Towards an effective string theory for the flux-tube spectrum

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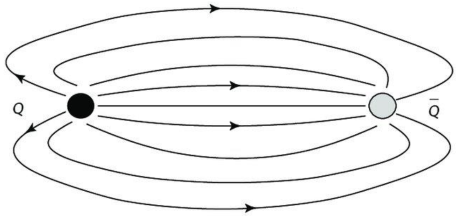
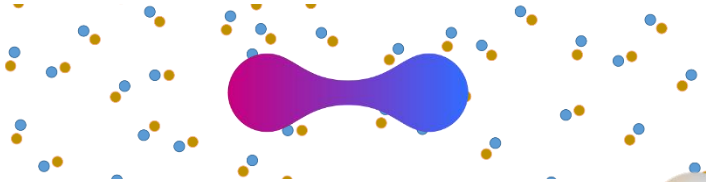
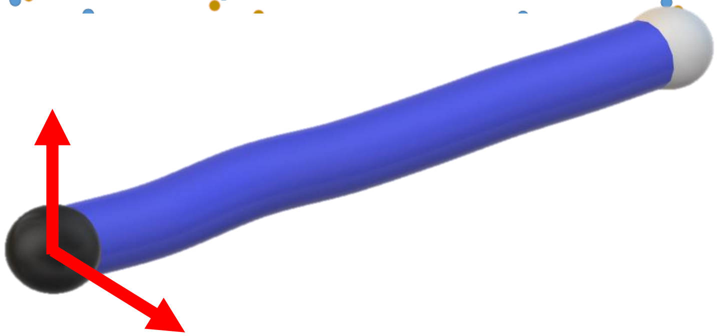
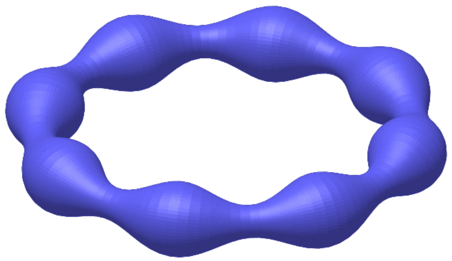
Are flux tubes strings?

- In QCD quarks are confined in bound states called flux-tubes
- Long flux tubes behave pretty much like thin strings
  - Energy increases with separation: $V \approx \sigma r, \;\; \sqrt{\sigma} \approx 440$ MeV
  - At some point the string breaks – String breaking –
  - We need dynamical fermions to observe string breaking
  - We work in pure gauge theory
- There are $\boldsymbol{D-2}$ massless Goldstone modes from broken translation invariance in the $\boldsymbol{D-2}$ directions
- There should be a Low Energy Effective String Theory model describing the energy spectrum of the flux tube
- Questions to be addressed
  - What is this effective string theory?
  - How good an approximation such an effective string theory is?
  - Are there additional massive excitations along the flux-tube?









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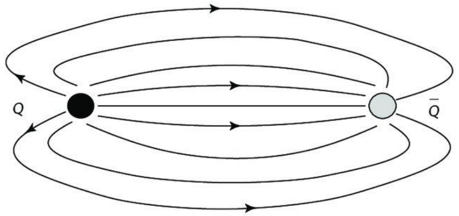

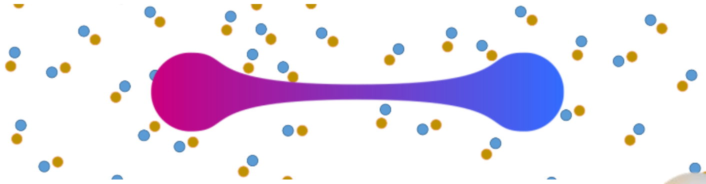

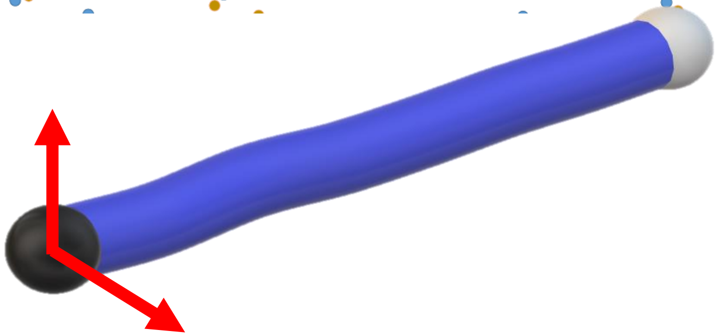

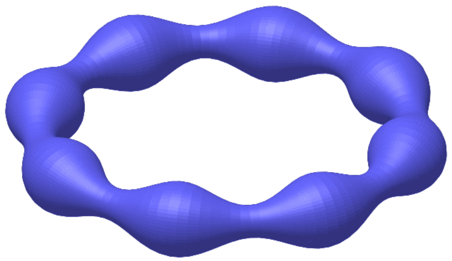

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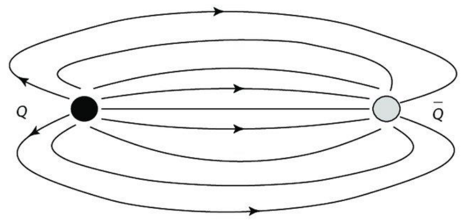

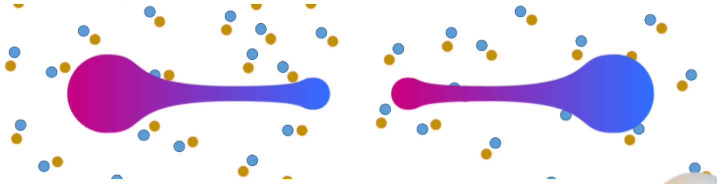

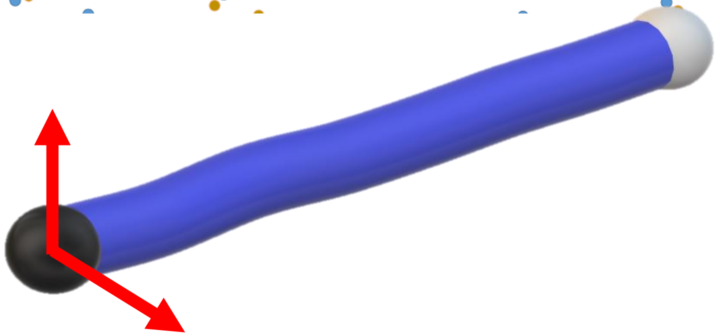

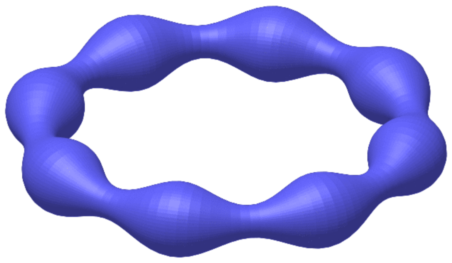

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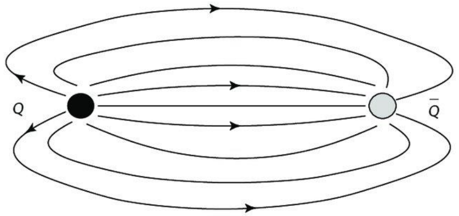

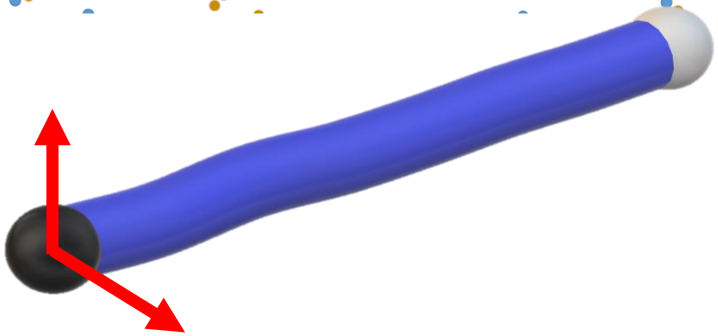

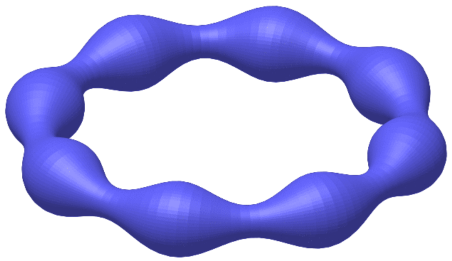

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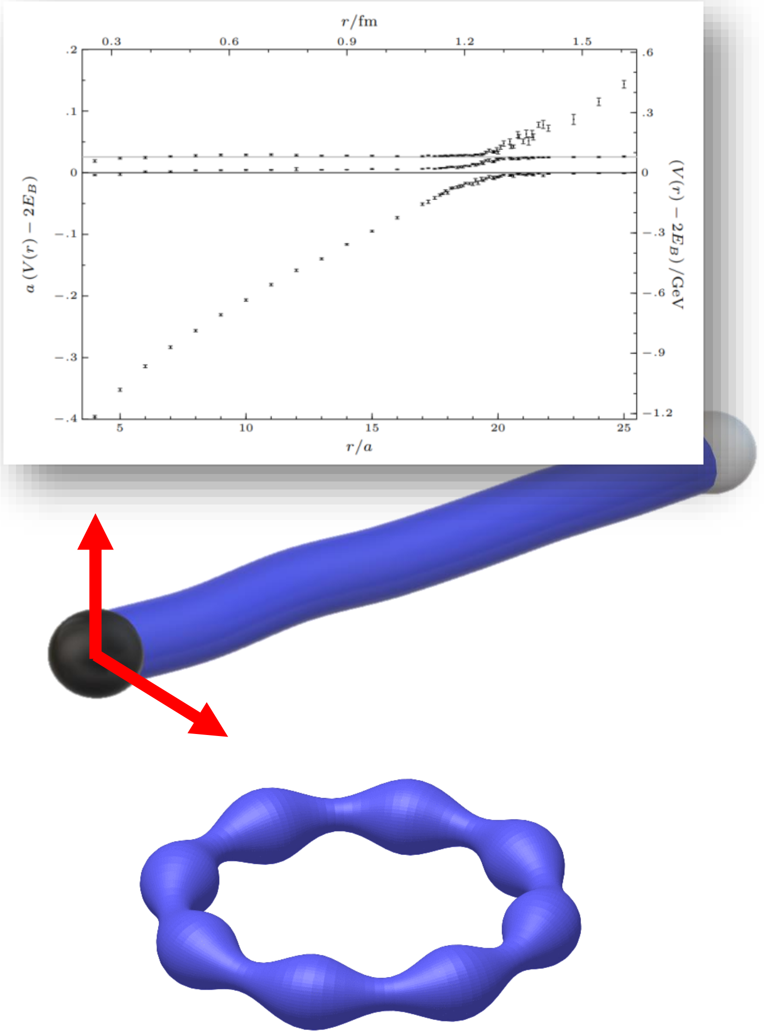
J. Bulava et al, Phys.Lett.B 793 (2019), 493-498



