Introduction to topological data analysis for lattice field theory

Jeffrey Giansiracusa
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Durham University

Collaboration with:
Biagio Lucini, Tin Sulejmanpasic, Xavier Crean, Nick Sale

1. Homology

Counts holes / voids

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Counts holes / voids

2. Persistent homology

Shows how the counts evolve with a parameter

Counting holes

Counting holes

0

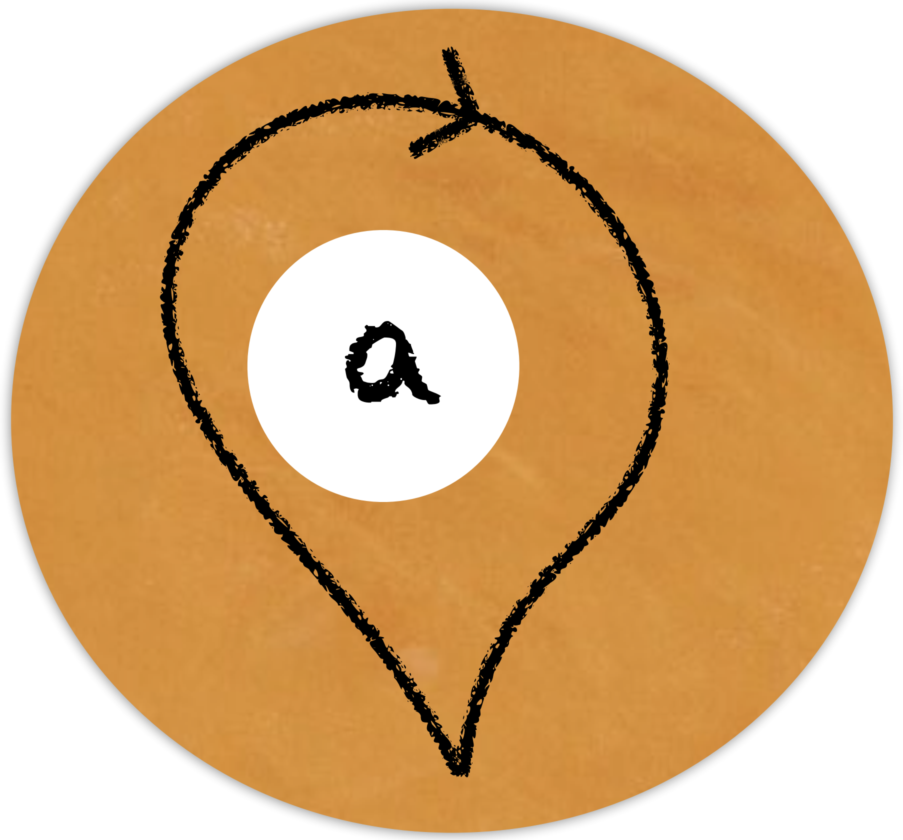
Counting holes

0

1

Counting holes

0



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Counting holes

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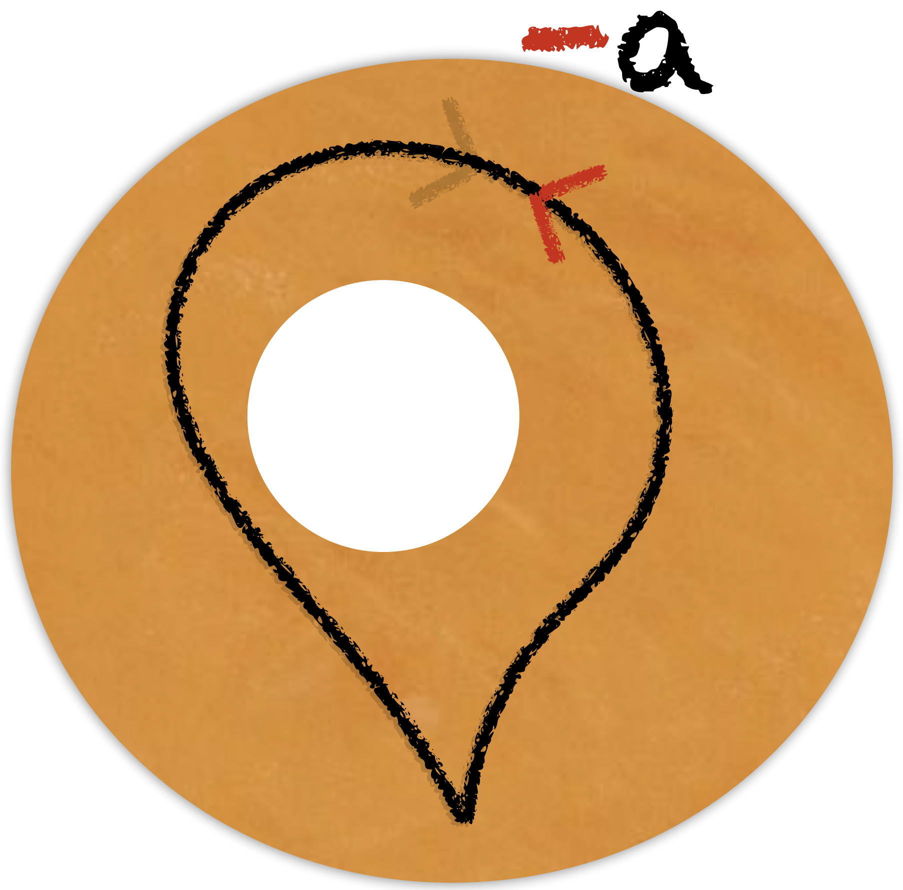

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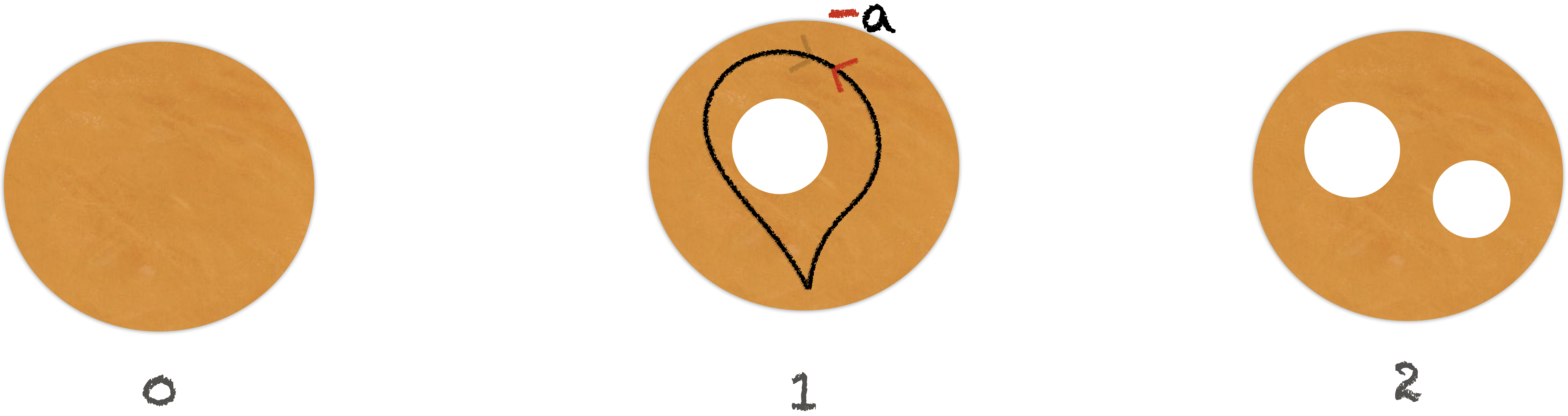
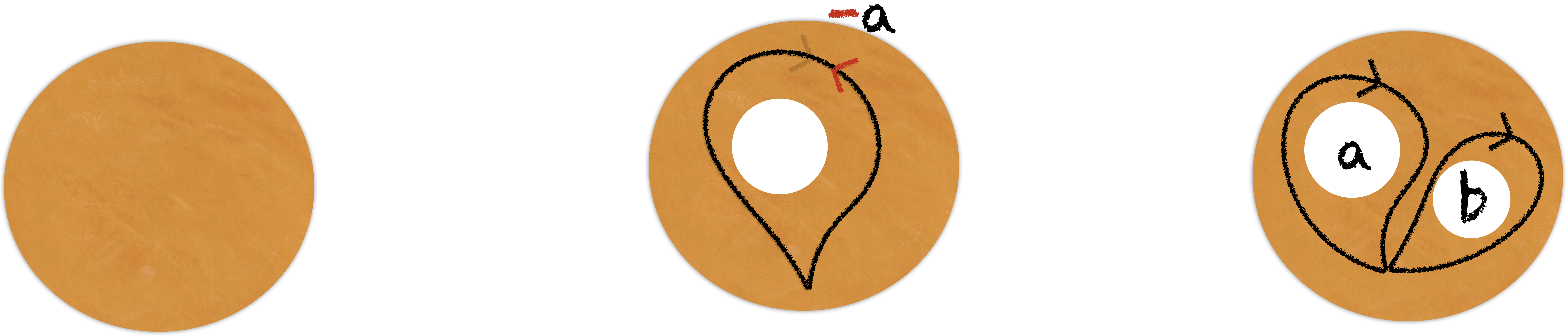
Counting holes



Counting holes



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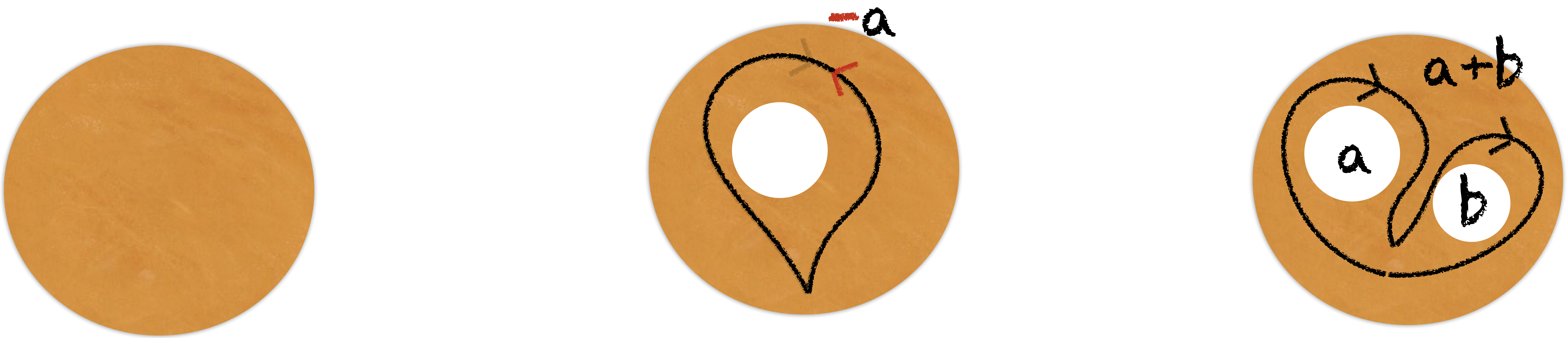
Counting holes



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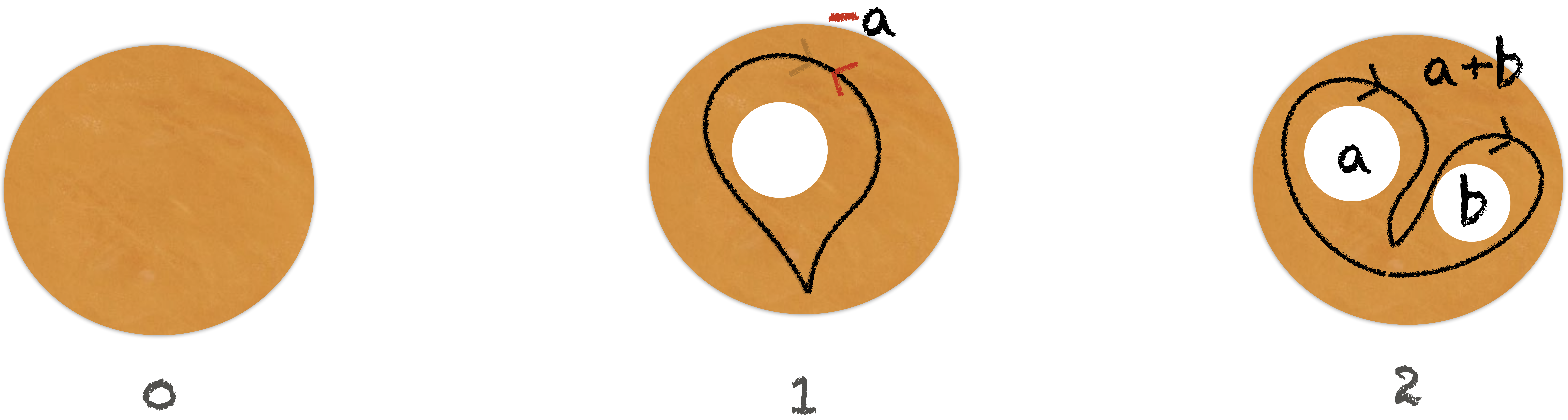
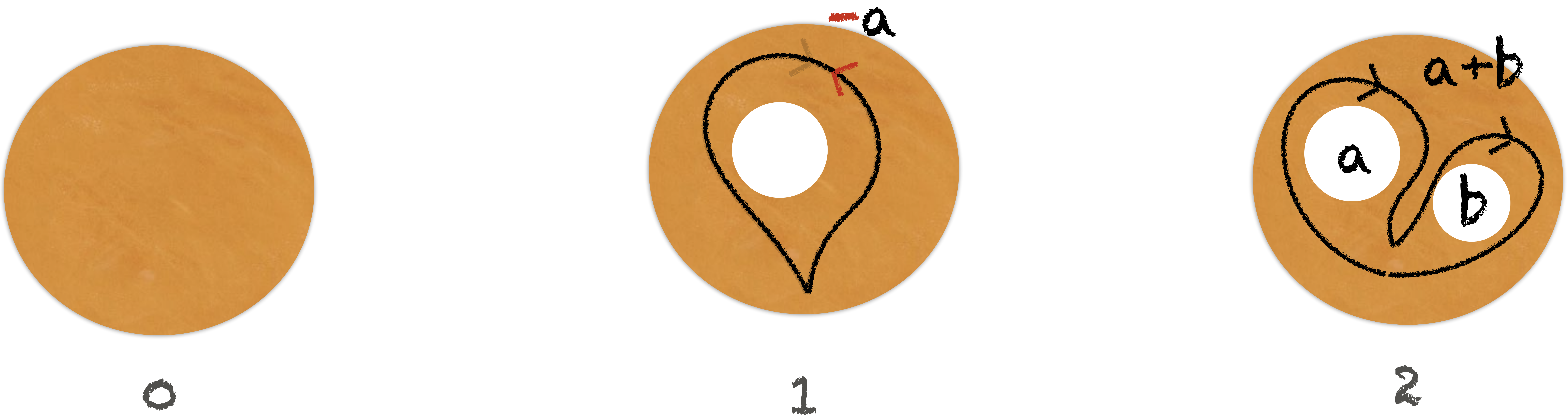
Counting holes



