mapsto \phi - \epsilon^{(0)}, \quad A_q^{(1)} \mapsto A_q^{(1)} + \delta\epsilon^{(0)}$$

$$\mathbb{Z}_q^{(3)}: \quad b^{(3)} \mapsto b^{(3)} - \epsilon^{(3)}, \quad D_q^{(4)} \mapsto D_q^{(4)} + \delta\epsilon^{(3)}$$

Where

$$\phi, b^{(3)} \in C^*(-,\mathbb{Z}_q), \quad \mathrm{A}_\mathrm{q}^{(1)}, D_q^{(4)} \in H^*(-,\mathbb{Z}_q), \quad \epsilon^{(*)} \in Z^*(-,\mathbb{Z}_q)$$

To make gauge invariant $\Gamma^{(5)}_{\mathrm{Mixed}}$,

$$\epsilon^{(3)} = -\underline{\Lambda^{(1)}} \cup \left(2qB_q^{(2)} + q^2\, \delta\Lambda^{(1)} \right).$$

↑ Gauge trsf. param. for 1-form center $\mathbb{Z}_q^{(0)}$ trsf.

Self-anomaly $\left[\mathbb{Z}_l^{(0)}\right]^3$

<u>perturbative anomaly</u>

$$\mathcal{A}^{per}_{\left[\mathbb{Z}_l^{(0)}\right]^3} = \frac{\dim R}{3l \cdot 8\pi^2} \int_M dA_l^{(1)} \wedge dA_l^{(1)} = 1 \bmod 9$$

<u>Non-perturbative calculation</u>

Anomaly-free condition $\mathcal{A}^{non-per}_{\left[\mathbb{Z}_l^{(0)}\right]^3} = (N^2 + 3N + 2)\Delta s_3 = 0 \mod 6n$

Where $\Delta s_3 = \sum_L s_L^3 - \sum_R S_R^3 = 20$

$$\mathcal{A}^{non-per}_{\left[\mathbb{Z}_l^{(0)}\right]^3} = 1 \bmod 9$$

Self-anomaly $\left[\mathbb{Z}_l^{(0)}\right]^3$

[Chang-Tse Hsieh (2018)]

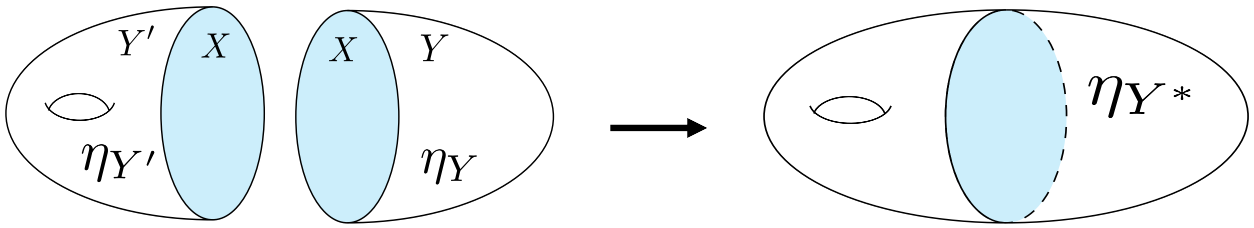
$\left[\mathbb{Z}_{l=6}^{(0)}\right]^3$ self-anomaly is a kind of global anomalies.

⇒ The non-triviality of extending a 4-dim. closed mfd. to a 5-dim. mfd.

= The non-triviality depends on the class of the 5-dimensional bordism group.

= $\Omega_5^{spin}(B\mathbb{Z}_6) \simeq \mathbb{Z}_9$

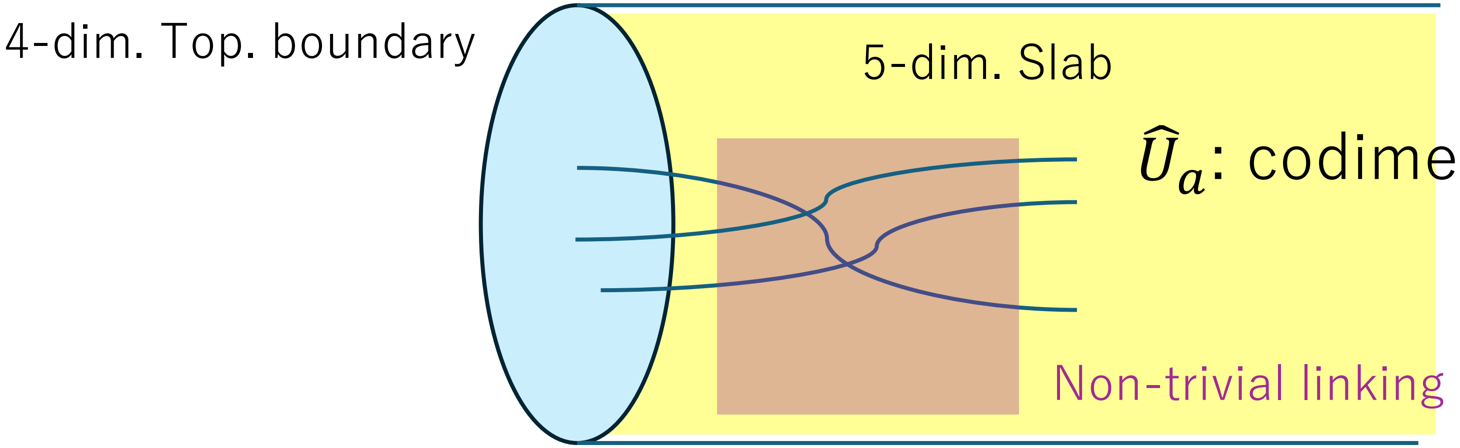
$Y^* = Y \cup Y'$



Nontrivial Linking

[J. Kaidi etc. (2023)]

Assume $$\Gamma^{(5)}_{\mathrm{Self}} \ni \frac{2\pi}{3!\,(2\pi)^3}\,\mathrm{dim}R \int_{N^5} \frac{1}{2} A_q \wedge dA_q \wedge dA_q$$



$\widehat{U}_a$: codimension - 3 ’t Hooft operator

$$< \widehat{U}_a(M^3)\widehat{U}_a(M'^3)\widehat{U}_a(M''^3) > \sim \exp\left[-2\pi i \frac{dimR}{l^3}\, Link(M^3, M'^3, M''^3)\right]$$

The existence of the non-trivial linking in the 5-dim. slab leads the anomaly in 4-dim. mfd.

Like CS term

$$\mathbb{Z}_3 \xrightarrow{i} \mathbb{Z}_9$$

$$\begin{array}{ccc} H^5(B\mathbb{Z}_3, \mathbb{Z}_3) & \xrightarrow{i^*} & H^5(B\mathbb{Z}_3, \mathbb{Z}_9) \\ \wr\| & & \wr\| \\ \mathbb{Z}_3 & \simeq & \mathbb{Z}_3 \end{array}$$

Generator : $A_3\beta_3A_3\beta_3A_3 \xrightarrow{i^*} \beta_9(\beta_3A_3\beta_3A_3)$