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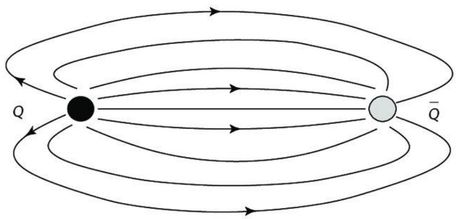
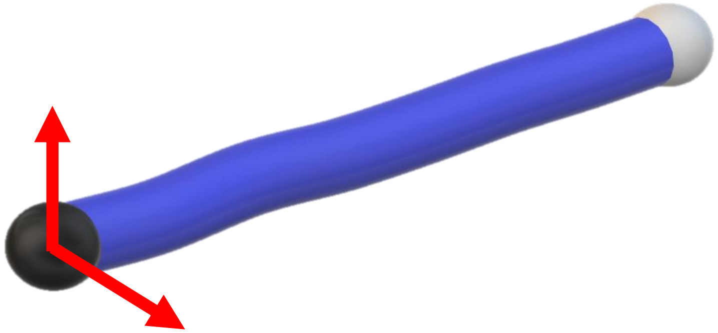
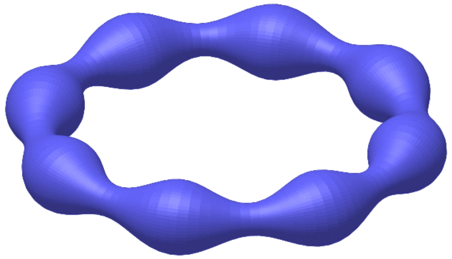
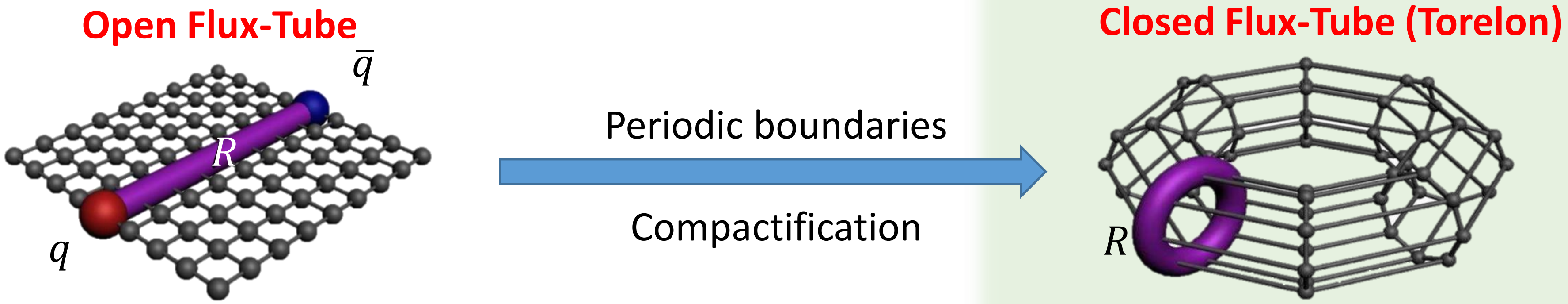






How to investigate

- Questions:
  - Is there a theoretical description in agreement with Lattice data for the flux-tube?
  - Is there a group of lattice data in striking disagreement with the theory?
  - What does this disagreement teach about the theory? Can it be extended?
- Choose a flux-tube set up:



[Caselle, Bicudo, Sharifian, Brandt, Kuti, AA]
- One needs to deal with the boundary terms
- Computationally not so expensive

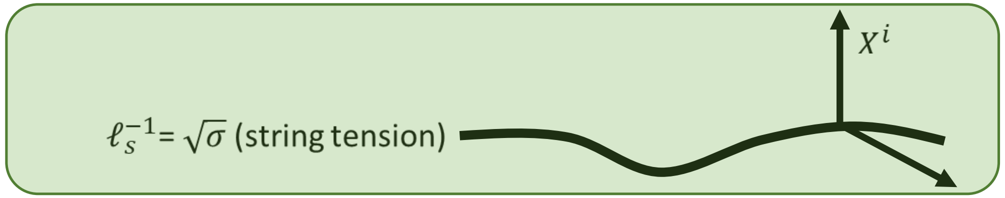
[Teper, AA, Caselle]
- No boundary terms
- Computationally expensive (length=lattice extent)
- Richer spectrum due to flux-compactification

- Effective string theories cannot capture pure gauge phenomena
- We extracted the spectrum of closed flux-tubes in the Large$-N$ limit

The effective string theory of long strings

- Universal properties of the QCD string studied extensively [Dubovsky, Gorbenko, Aharony]
- Re-parametrisation invariance and D-dimensional target space Poincaré symmetry

$$S = -\int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\,\left[\ell_s^{-2} + \mathcal{R} + a_2 K^2 + b_2 K^{\mu}_{\alpha\beta} K^{\alpha\beta}_{\mu} + O(\ell_s^2)\right]$$



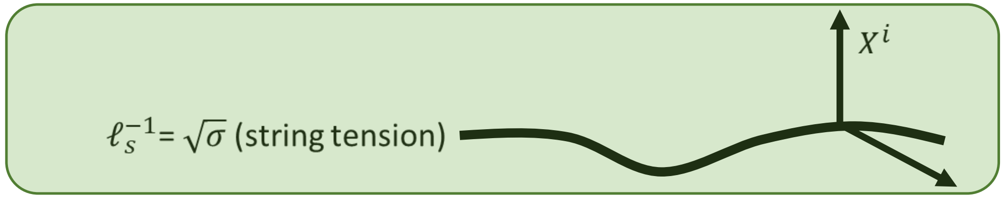
- First non-trivial subleading correction starts at ℓ_s^2 level
- Hence, the perturbative perspectives of the theory are universally determined by Nambu-Goto action

$$S_{\rm NG} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\,, \qquad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$$

The Goddard-Goldstone-Rebbi-Thorn spectrum

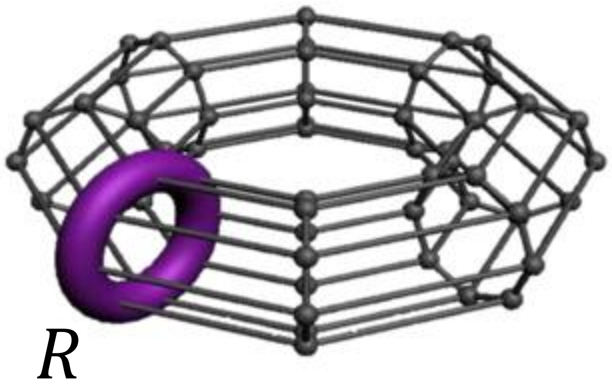
- We quantize the Closed bosonic string (Nambu-Goto)

$$S_{\rm NG} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}$$



- The spectrum of a closed bosonic string compactified around a torus is given by:

$$E_{\rm GGRT}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right) + \left(\frac{2\pi q}{R}\right)^2}$$



- The spectrum is described by

1. The winding momentum $p_{||} = 2\pi q/R$ with $q = 0, \pm 1, \pm 2, \ldots$
2. The total contribution of $N_L = \sum_{k>0} n_L(k)k$ and $N_R = \sum_{k'>0} n_R(k')k'$
3. Level matching Constrain: $N_L - N_R = q$

The expansion in $1/Rl_s^{-1}$

- Topic received contributions since the early 80s

[M. Lüscher ’81, J. Polchinski & A. Strominger ‘90, M. Lüscher & P. Weisz ’04, O. Aharony et al ’07 – 11, S. Dubovsky et al ’12 – 19]

$$\begin{aligned} E_n(R) &= l_s^{-1}(Rl_s^{-1}) \\ &+ \frac{4\pi l_s^{-1}}{(Rl_s^{-1})}\left(n - \frac{D-2}{24}\right) \\ &- \frac{8\pi^2 l_s^{-1}}{(Rl_s^{-1})^3}\left(n - \frac{D-2}{24}\right)^2 \\ &+ \frac{32\pi^3 l_s^{-1}}{(Rl_s^{-1})^5}\left(n - \frac{D-2}{24}\right)^3 + \left(\frac{1}{(Rl_s^{-1})^7}\right) \end{aligned}$$