- Can do arithmetic with holes

Counting holes



- Can do arithmetic with holes
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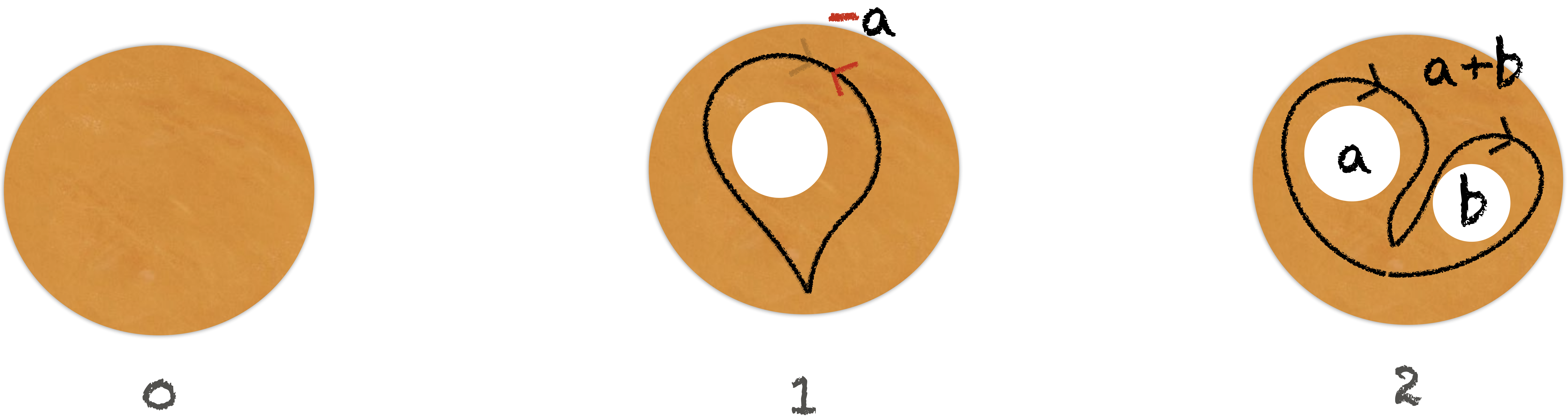

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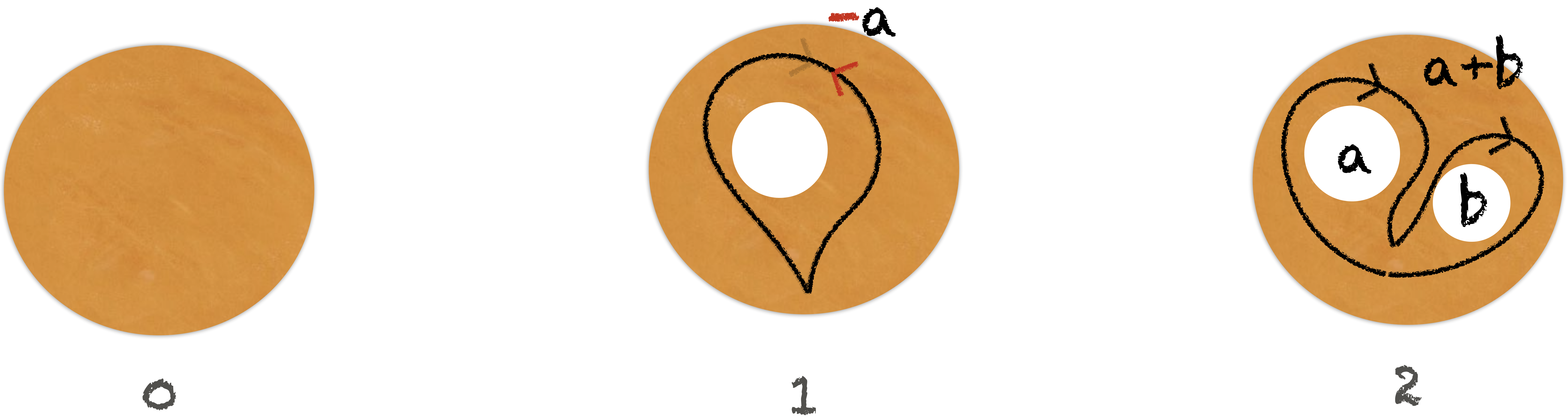

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Homology

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Output: vector spaces

$$H_0(X), \quad H_1(X), \quad H_2(X), \quad H_3(X), \ \ldots$$

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Connected components — Holes — Voids — Higher voids

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Connected components | Holes | Voids | Higher voids

Dimension = number of holes/voids

Homology

Input: a geometric object X ← What is X?

Output: vector spaces

$$H_0(X), \quad H_1(X), \quad H_2(X), \quad H_3(X), \quad \ldots$$

Connected components — Holes — Voids — Higher voids

Dimension = number of holes/voids

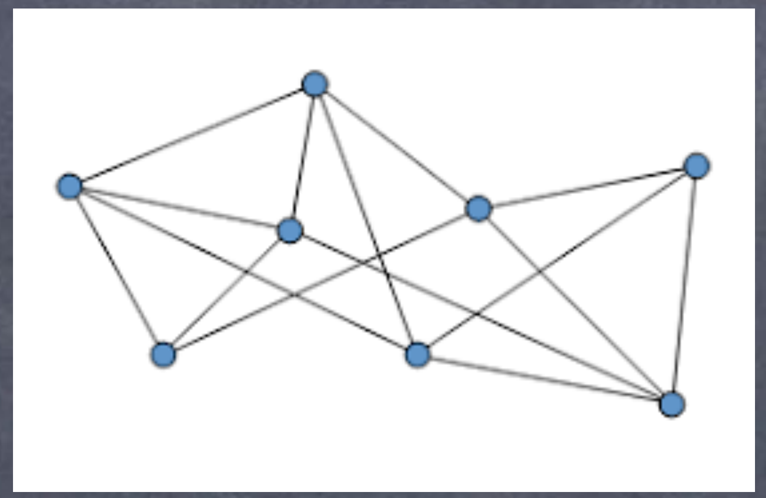
Examples

Dim H_0	
Dim H_1	
Dim H_2	
Dim H_3	

Examples

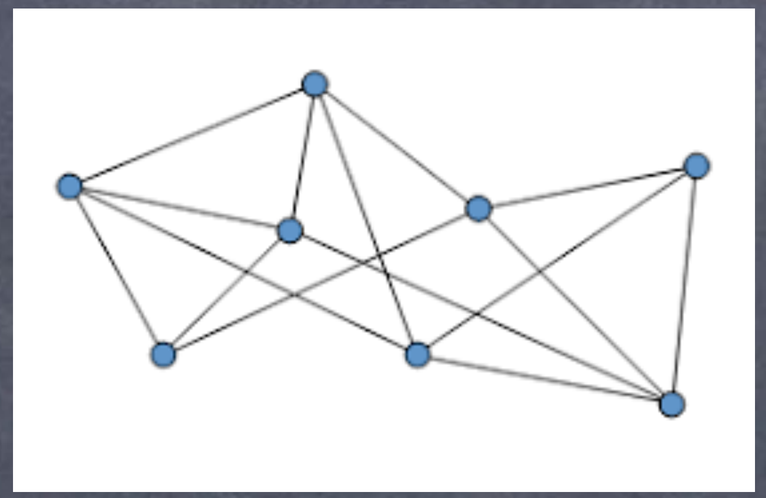
	Circle
	
Dim H_0	1
Dim H_1	1
Dim H_2	0
Dim H_3	$\vdots$

Examples

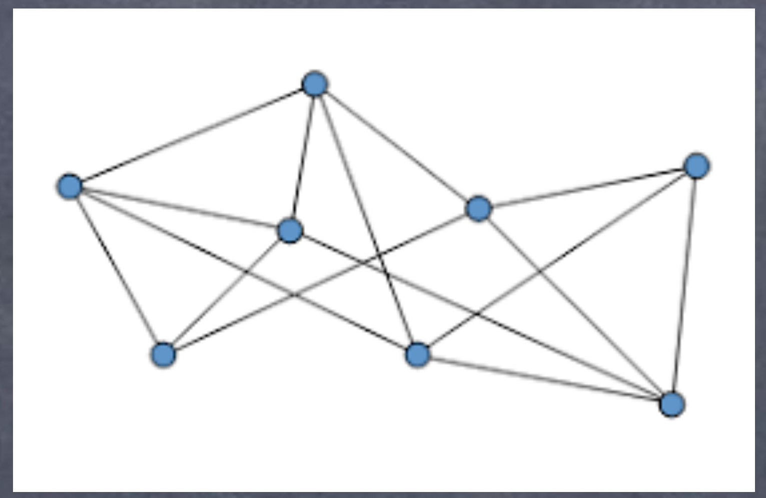


	Circle	Graph
Dim H_0	1	Number of components
Dim H_1	1	Number of loops
Dim H_2	0	0
Dim H_3	$\vdots$	$\vdots$

Examples

	Circle	Graph	Sphere
			
Dim H_0	1	Number of components	1
Dim H_1	1	Number of loops	0
Dim H_2	0	0	1
Dim H_3	$\vdots$	$\vdots$	0

Examples

	Circle	Graph	Sphere	surface of genus g
				
Dim H_0	1	Number of components	1	1
Dim H_1	1	Number of loops	0	2g
Dim H_2	0	0	1	1
Dim H_3	⋮	⋮	0	0

Given a statistical system on a lattice

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Boltzmann distribution
+ Monte Carlo
→ configurations $\phi_1, \phi_2, \phi_3, \ldots$

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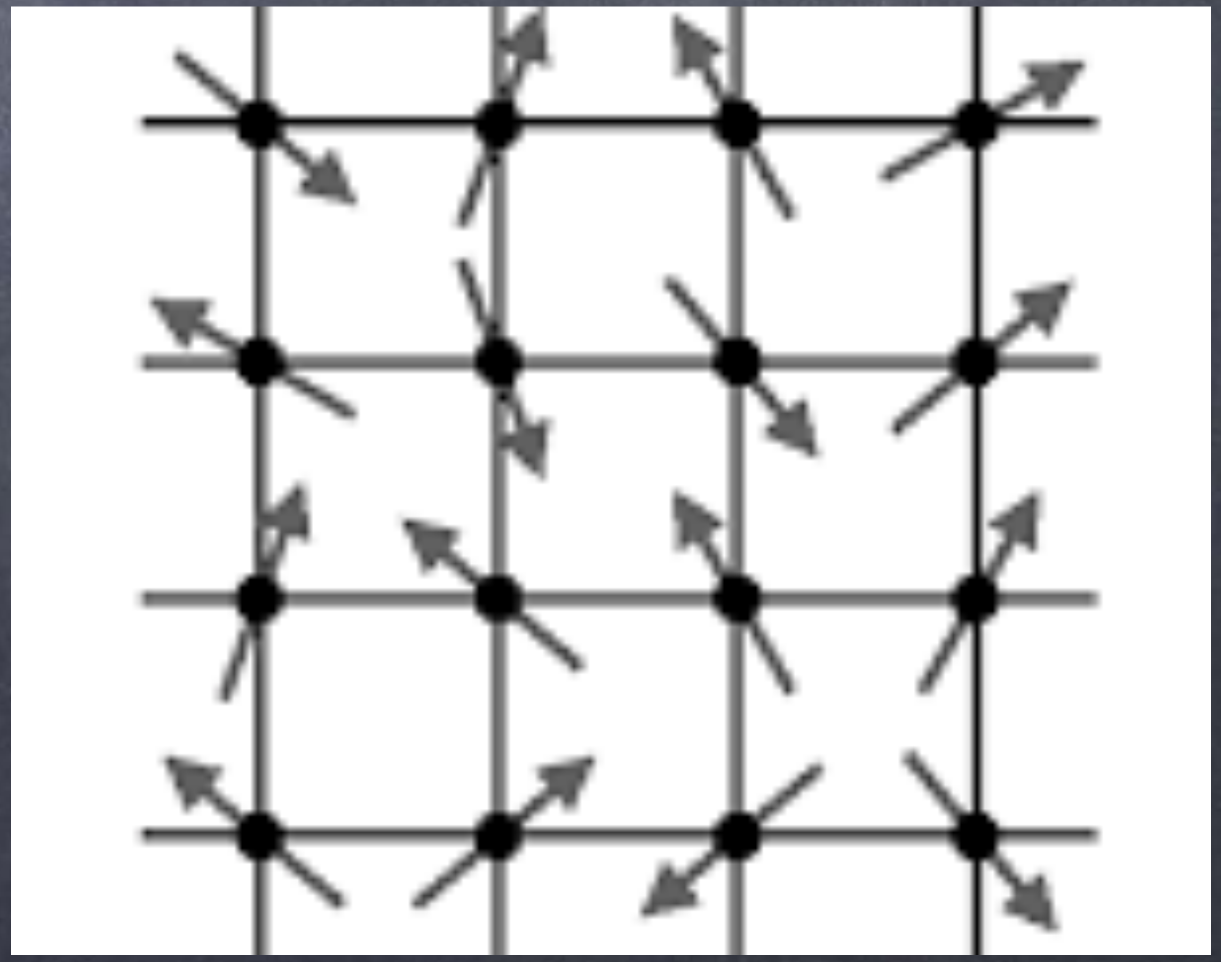
E.g., 2d XY-model on an $L \times L$ lattice

$\phi_i \in U(1)^{L^2}$



Methodology 1:

Quantify the topology of the (the dense part of) Boltzmann distribution on configuration space.

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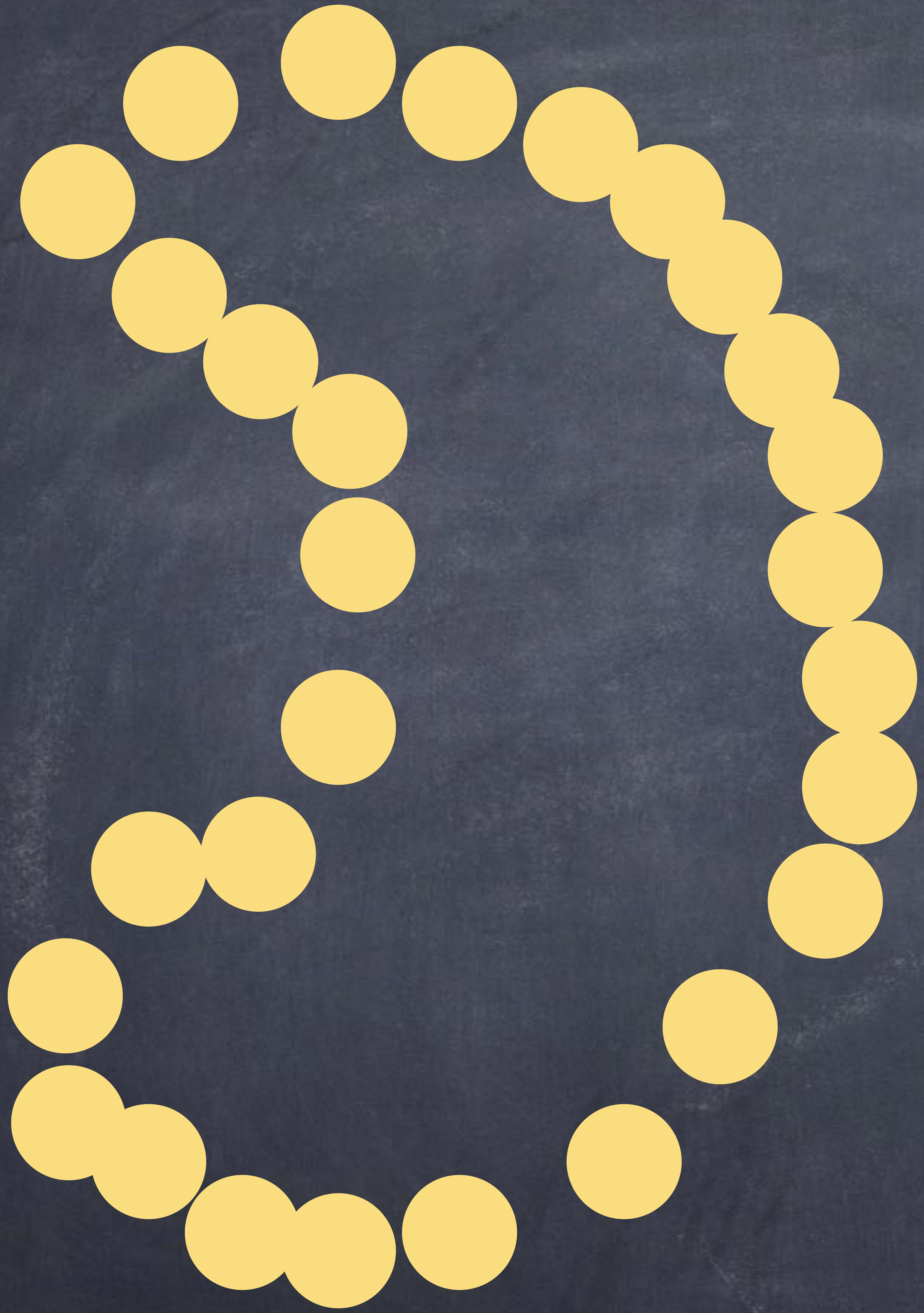
Choose a scale and compute homology $H_*(X)$



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Which scale?
Use them all!
Persistent homology

Methodology 2:

Use homology as an observable.

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Configuration ϕ

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Configuration ϕ

$$\downarrow$$

Geometric object $X(\phi)$

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Homology $H_*(X(\phi))$

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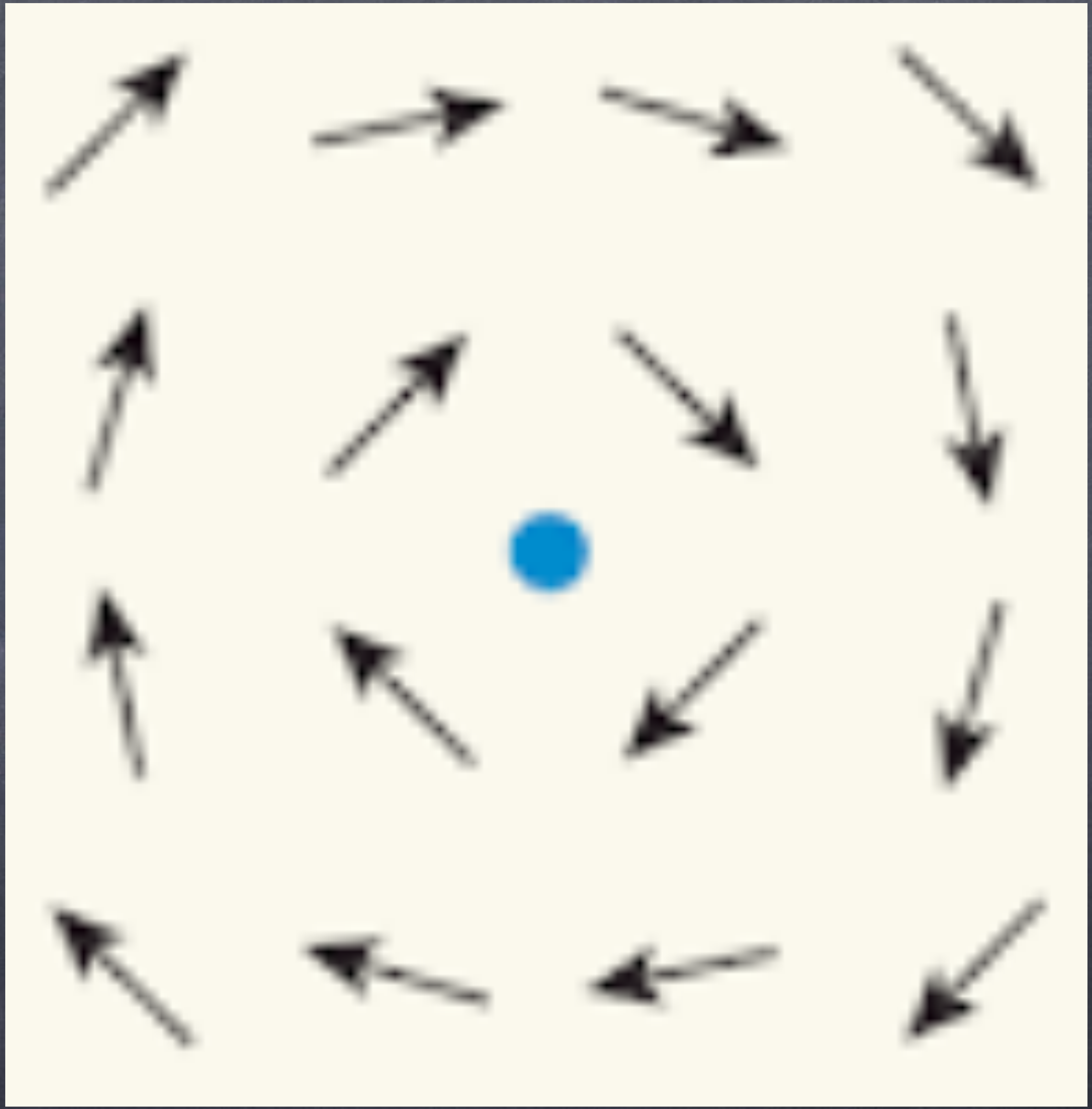