Linear Confinement

Lüscher 1980, Polchinski&Strominger 1991

Lüscher&Weisz 2004, Drummond 2004

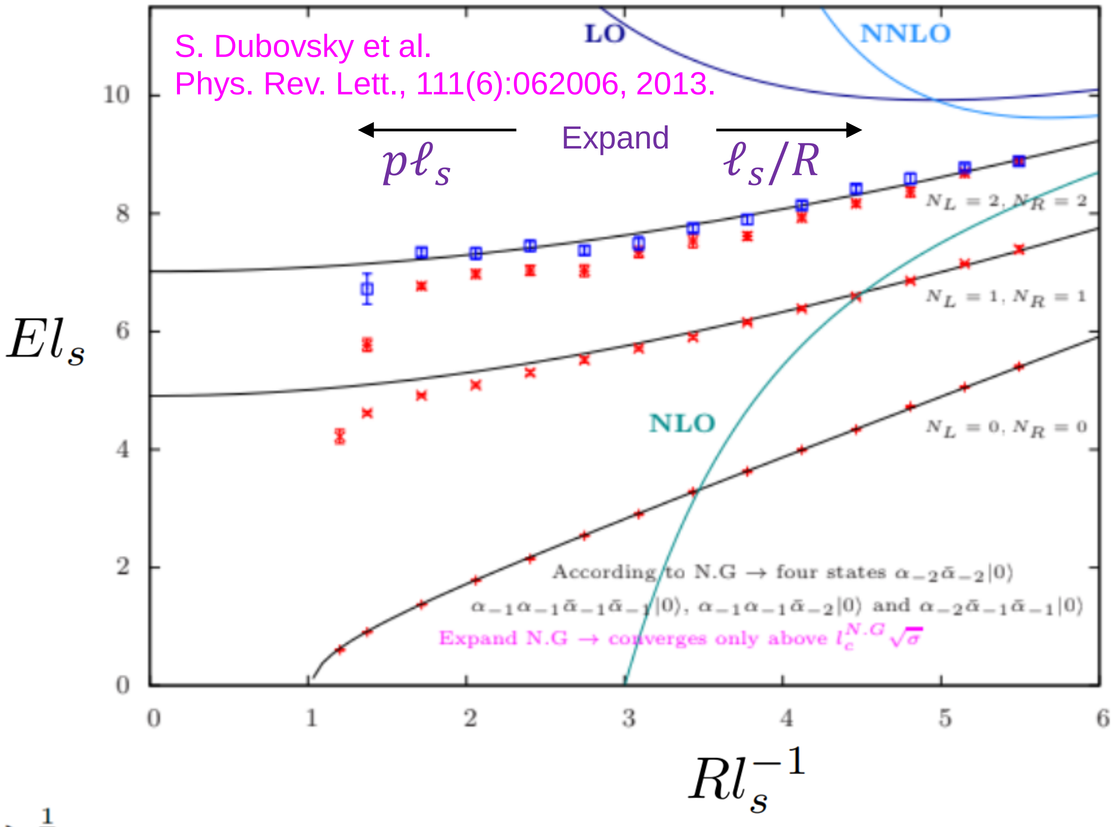
Aharony&Karzbrun 2009

- Relation to the GGRT spectrum:

$$\begin{aligned} E_n(R) &= l_s^{-1}(Rl_s^{-1}) + \frac{l_s^{-1}c_1^{\mathrm{GGRT}}}{(Rl_s^{-1})} + \frac{l_s^{-1}c_2^{\mathrm{GGRT}}}{(Rl_s^{-1})^3} + \frac{l_s^{-1}c_3^{\mathrm{GGRT}}}{(Rl_s^{-1})^5} + \left(\frac{1}{(Rl_s^{-1})^7}\right) \\ &= E_{\mathrm{GGRT}}(N_L = 0, N_R = 0, R) + \left(\frac{1}{(Rl_s^{-1})^7}\right) \end{aligned}$$

The expansion in $1/Rl_s^{-1}$ - the $D = 2 + 1$ case

N_L, N_R	q	$P_\perp$	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$



Expansion valid for $Rl_s^{-1} >> R_C^{\mathrm{GGRT}} l_s^{-1} = \left\{8\pi\left(n - \frac{1}{12}\right)\right\}^{\frac{1}{2}}$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

- In $D = 2 + 1$ only string like states have been observed so far
- $D = 3 + 1$ case is more complex and interesting to look for such states

Spectrum from world-sheet scattering

- Thermodynamic Bethe Ansatz (TBA)

⟶ finite volume spectrum of a $(1+1)-D$ integrable theory from 2 → 2 scattering

- Leading spectrum is given by integrable theory of $D-2$ scalars with phase shift:

$$e^{2i\delta(s)} = e^{is\ell_s^2/4}$$

- Using TBA leads to the GGRT spectrum
- Phonon scattering amplitudes can be calculated with perturbation theory
- Diagonalization of the S −matrix:

$$2\delta_{sym} = \frac{\ell_s^2 s}{4} - \frac{11\ell_s^4 s^2}{192\pi} + \frac{A-B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 2^{\pm}$$

$$2\delta_{anti} = \frac{\ell_s^2 s}{4} + \frac{11\ell_s^4 s^2}{192\pi} - \frac{A+B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 0^{-}$$

$$2\delta_{sing} = \frac{\ell_s^2 s}{4} + \frac{11\ell_s^4 s^2}{192\pi} + \frac{3A-B}{256\pi^2}\ell_s^6 s^3 + O(s^4) \quad \longleftarrow \quad 0^{+}$$

Spectrum from world-sheet scattering

- TBA formulates with the generalized quantization condition around the circle:

$$p_{li}R + \sum_j 2\delta_{a_i a_j}\left(p_{li}, p_{rj}\right) - i\sum_b \int_0^\infty \frac{dq}{2\pi}\frac{d2\delta_{a_i b}\left(ip_{li}, q\right)}{dq}\ln\left(1 - e^{-R\epsilon_r^b(q)}\right) = 2\pi N_{li}\,,$$

$$p_{ri}R + \sum_j 2\delta_{a_j a_i}\left(p_{ri}, p_{lj}\right) + i\sum_b \int_0^\infty \frac{dq}{2\pi}\frac{d2\delta_{b a_i}\left(-ip_{ri}, q\right)}{dq}\ln\left(1 - e^{-R\epsilon_l^b(q)}\right) = 2\pi N_{ri}\,,$$

- The pseudo-energies $\epsilon^a_{l(r)}$ satisfy:

$$\epsilon_l^a(q) = q + \frac{i}{R}\sum_i 2\delta_{ab_i}\left(q, -ip_{ri}\right) + \frac{1}{2\pi R}\sum_b \int_0^\infty dq' \frac{d2\delta_{ab}\left(q, q'\right)}{dq'}\ln\left(1 - e^{-R\epsilon_r^b(q')}\right)$$

$$\epsilon_r^a(q) = q - \frac{i}{R}\sum_i 2\delta_{b_i a}\left(q, ip_{li}\right) + \frac{1}{2\pi R}\sum_b \int_0^\infty dq' \frac{d2\delta_{ba}\left(q, q'\right)}{dq'}\ln\left(1 - e^{-R\epsilon_l^b(q')}\right)$$

- The energy of a state is written as

$$\Delta E = \sum_i p_{li} + \sum_i p_{ri} + \frac{1}{2\pi}\sum_a \int_0^\infty dq\ln\left(1 - e^{-R\epsilon_l^a(q)}\right) + \frac{1}{2\pi}\sum_a \int_0^\infty dq\ln\left(1 - e^{-R\epsilon_r^a(q)}\right)$$

- + Approximations

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ABA

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$$\epsilon_r^a(q)=q-\frac{i}{R}\sum_i 2\delta_{b_ia}\left(q,ip_{li}\right)+\frac{1}{2\pi R}\sum_b\int_0^\infty dq'\frac{d2\delta_{ba}\left(q,q'\right)}{dq'}\ln\left(1-e^{-R\epsilon_l^b(q')}\right)$$

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$$\Delta E=\sum_i p_{li}+\sum_i p_{ri}+\frac{1}{2\pi}\sum_a\int_0^\infty dq\ln\left(1-e^{-R\epsilon_l^a(q)}\right)+\frac{1}{2\pi}\sum_a\int_0^\infty dq\ln\left(1-e^{-R\epsilon_r^a(q)}\right)$$

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Spectrum from world-sheet scattering

- $T\bar{T}$ simplifies TBA because the undressed theory is free in the leading order

$$E(R,\ell_s) = \frac{1}{\mathcal{R}_0}\left(R + \frac{\ell_s^2}{2}E(R,\ell_s)\right)E\left(\mathcal{R}_0,0\right) + \frac{\ell_s^2}{2\mathcal{R}_0}P(R)P\left(\mathcal{R}_0\right)$$

- We neglect all the winding corrections in the undressed spectrum
- Momentum Quantization Condition becomes the Asymptotic Bethe Ansatz (ABA)
- We can investigate the spectrum of phonons – massive excitations using ABA + $T\bar{T}$ deformations. [Chen, Conkey, Dubovsky, Hernández-Chifflet 2018]

- Recipe
  - Start with a world-sheet theory of free phonons (phonons can interact at subleading order in low energy limit) and massive particles.
  - Compute the finite volume spectrum of this theory using ABA.
  - Deform the theory by $T\bar{T}$ operator to the string scale, which will automatically incorporate the axion-interaction at leading order.

Theory – The Axionic String Ansatz (ASA)

- Lattice calculations demonstrate that there is one massive resonance
- We add a massive resonance [Dubovsky et al 2013]

$$S_a = \int d^2\sigma \sqrt{-h} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{Q_\phi}{4} h^{\alpha\beta}\epsilon_{\mu\nu\lambda\rho}\partial_\alpha t^{\mu\nu}\partial_\beta t^{\lambda\rho}\phi \right)$$

- ϕ is a pseudoscalar – the world-sheet axion $t^{\mu\nu} = \frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}\partial_\alpha X^\mu \partial_\beta X^\nu$
- From Monte-Carlo data of $4D\ SU(3)$ Yang-Mills: $Q_\phi \approx 0.38 \pm 0.04, \quad m \approx 1.85^{+0.02}_{-0.03}\ell_s^{-1}$
- Integrable coupling: $Q_{\text{integrable}} = \sqrt{7/(16\pi)} \approx 0.373$
- Can we describe all states in $D = 3 + 1$ with the worldsheet fields only?