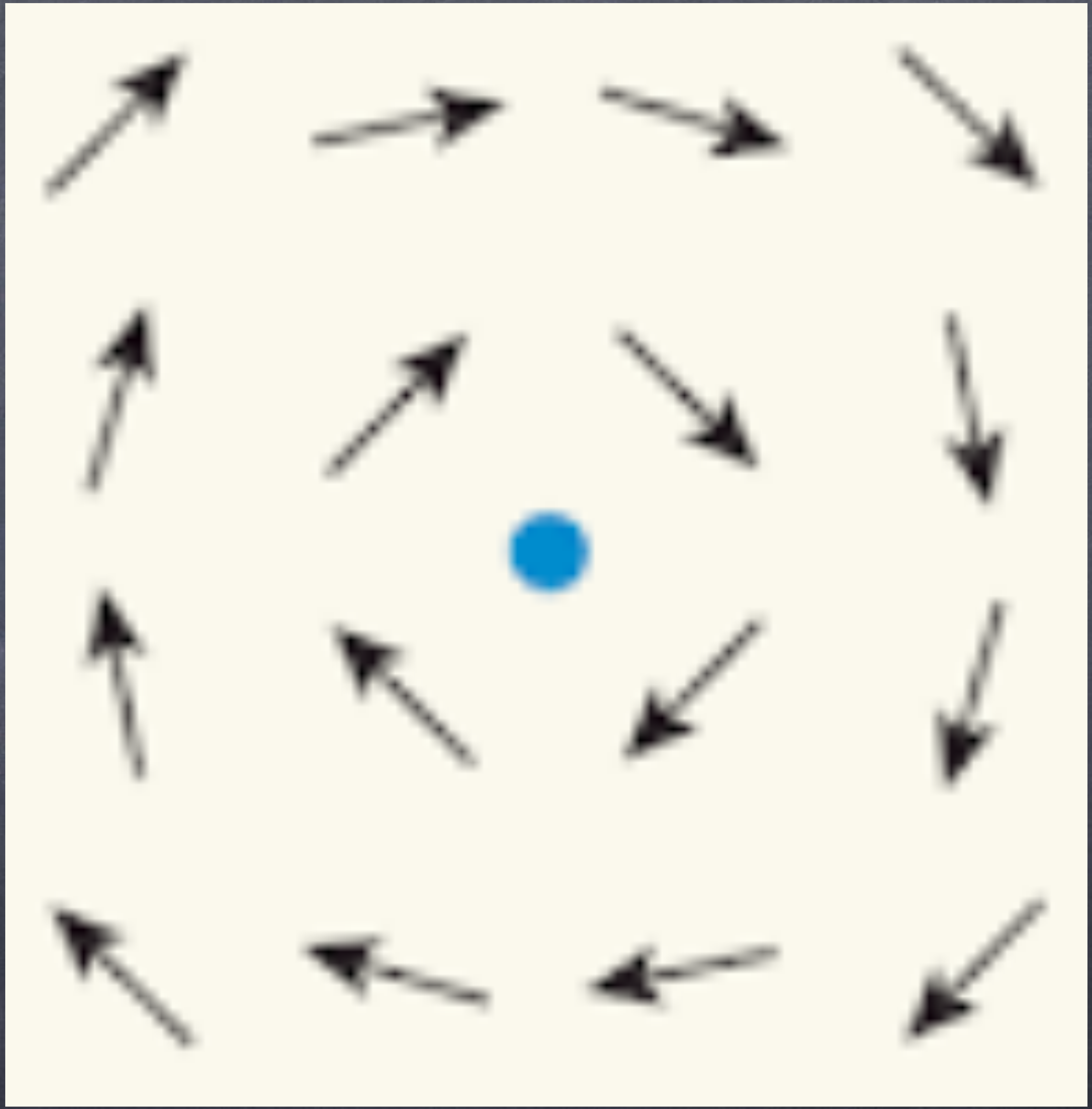
Configuration ϕ

$\downarrow$

Geometric object $X(\phi)$

XY-model example

1. Fill in an edge if the spins are close.



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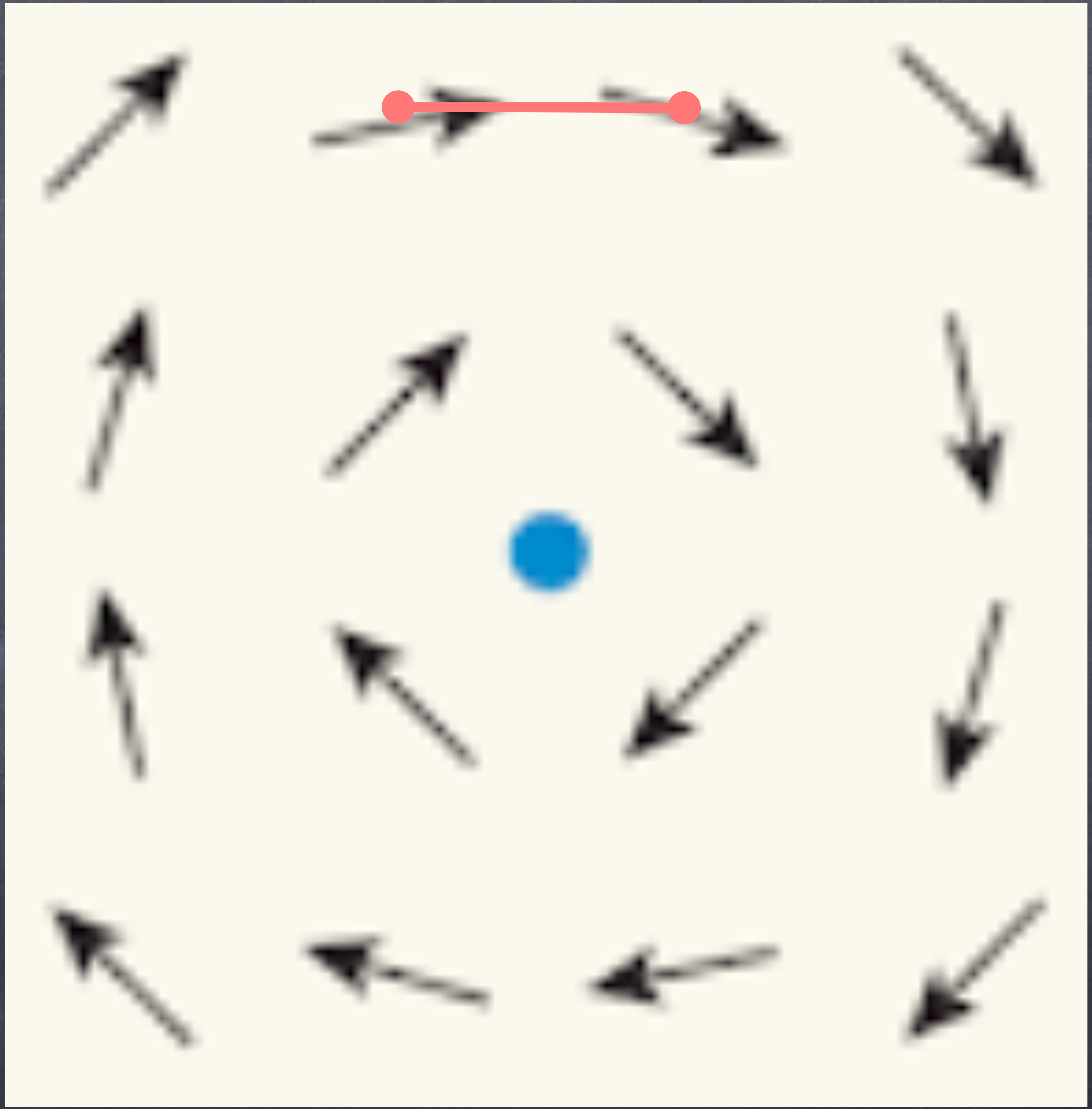

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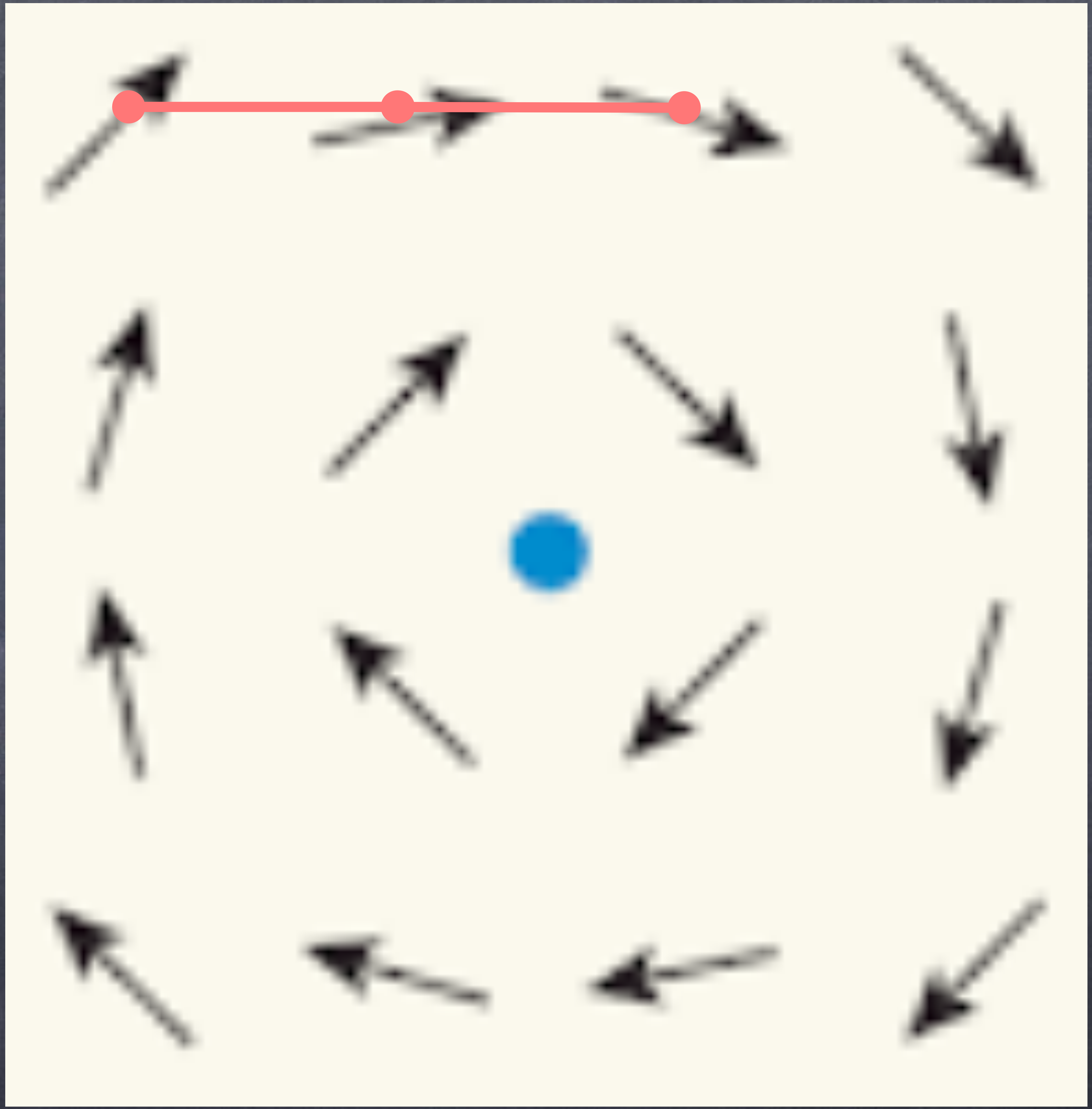

Configuration ϕ

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Geometric object $X(\phi)$

XY-model example

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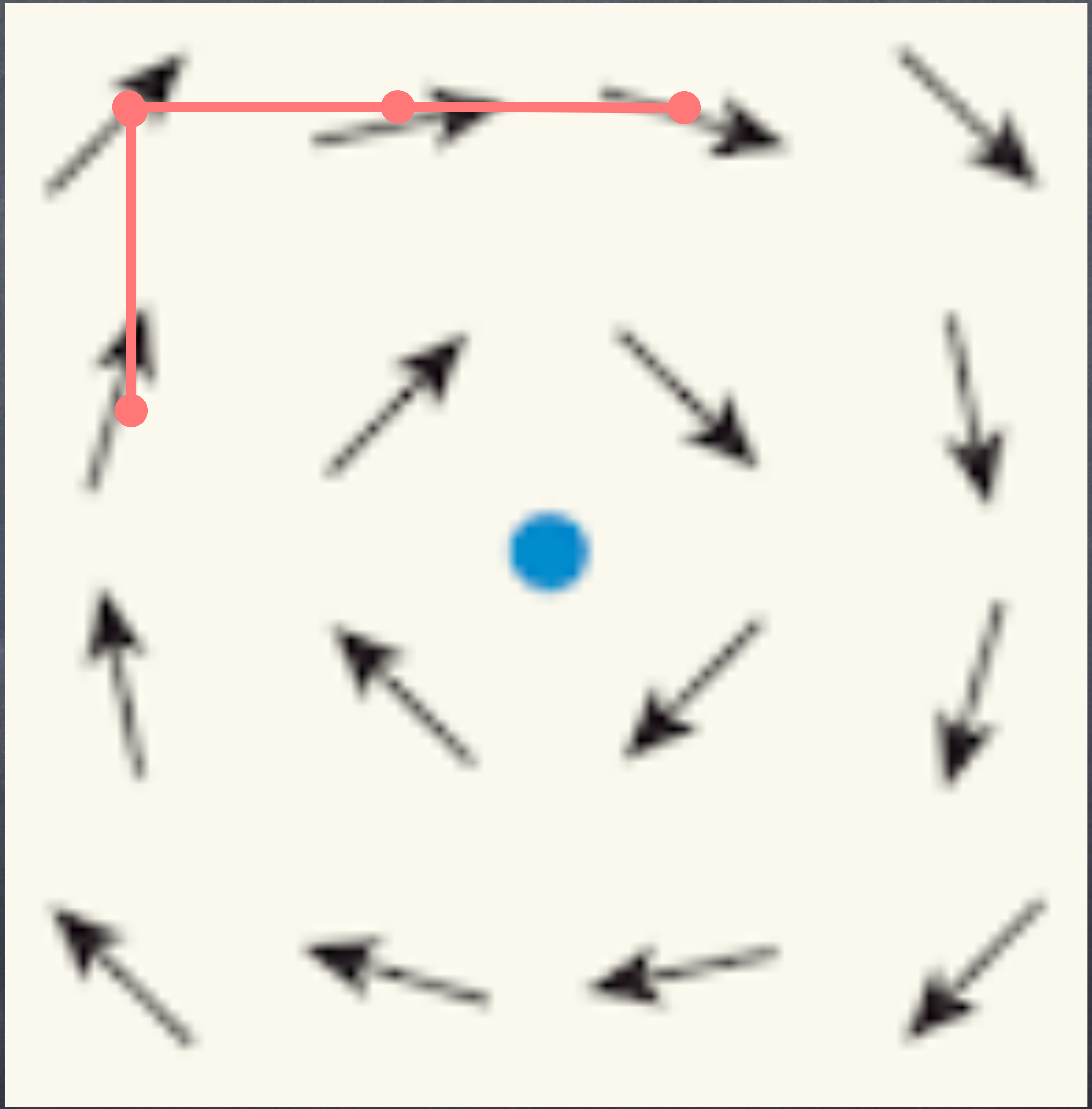