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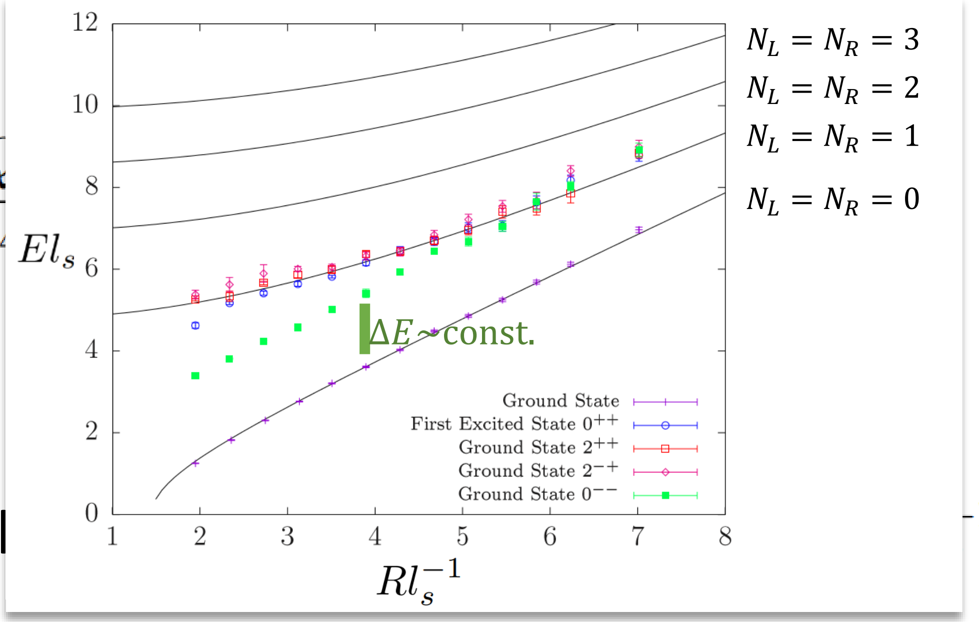

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- From Monte-Carlo data of $4D$ $SU(3)$ Yang-Mill

-1

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- Lattice calculations demonstrate that there is one massive resonance
- We add a massive resonance [Dubovsky et al 2013]

$$S_a = \int d^2\sigma \sqrt{-h} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{Q_\phi}{4} h^{\alpha\beta} \epsilon_{\mu\nu\lambda\rho} \partial_\alpha t^{\mu\nu} \partial_\beta t^{\lambda\rho} \phi \right)$$

- ϕ is a pseudoscalar – the world-sheet axion $t^{\mu\nu} = \frac{\epsilon^{\alpha\beta}}{\sqrt{-h}} \partial_\alpha X^\mu \partial_\beta X^\nu$

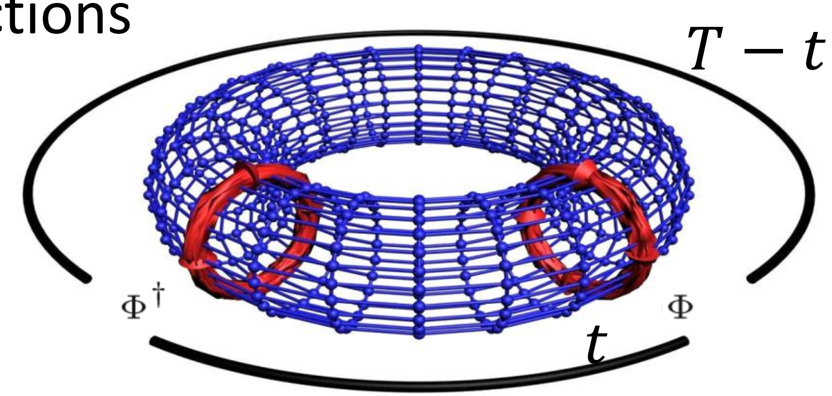
- From Monte-Carlo data of $4D$ $SU(3)$ Yang-Mills: $Q_\phi \approx 0.38 \pm 0.04, \quad m \approx 1.85^{+0.02}_{-0.03} \ell_s^{-1}$

- Integrable coupling: $Q_{\text{integrable}} = \sqrt{7/(16\pi)} \approx 0.373$

- Can we describe all states in $D = 3 + 1$ with the worldsheet fields only?

Lattice Calculation in $D = 3 + 1$ – Extracting the spectrum

- The spectrum is extracted using torelon correlation functions



$$\langle \phi^\dagger(t = an_t)\phi(0)\rangle = \langle \phi^\dagger e^{-Han_t}\phi\rangle = \sum_i |c_i|^2 e^{-aE_i n_t}$$

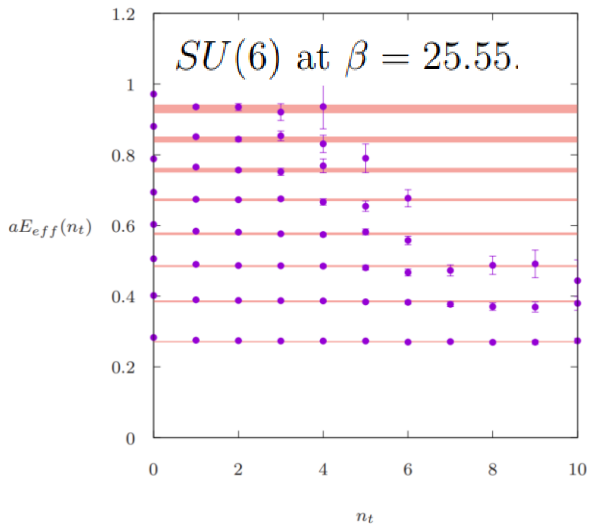
$$\stackrel{t\to\infty}{=} |c_0|^2 e^{-aE_0 n_t}$$

- Using the effective mass

Provides the ground state

$$\lim_{t\to\infty}\left[-\ln\left(\frac{C(t)}{C(t-a)}\right)\right] = aE_0$$

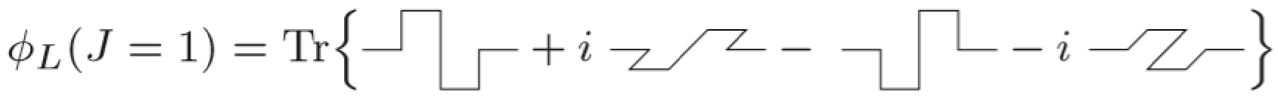
Example of effective masses →

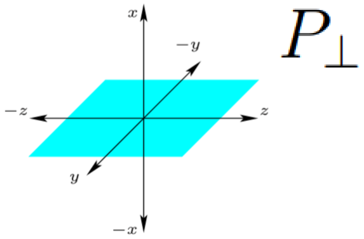


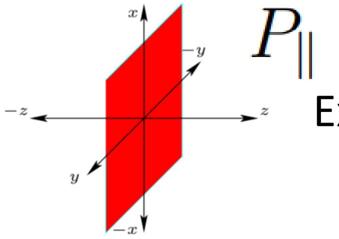
- For excitation spectrum we use GEVP

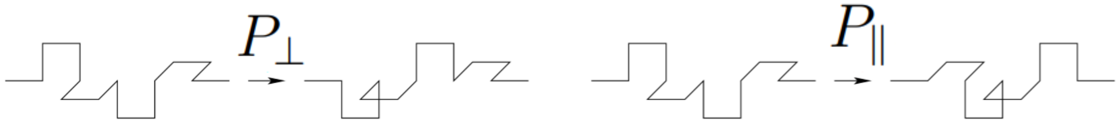
Lattice Calculation in $D = 3 + 1$ – The Quantum Numbers

Quantum numbers

- Spin J: $\phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$ Example 

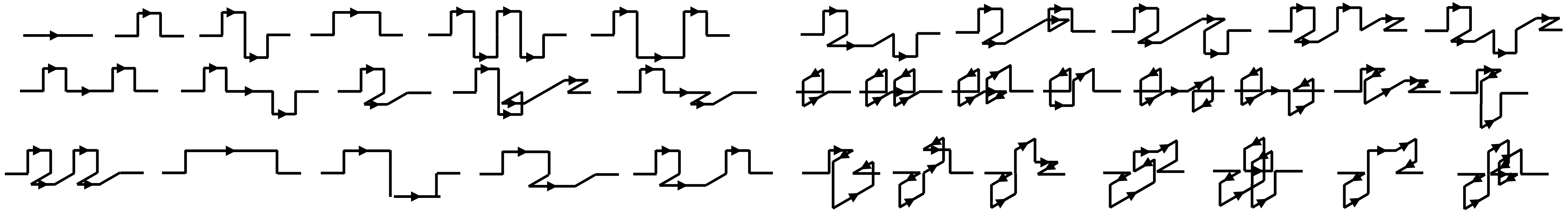
- Parity: 



Example 

We build operators described by the quantum numbers of $J, P_\perp, P_\parallel$

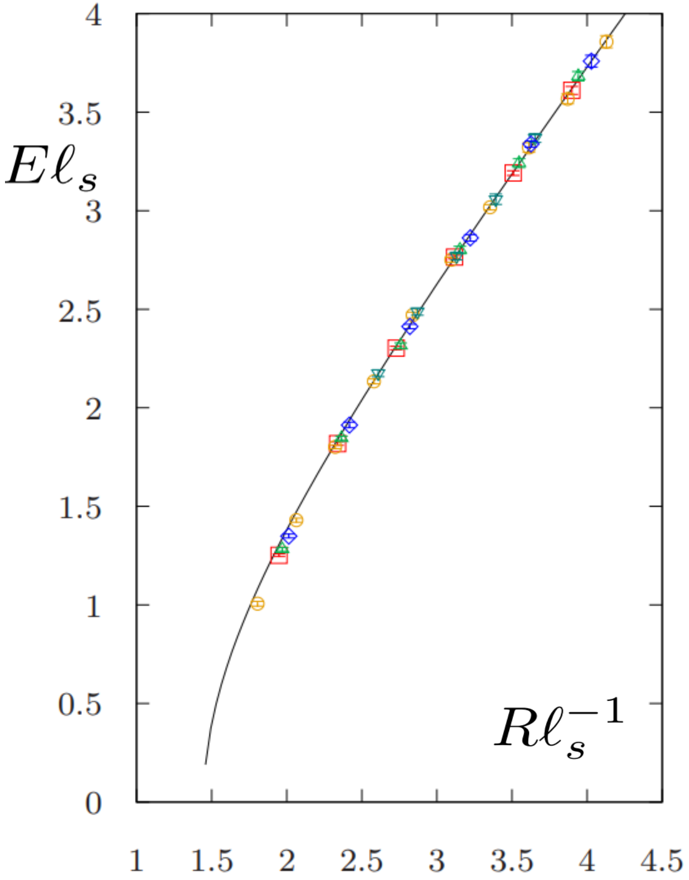
- We use a large basis of operators with transverse deformations:



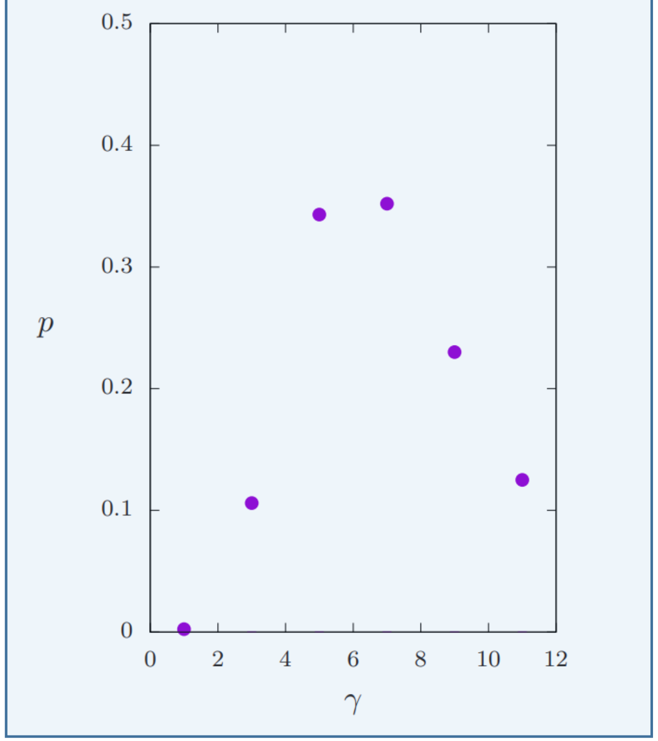
Results: the absolute ground state $N_L = N_R = 0$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0\rangle$
	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0\rangle$
	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	$-$	$-$	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
	2	$+$	$+$	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	$-$	$+$	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	$+$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	$+$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	$-$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	$-$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	3	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	3	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	4	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$
	4	$-$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$

$$E_{\text{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2 (N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$



$$E(R) = E_{\text{GGRT}}(R) - \frac{\tilde{c}\, l_s^{-1}}{(R l_s^{-1})^{\gamma}}$$