Configuration ϕ

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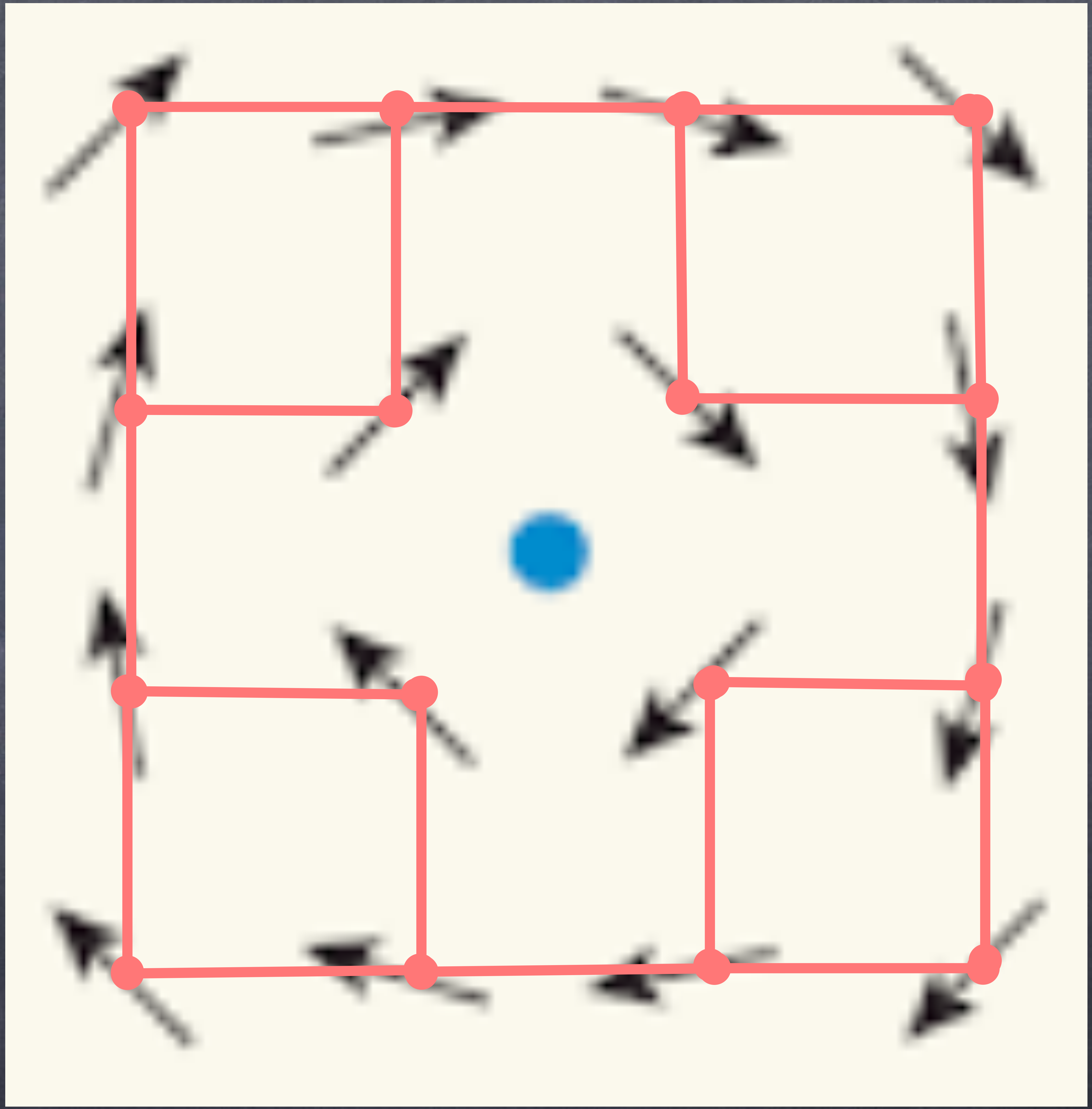


Configuration ϕ

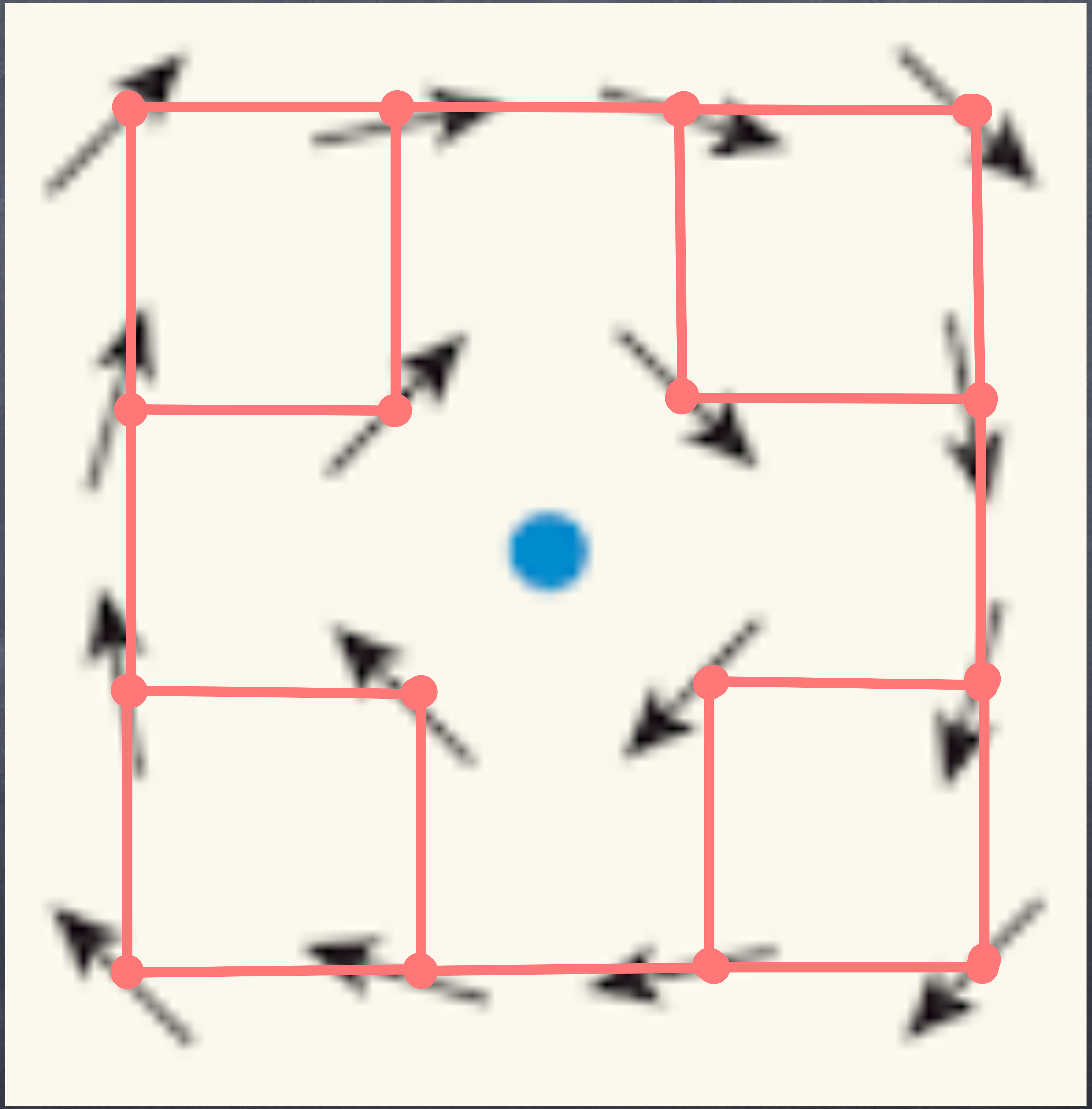
$\downarrow$

Geometric object $X(\phi)$

XY-model example

1. Fill in an edge if the spins are close.

2. Fill in a plaquette if all the edges are present.



Configuration ϕ

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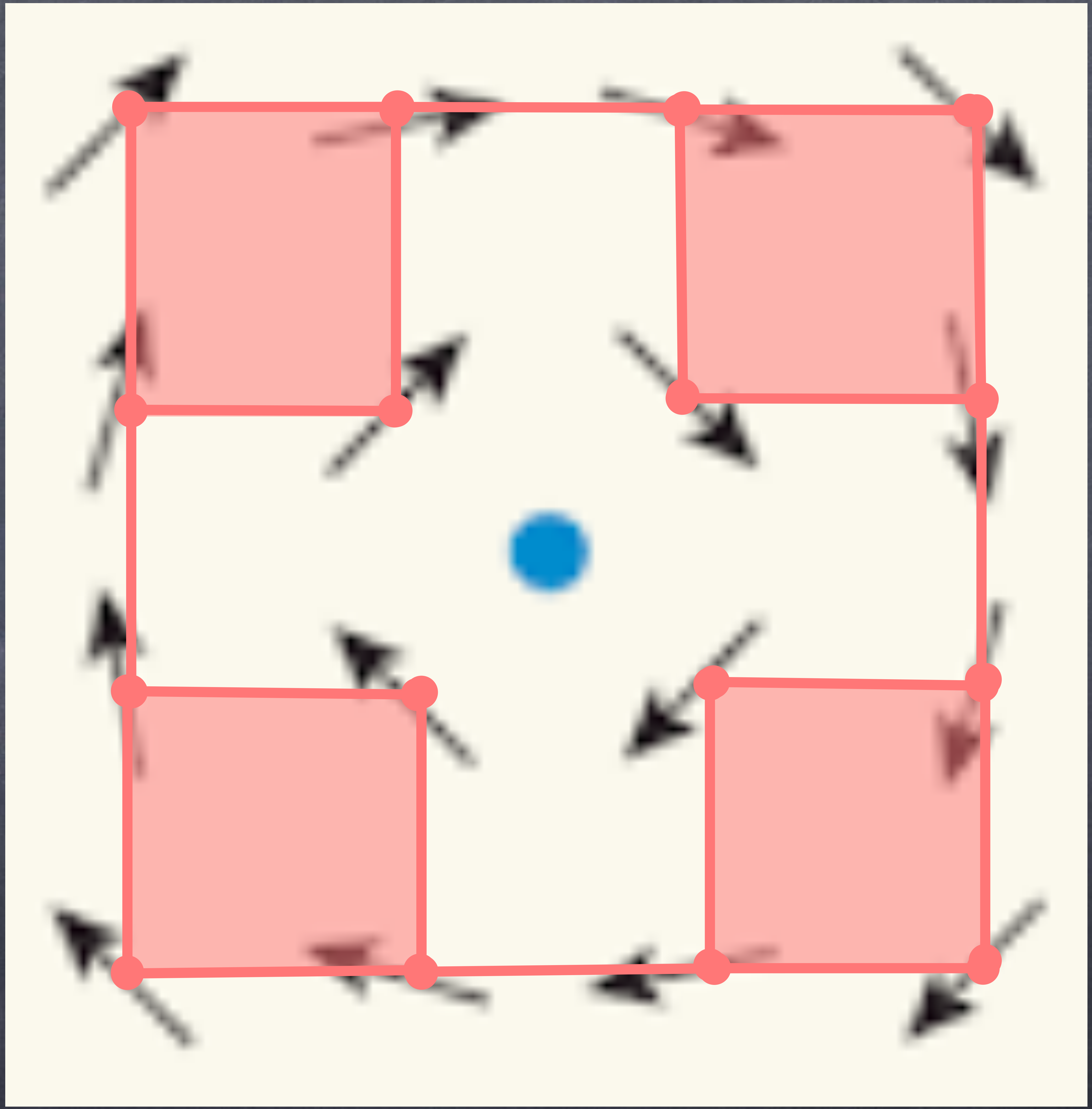

Configuration ϕ

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Geometric object $X(\phi)$

XY-model example

1. Fill in an edge if the spins are close.

threshold

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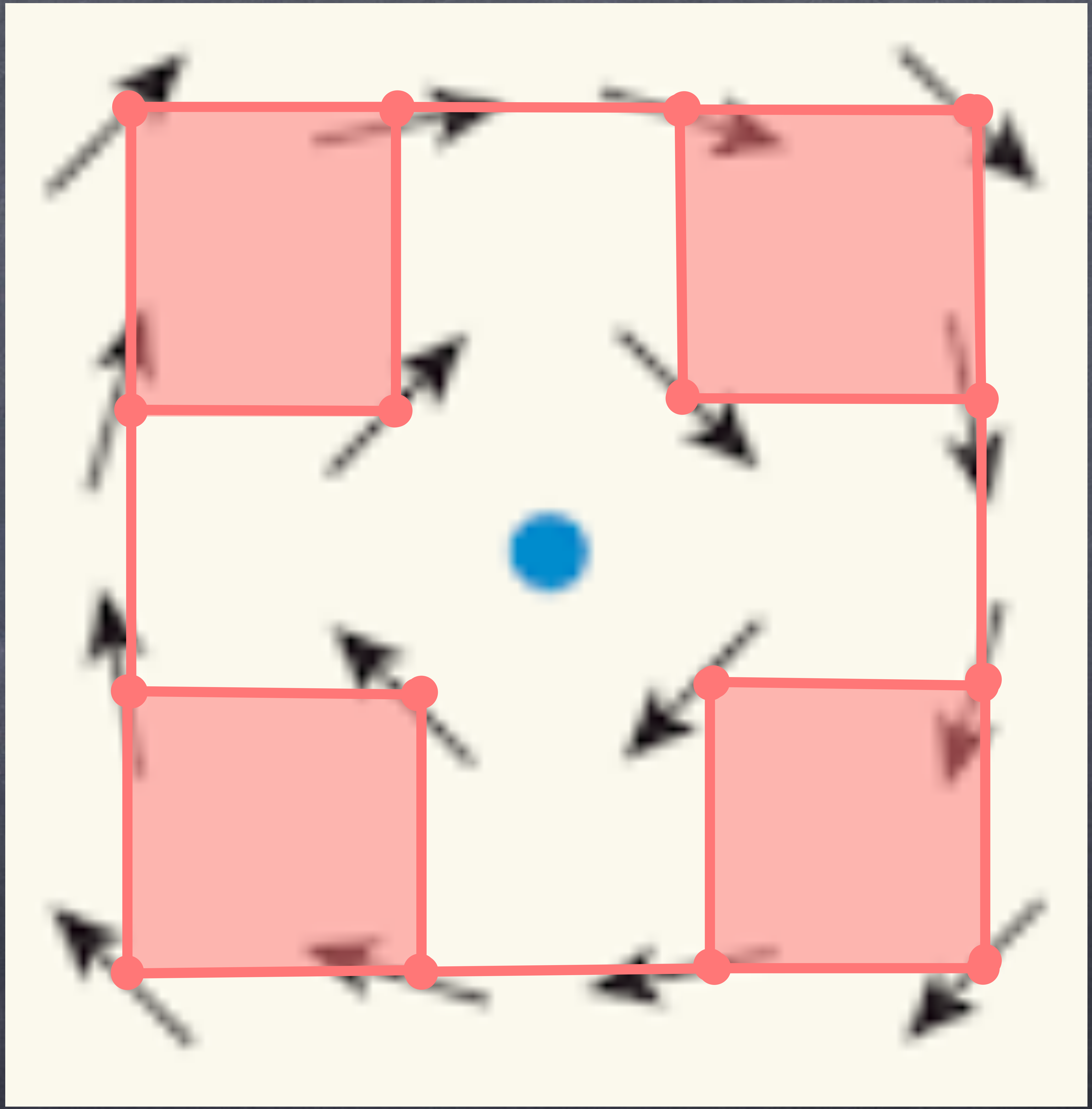

A continuous map of geometric objects

$$X \to Y$$

induces a linear map $H_*(X) \to H_*(Y)$

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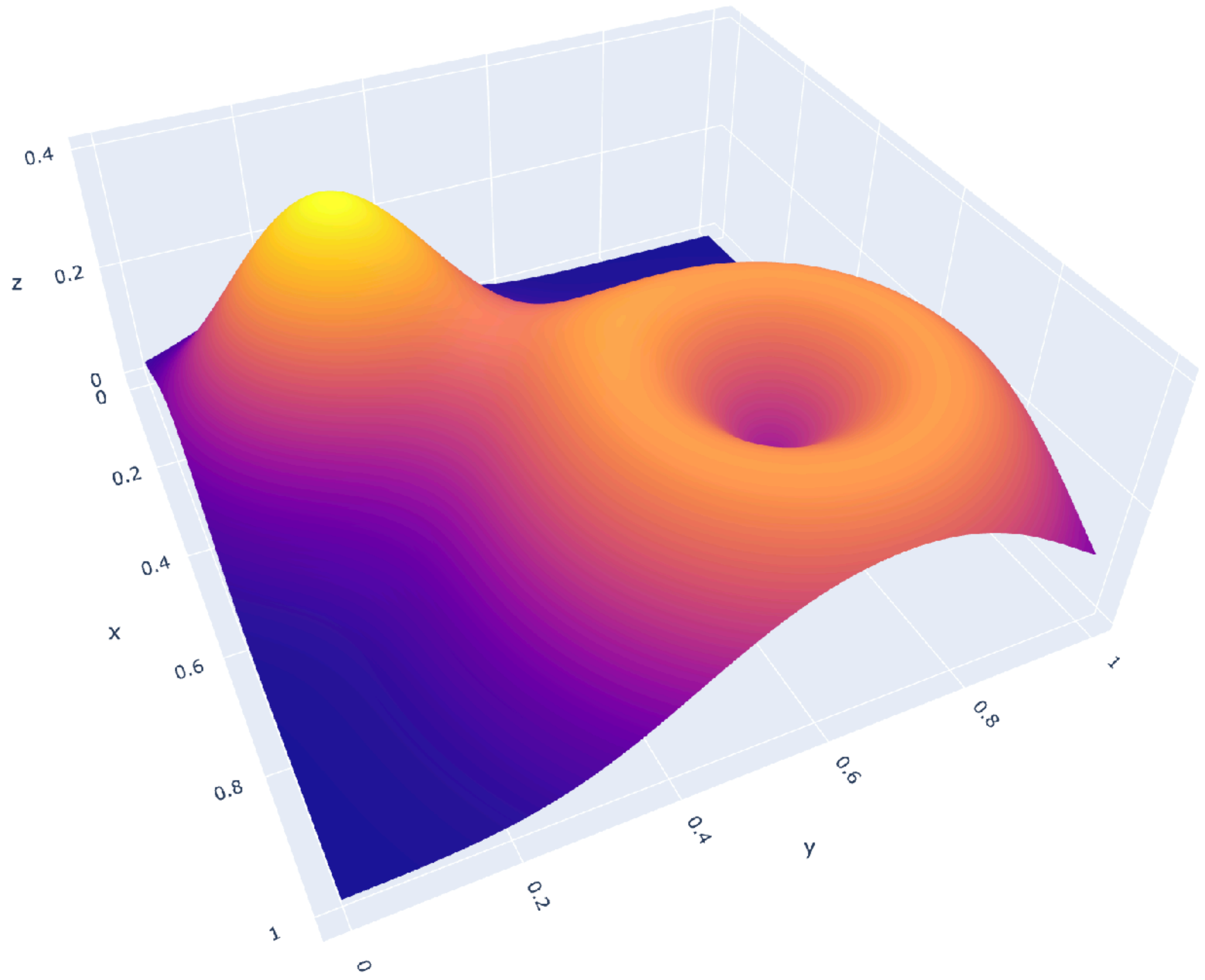
A sequence of geometric objects

$$X_1 \to X_2 \to \cdots$$

induces a sequence of vector spaces and linear maps

$$H_*(X_1) \to H_*(X_2) \to \cdots$$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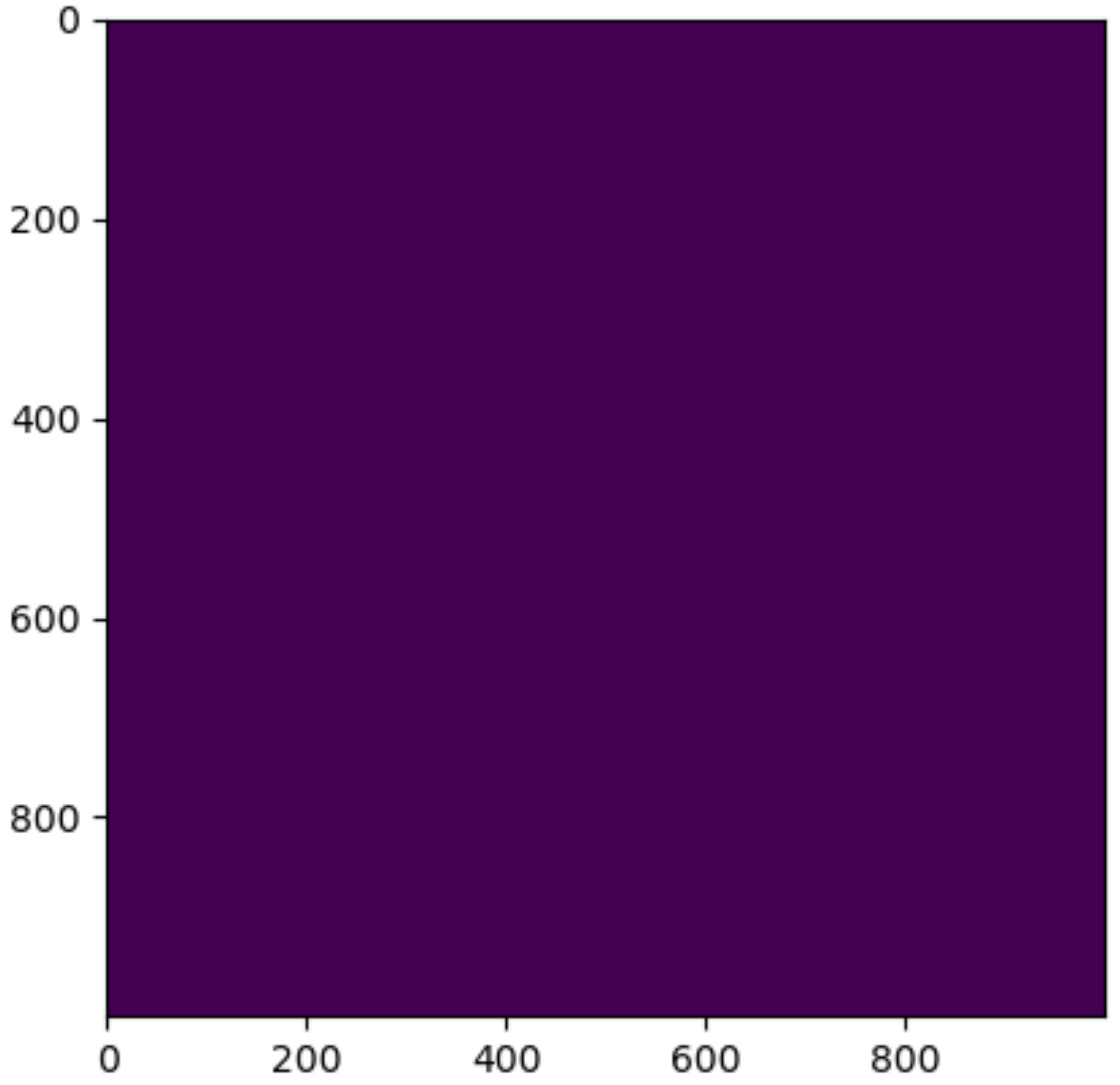
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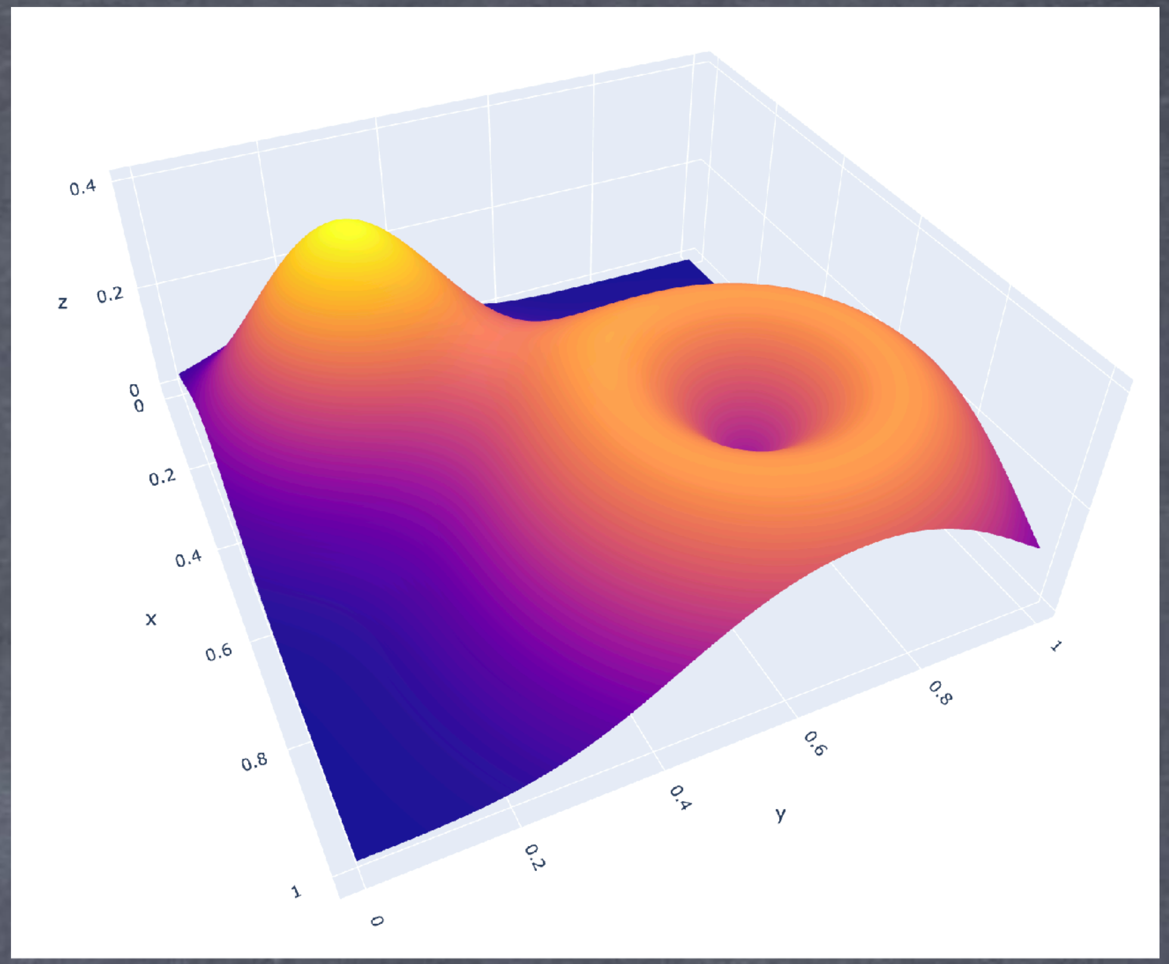