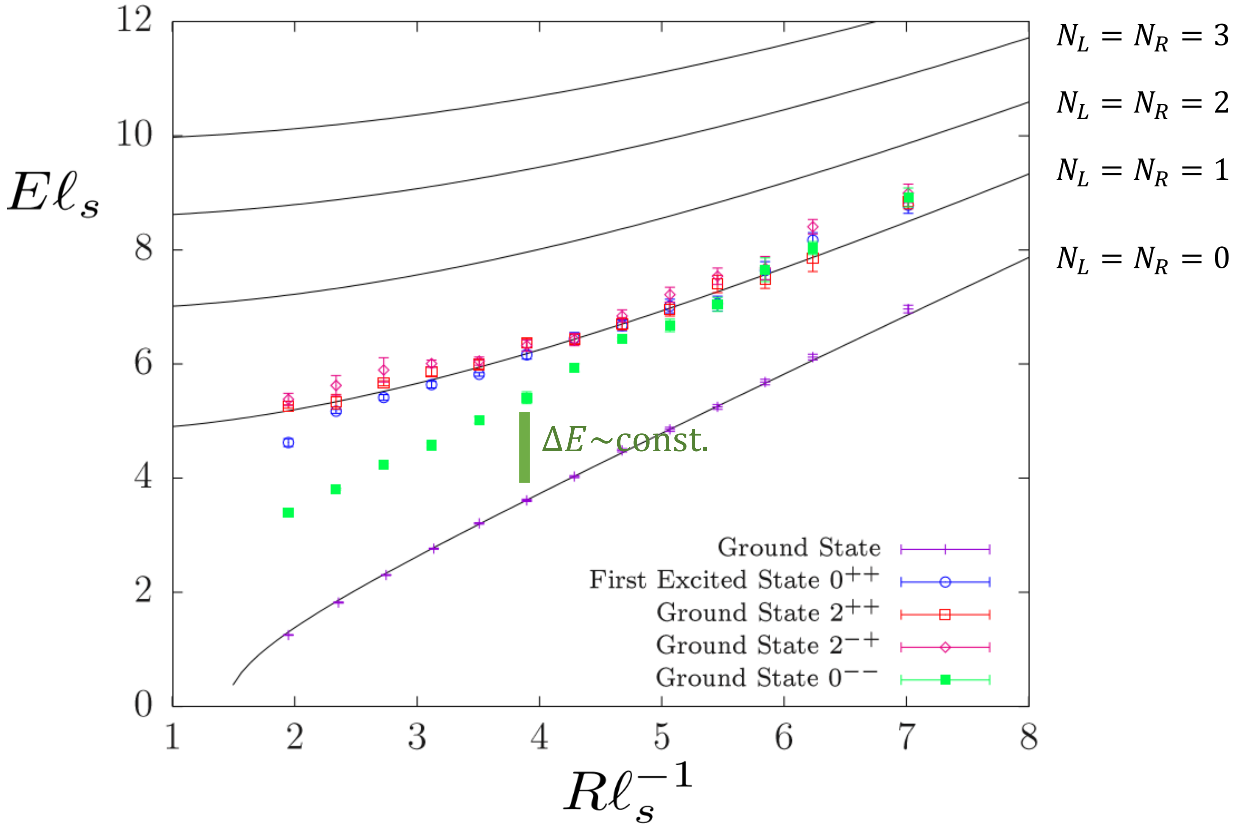


Results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.0625$

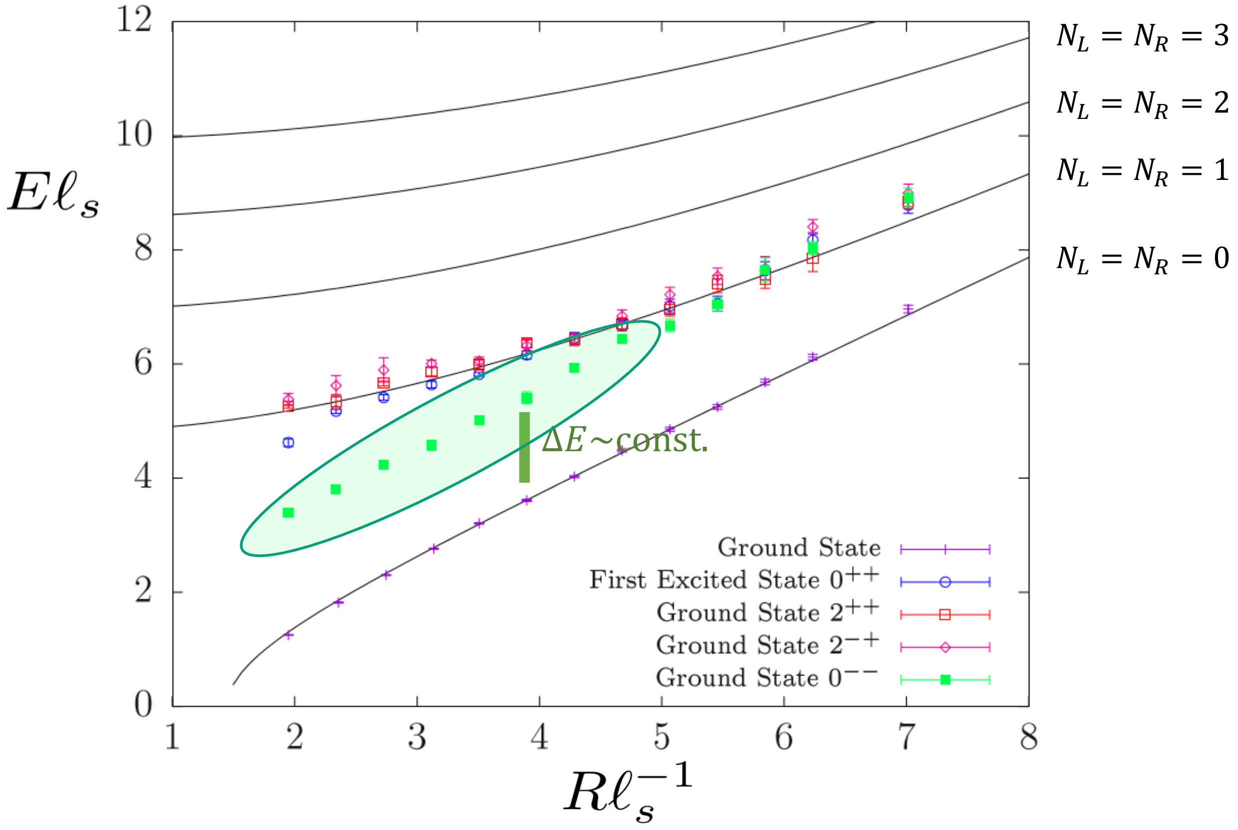


Results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.0625$

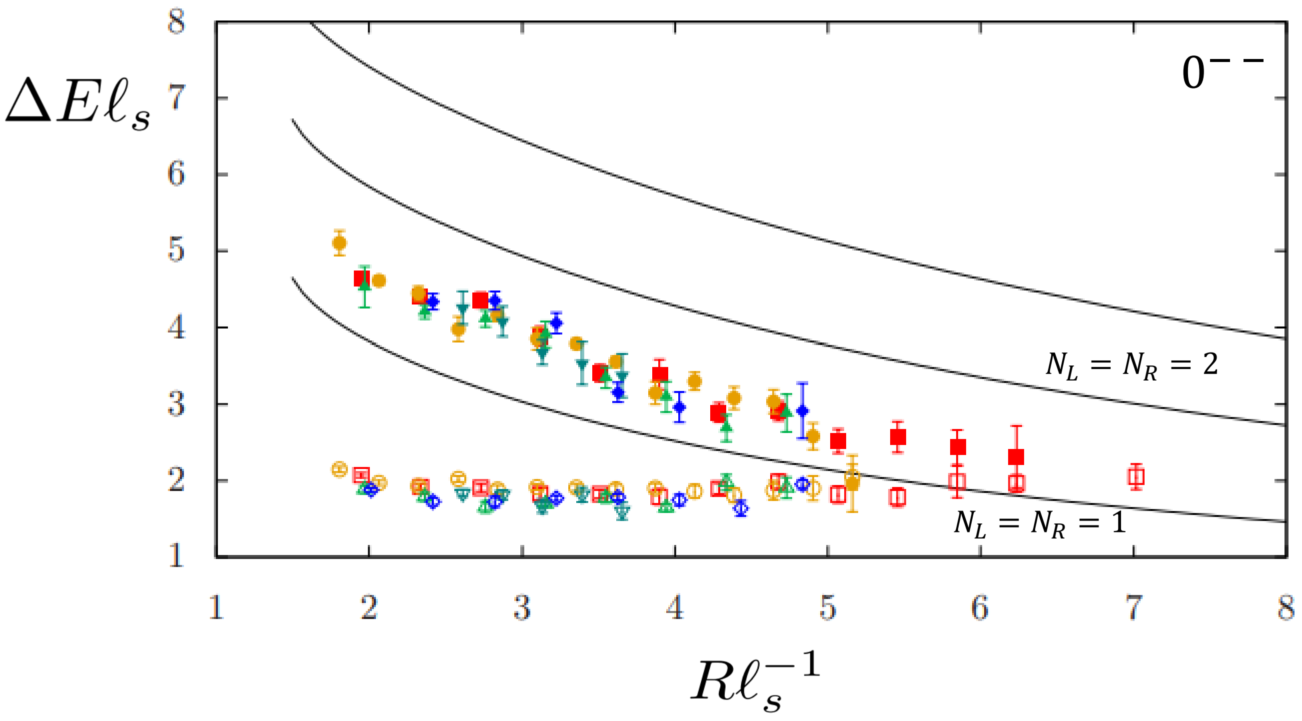


Results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	$\pm$	+	$[(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	1	$\pm$	−	$[(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	1	$\pm$	+	$[(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-)]\lvert 0\rangle$
	1	$\pm$	−	$[(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-)]\lvert 0\rangle$
	2	+	+	$(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-)\lvert 0\rangle$
	2	−	+	$(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-)\lvert 0\rangle$
	2	+	+	$[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	2	+	−	$[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	2	−	+	$[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) + (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	2	−	−	$[(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+) - (a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	3	$\pm$	+	$[(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-)]\lvert 0\rangle$
	3	$\pm$	−	$[(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-)]\lvert 0\rangle$
	4	+	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for all the ensembles



$SU(3), \beta = 6.0625$ $SU(5), \beta = 17.63$ $SU(6), \beta = 25.55$

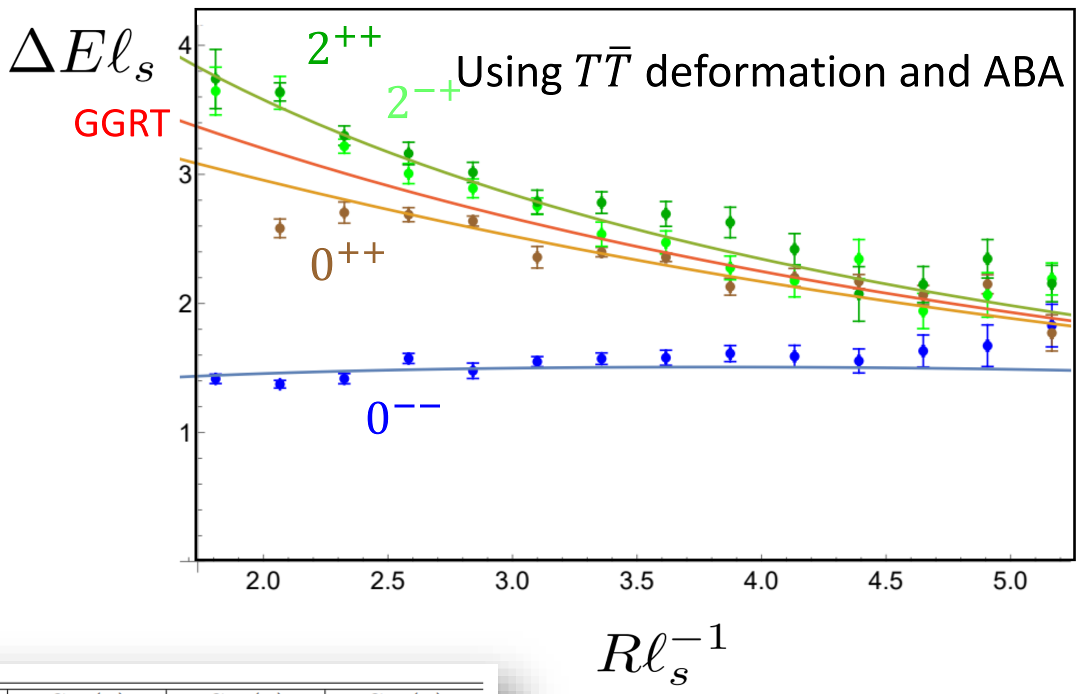
$SU(3), \beta = 6.338$ $SU(5), \beta = 18.375$

Results: the GGRT state with $N_L = N_R = 1$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\text{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



	$SU(3)$	$SU(5)$	$SU(6)$
2^{++}			
$m\ell_s$	1.812(16)	1.647(23)	1.653(17)
Q_ϕ	0.377(7)	0.387(10)	0.385(10)
2^{+-}			
$m\ell_s$	1.811(16)	1.648(23)	1.656(17)
Q_ϕ	0.354(6)	0.346(7)	0.337(10)

$$2\delta_{res}(p) = 2\sigma_2 \tan^{-1}\left(\frac{8Q_\phi^2 \ell_s^4 p^6}{m^2 - 4p^2}\right) + \sigma_1 \frac{8Q_\phi^2 \ell_s^4 p^6}{m^2 + 4p^2}$$

The Axionic States

N_L, N_R	$\|J\|$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\|0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\|0\rangle$
	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\|0\rangle$
	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\|0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\|0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\|0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\|0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\|0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\|0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\|0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\|0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\|0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\|0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\|0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\|0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\|0\rangle$
	4	−	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\|0\rangle$

N_L, N_R	$\|J\|$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\|0\rangle$
	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\|0\rangle$
	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\|0\rangle$
	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\|0\rangle$
	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\|0\rangle$
	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\|0\rangle$
	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\|0\rangle$

Operator that creates a massive excitation with momentum $\frac{2\pi k}{R}$

$$A_k \approx a_{k+1}^+ a_{-1}^- - a_{k+1}^- a_{-1}^+ \text{ for } k \geq 0.$$

N_L, N_R	$\|J\|$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0\|0\rangle$
$N_L = 2, N_R = 1$	0	−		$A_1\|0\rangle$
	1	±		$A_0\left(a_1^+ \mp a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\|0\rangle$
	0	−	−	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
	1	±	+	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\|0\rangle$
	1	±	−	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\|0\rangle$
	2	+	−	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\|0\rangle$
	2	−	−	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = 3, N_R = 1$	0	−		$A_2\|0\rangle$
	0	−		$A_0 a_1^+ a_1^-\|0\rangle$
	1	±		$A_1\left(a_1^+ \mp a_1^-\right)\|0\rangle$
	1	±		$A_0\left(a_2^+ \mp a_2^-\right)\|0\rangle$
	2	+		$A_0\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\|0\rangle$
	2	−		$A_0\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\|0\rangle$
$N_L = 3, N_R = 2$	0	+		$A_1 A_0\|0\rangle$
	0	−		$A_{-1} a_1^+ a_1^-\|0\rangle$
	0	−		$A_1\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
	0	−		$A_0\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 3, N_R = 3$	0	+	+	$A_1 A_{-1}\|0\rangle$
	0	−	−	$A_0 A_0 A_0\|0\rangle$
	0	+	+	$A_0\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\|0\rangle$
	0	−	−	$A_0\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\|0\rangle$
	0	−	+	$\left[A_{-1}\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) - A_1\left(a_{-2}^+ a_1^- + a_{-2}^- a_1^+\right)\right]\|0\rangle$
	0	−	−	$\left[A_{-1}\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) + A_1\left(a_{-2}^+ a_1^- + a_{-2}^- a_1^+\right)\right]\|0\rangle$
	0	−	+	$\left(A_2 a_{-1}^+ a_{-1}^- - A_{-2} a_1^+ a_1^-\right)\|0\rangle$
	0	−	−	$\left(A_2 a_{-1}^+ a_{-1}^- + A_{-2} a_1^+ a_1^-\right)\|0\rangle$

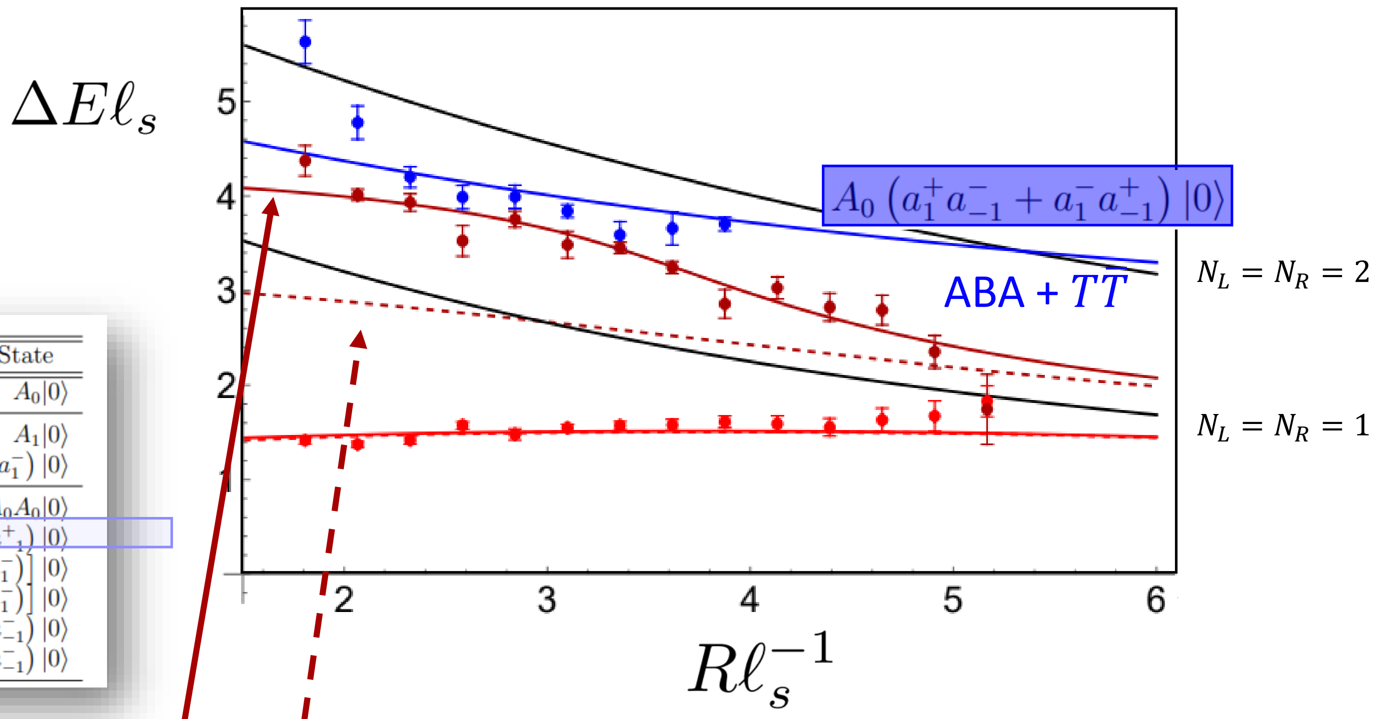
Results: the 0^{--} state and the axion

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0 \rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+) \lvert 0 \rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+) \lvert 0 \rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-) \lvert 0 \rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-) \lvert 0 \rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+) \lvert 0 \rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+) \lvert 0 \rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+) \lvert 0 \rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+) \lvert 0 \rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0 \rangle$
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-) \lvert 0 \rangle$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0 \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	−		$A_1 \lvert 0 \rangle$
	1	±		$A_0 (a_1^+ \mp a_1^-) \lvert 0 \rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0 \lvert 0 \rangle$
	0	−	−	$A_0 (a_1^+ a_{-1}^- + a_1^- a_{-1}^+) \lvert 0 \rangle$
	1	±	+	$[A_1 (a_{-1}^+ \mp a_{-1}^-) - A_{-1} (a_1^+ \mp a_1^-)] \lvert 0 \rangle$
	1	±	−	$[A_1 (a_{-1}^+ \mp a_{-1}^-) + A_{-1} (a_1^+ \mp a_1^-)] \lvert 0 \rangle$
	2	+	−	$A_0 (a_1^+ a_{-1}^+ - a_1^- a_{-1}^-) \lvert 0 \rangle$
	2	−	−	$A_0 (a_1^+ a_{-1}^+ + a_1^- a_{-1}^-) \lvert 0 \rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2 (N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



TBA with no high orders in s

TBA with higher orders $s^4 l_s^{-8}$ $\quad 2\delta_{anti}(s) = \frac{s}{4} + \frac{11 s^2}{192\pi} + a_3 s^3 + a_4 s^4 + \cdots$

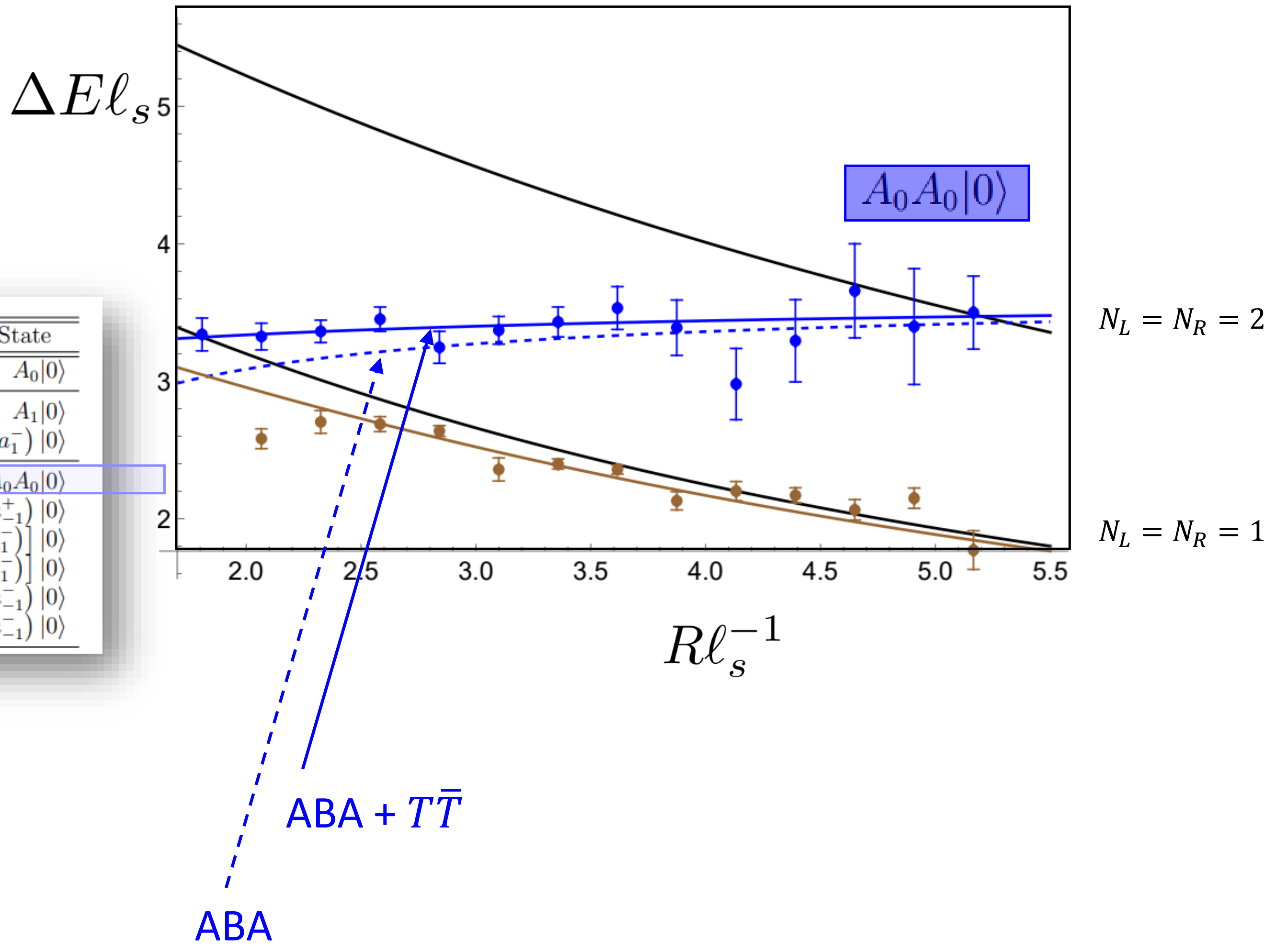
Results: the 0^{++} states and the axion

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1			
	1			
	1			
	1			
	2			
	2			
	2			
	2			
	2			
	2			
	3			
	3			
	4			
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	−	−	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$A_1\lvert 0\rangle$
	1	±		$A_0\,(a_1^+ \mp a_1^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\lvert 0\rangle$
	0	−	−	$A_0\,(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	1	±	+	$[A_1\,(a_{-1}^+ \mp a_{-1}^-) - A_{-1}\,(a_1^+ \mp a_1^-)]\lvert 0\rangle$
	1	±	−	$[A_1\,(a_{-1}^+ \mp a_{-1}^-) + A_{-1}\,(a_1^+ \mp a_1^-)]\lvert 0\rangle$
	2	+	−	$A_0\,(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	−	$A_0\,(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



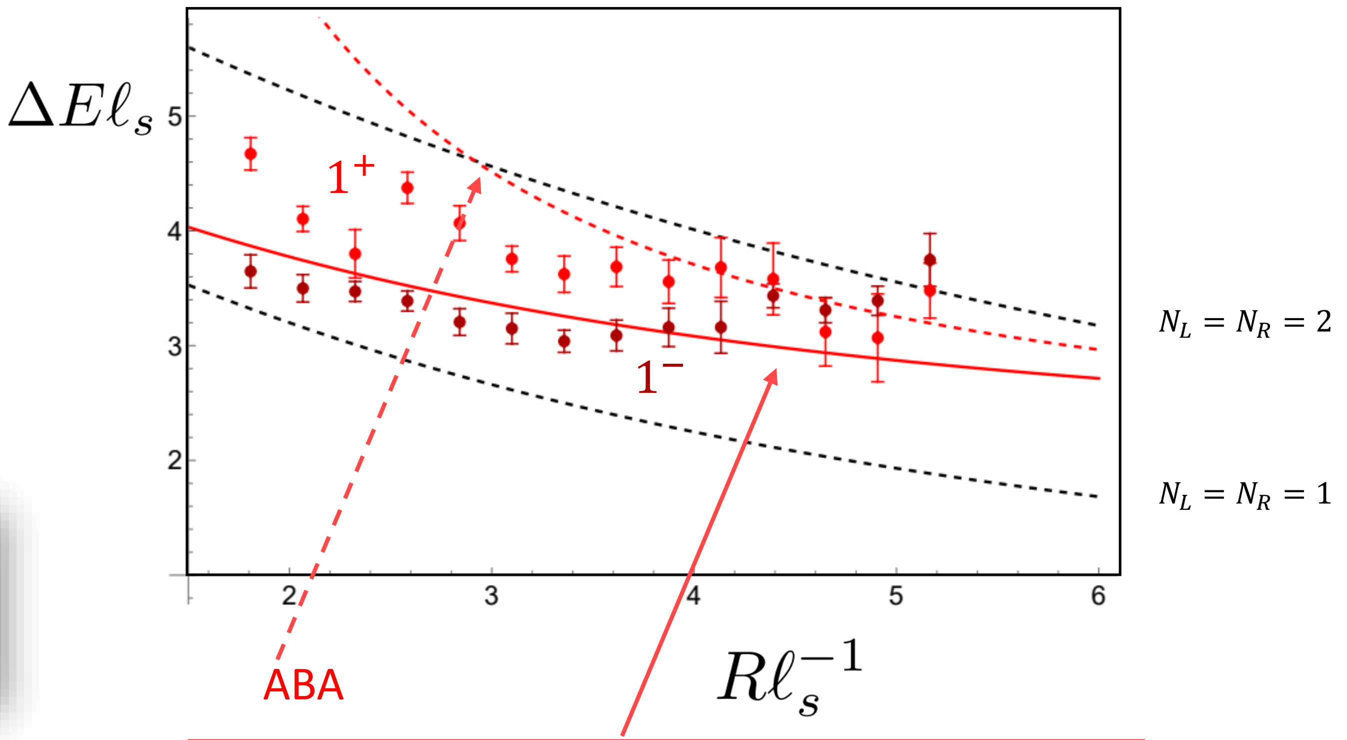
Results: the 1^+, 1^- ground states

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$(a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_{-1}^- - a_1^- a_{-1}^+)\lvert 0\rangle$
	2	+	+	$(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	+	$(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$(a_2^+ a_{-2}^- + a_2^- a_{-2}^+)\lvert 0\rangle$
	0	−	−	$(a_2^+ a_{-2}^- - a_2^- a_{-2}^+)\lvert 0\rangle$
	0	+	+	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	−	−	$(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$[(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	1	±	−	$[(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+) \pm (a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+)]\lvert 0\rangle$
	1	±	+	$[(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-)]\lvert 0\rangle$
	1	±	−	$[(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+) \pm (a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-)]\lvert 0\rangle$
	2	+	+	$(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-)\lvert 0\rangle$
	2	−	+	$(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-)\lvert 0\rangle$
	4	−	+	$(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-)\lvert 0\rangle$