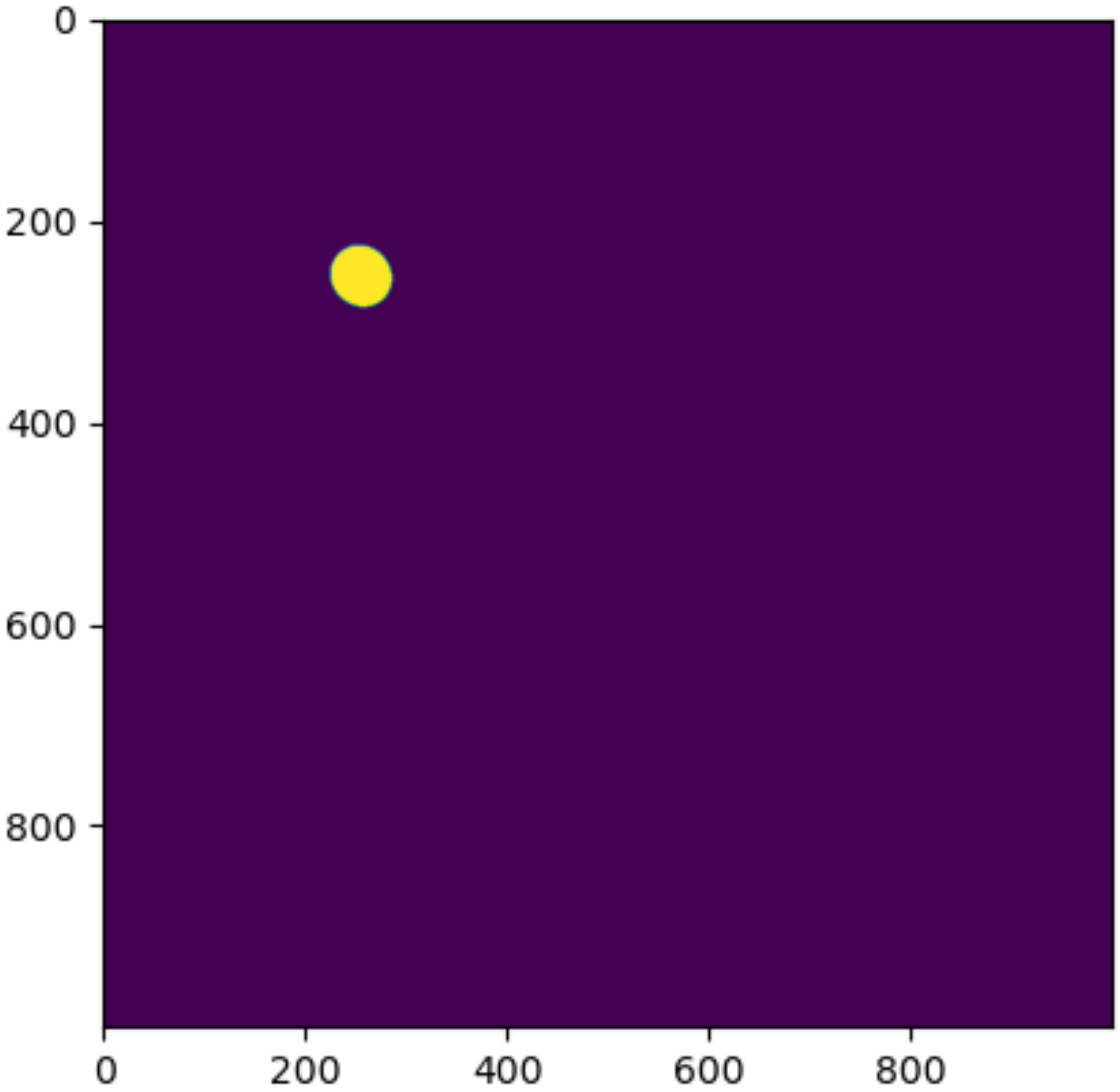
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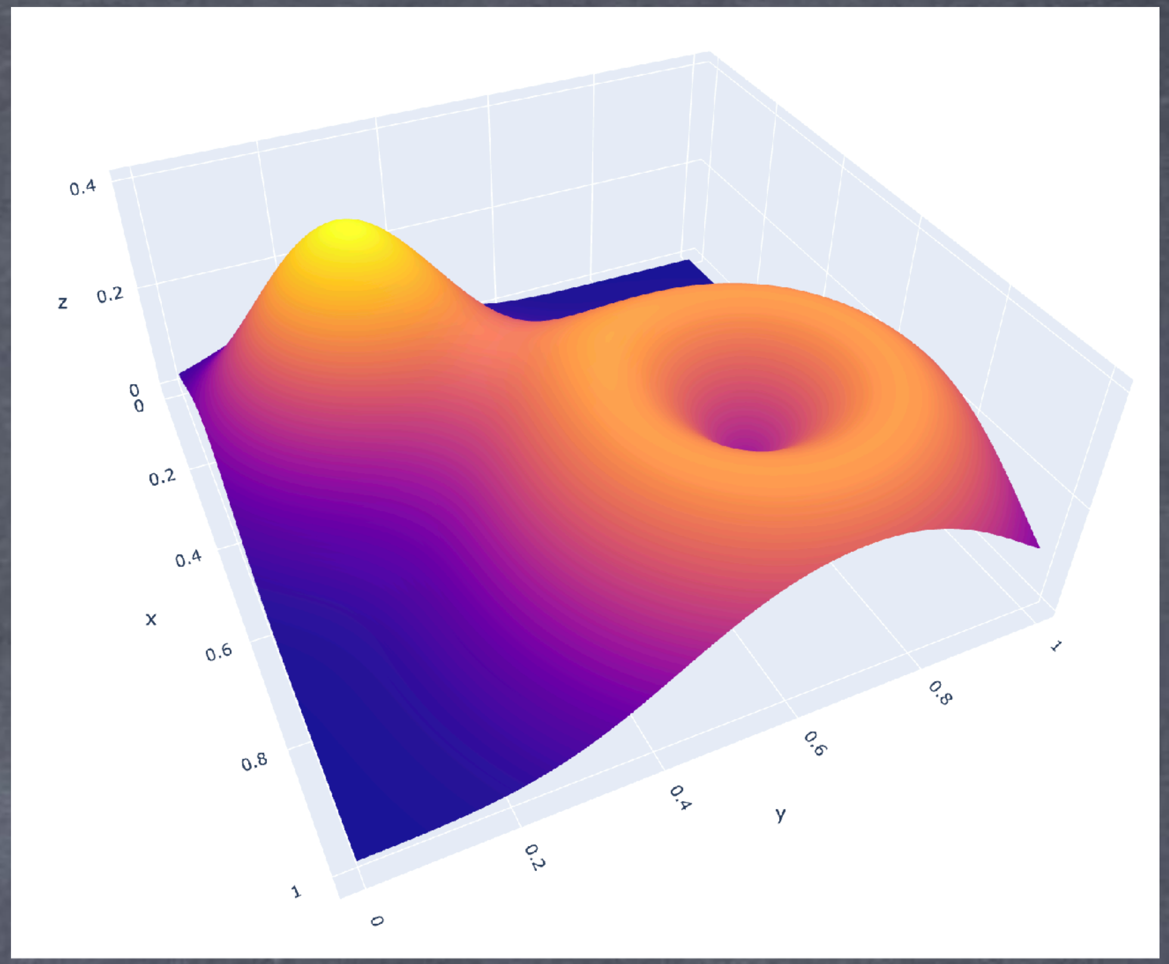

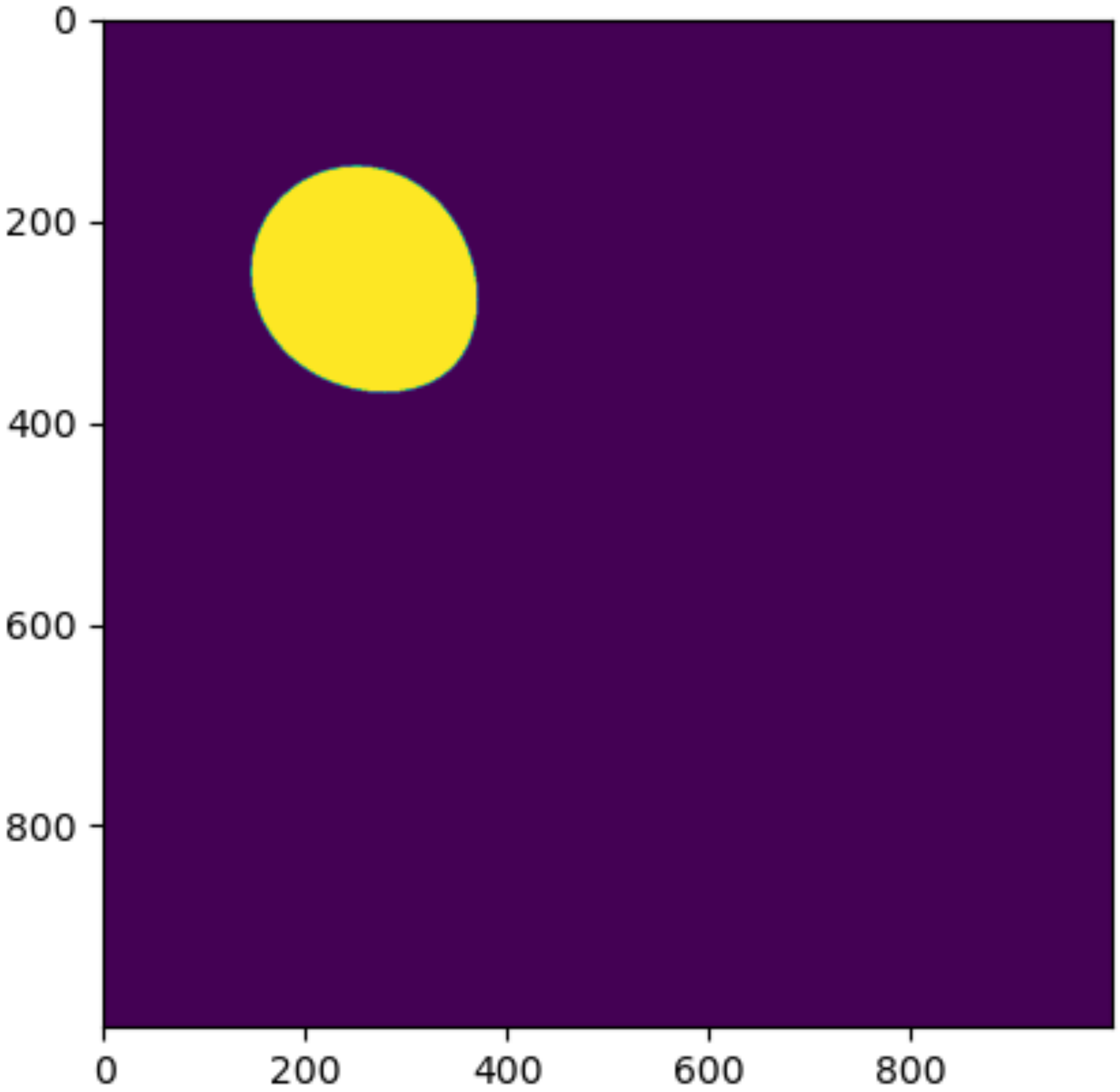
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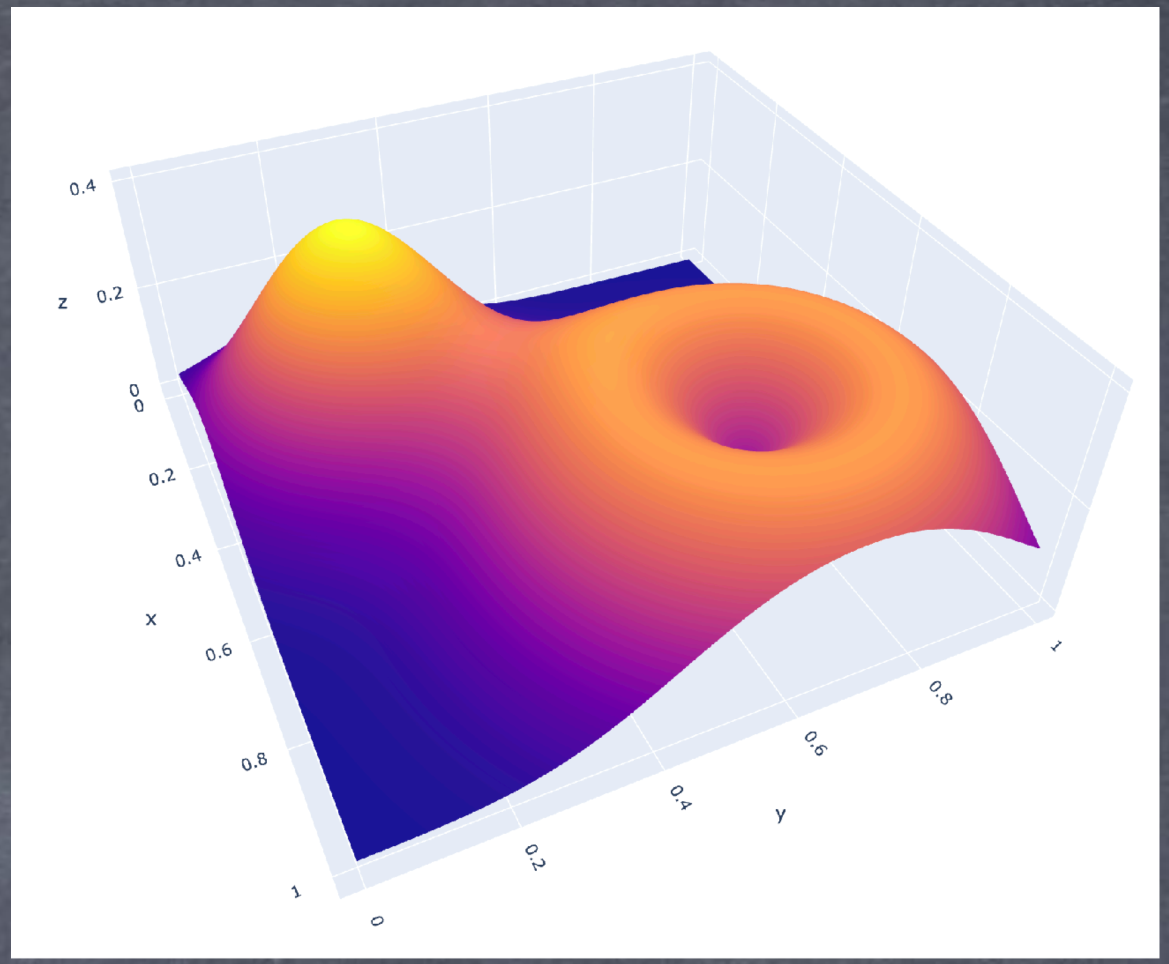