$N_L = 2, N_R = 2$	0	+	+	$A_0 A_0\lvert 0\rangle$
	0	−	−	$A_0 (a_1^+ a_{-1}^- + a_1^- a_{-1}^+)\lvert 0\rangle$
	1	±	+	$[A_1 (a_{-1}^+ \mp a_{-1}^-) - A_{-1} (a_1^+ \mp a_1^-)]\lvert 0\rangle$
	1	±	−	$[A_1 (a_{-1}^+ \mp a_{-1}^-) + A_{-1} (a_1^+ \mp a_1^-)]\lvert 0\rangle$
	2	+	−	$A_0 (a_1^+ a_{-1}^+ - a_1^- a_{-1}^-)\lvert 0\rangle$
	2	−	−	$A_0 (a_1^+ a_{-1}^+ + a_1^- a_{-1}^-)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

Results for $SU(3)$ and $\beta = 6.338$



$$\left[(A_1 a_{-1}^+ \mp A_1 a_{-1}^-) \mp (A_{-1} a_1^+ \mp A_{-1} a_1^-)\right]\lvert 0\rangle$$

ABA + $T\overline{T}$

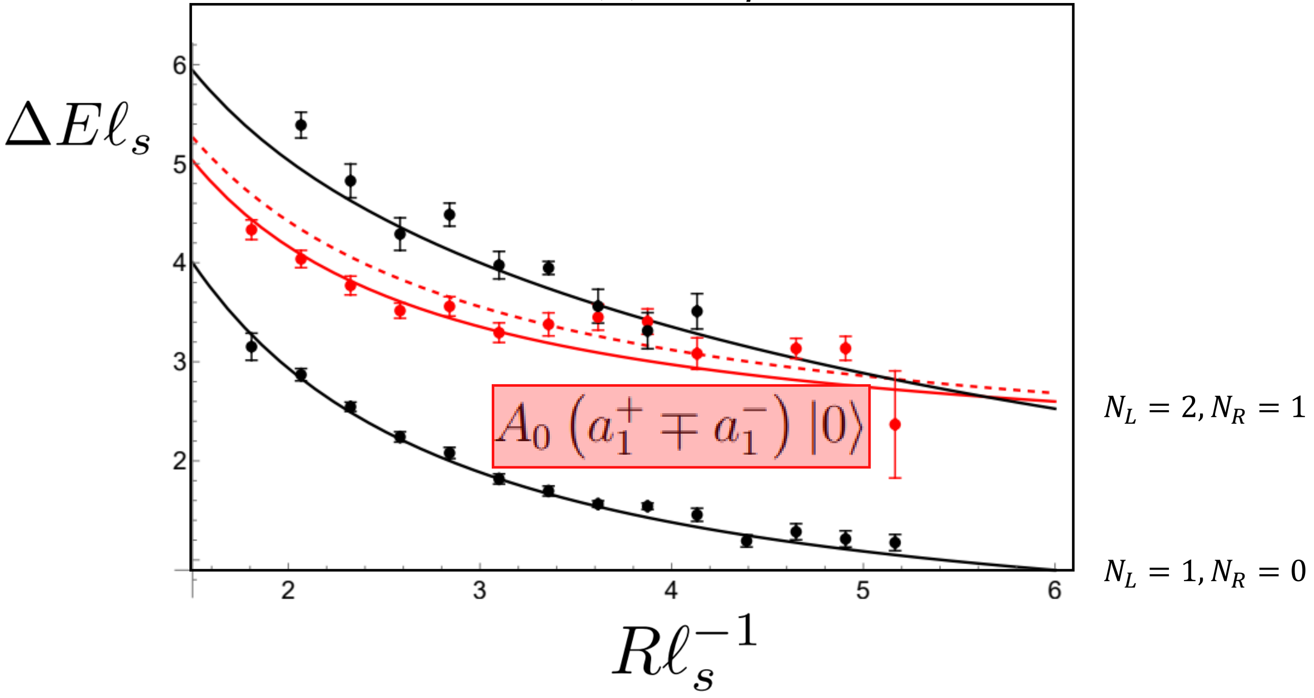
Results: The $J = 1, q = 1$ ground, 1^{st} and 2^{nd} excited states

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

N_L, N_R	$\lvert J\rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	$-$	$-$	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$A_1\lvert 0\rangle$
	1	$\pm$		$A_0\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
	0	$+$	$+$	$A_0 A_0\lvert 0\rangle$
	0	$-$	$-$	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
	1	$\pm$	$+$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	$\pm$	$-$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$	$-$	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right) + \left(\frac{2\pi q}{R}\right)^2}$$

Results for $SU(3)$ and $\beta = 6.338$



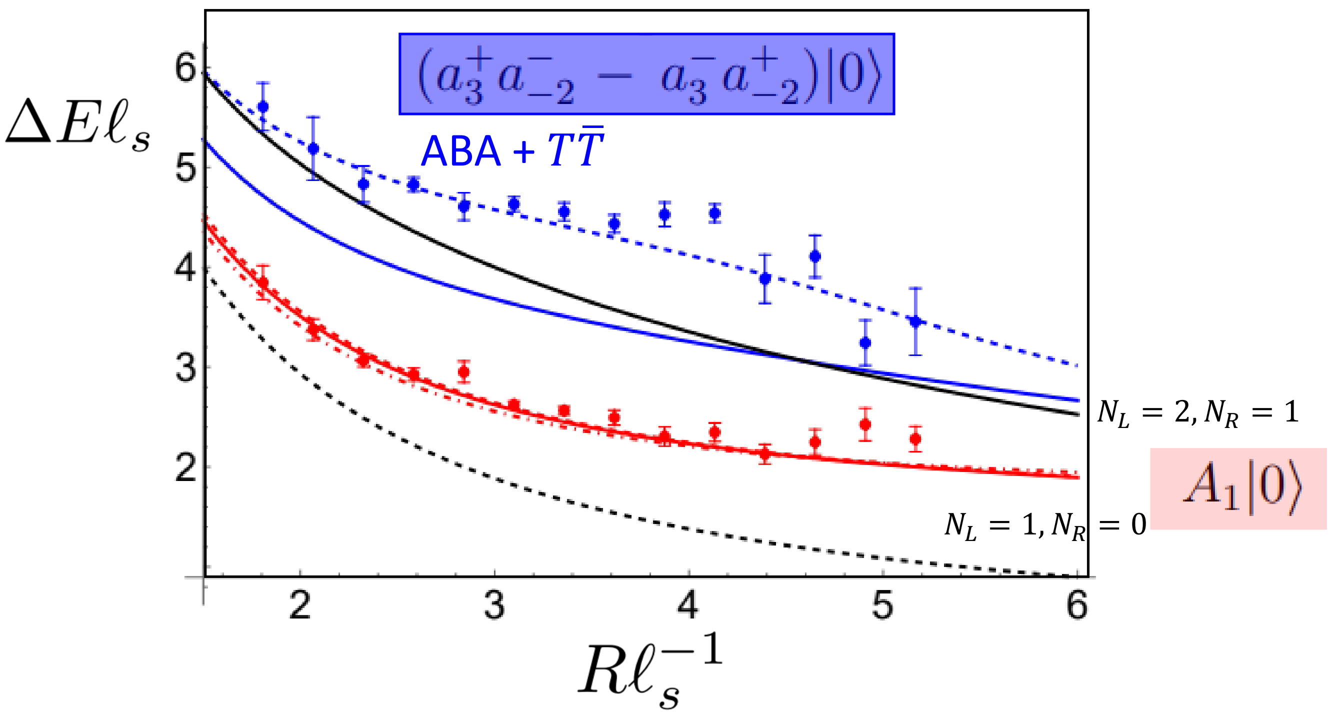
Results: The $0^-, q = 1$ ground and first excitations

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

N_L, N_R	$\lvert J \rvert$	$P_\perp$	$P_\parallel$	String State
$N_L = N_R = 1$	0	$-$	$-$	$A_0\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$A_1\lvert 0\rangle$
	1	$\pm$		$A_0\left(a_1^+ \mp a_1^-\right)\lvert 0\rangle$
	0	$+$	$+$	$A_0 A_0\lvert 0\rangle$
	0	$-$	$-$	$A_0\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	$\pm$	$+$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) - A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[A_1\left(a_{-1}^+ \mp a_{-1}^-\right) + A_{-1}\left(a_1^+ \mp a_1^-\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$A_0\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	2	$-$	$-$	$A_0\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$

$$E_{\mathrm{GGRT}}(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right) + \left(\frac{2\pi q}{R}\right)^2}$$

Results for $SU(3)$ and $\beta = 6.338$



Outlook

- Low-lying spectrum is given by an effective theory with one massive axion
- The interaction involving axions can be approximated by the $T\bar{T}$ deformation
- We observe two towers of states – (phonon) + (axions + phonon) states
- There is an axion in $4D$ $SU(N)$ Gauge Theories
- We have obtained the spectrum of closed flux-tubes in 4D $SU(N)$ Gauge Theories for all configurations of $\{J, P_{\perp}, P_{\parallel}\}$ using LGTs

- So far there is no effective string theory describing the open flux tube
- With collaborators we investigate the open flux tube [A. Sharifian poster]
- TBA Analysis is needed for the open flux tube

Thanks for your attention!!!