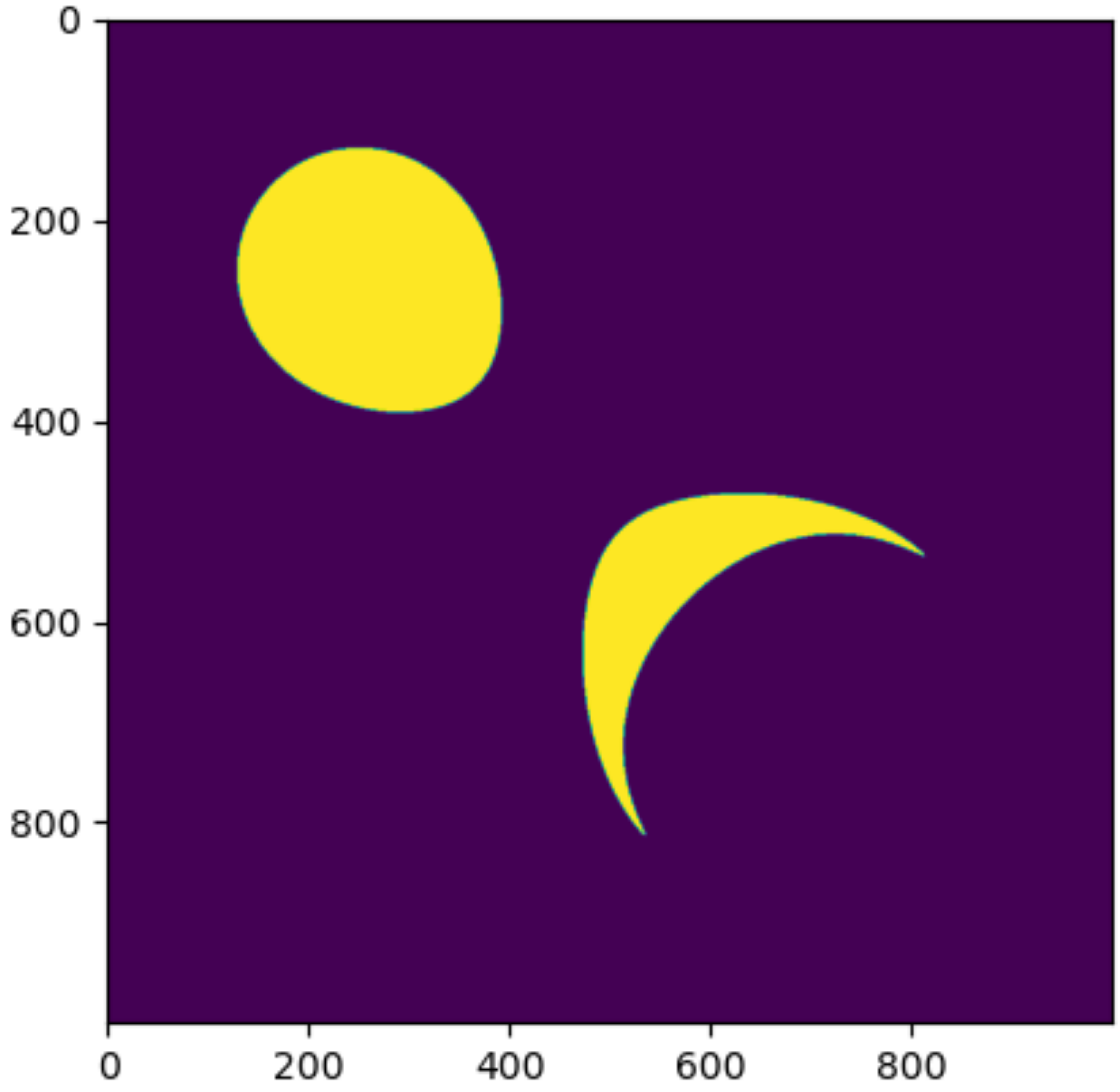

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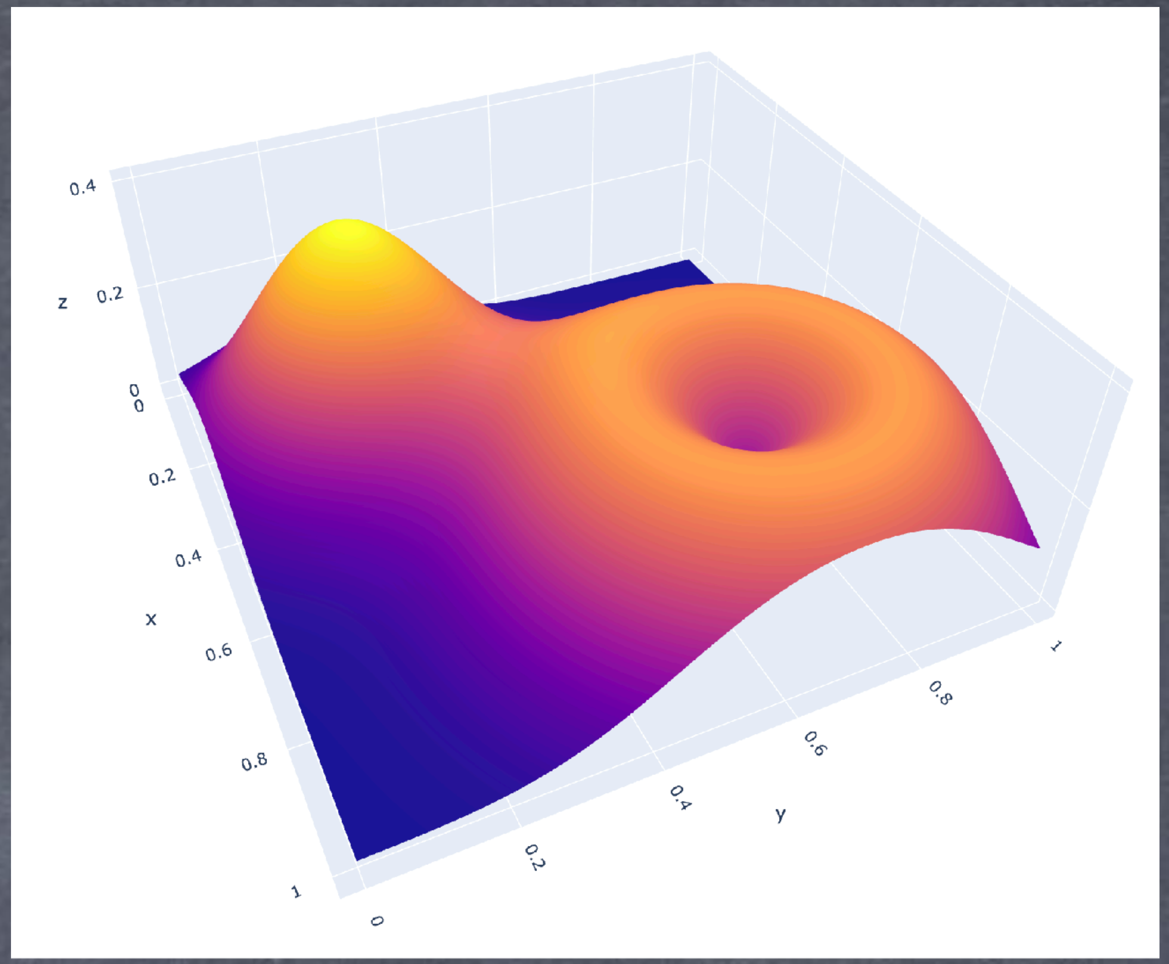

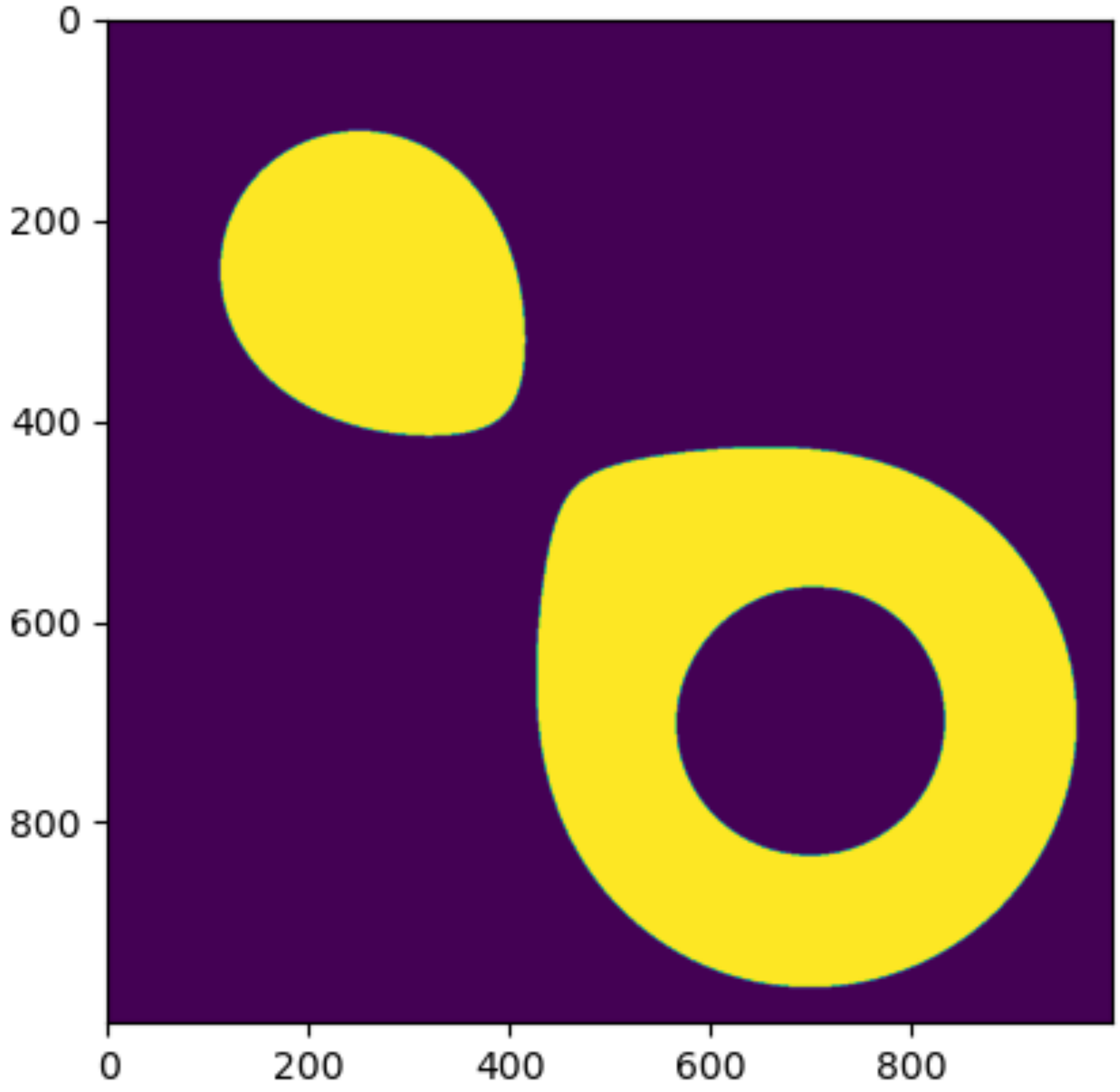
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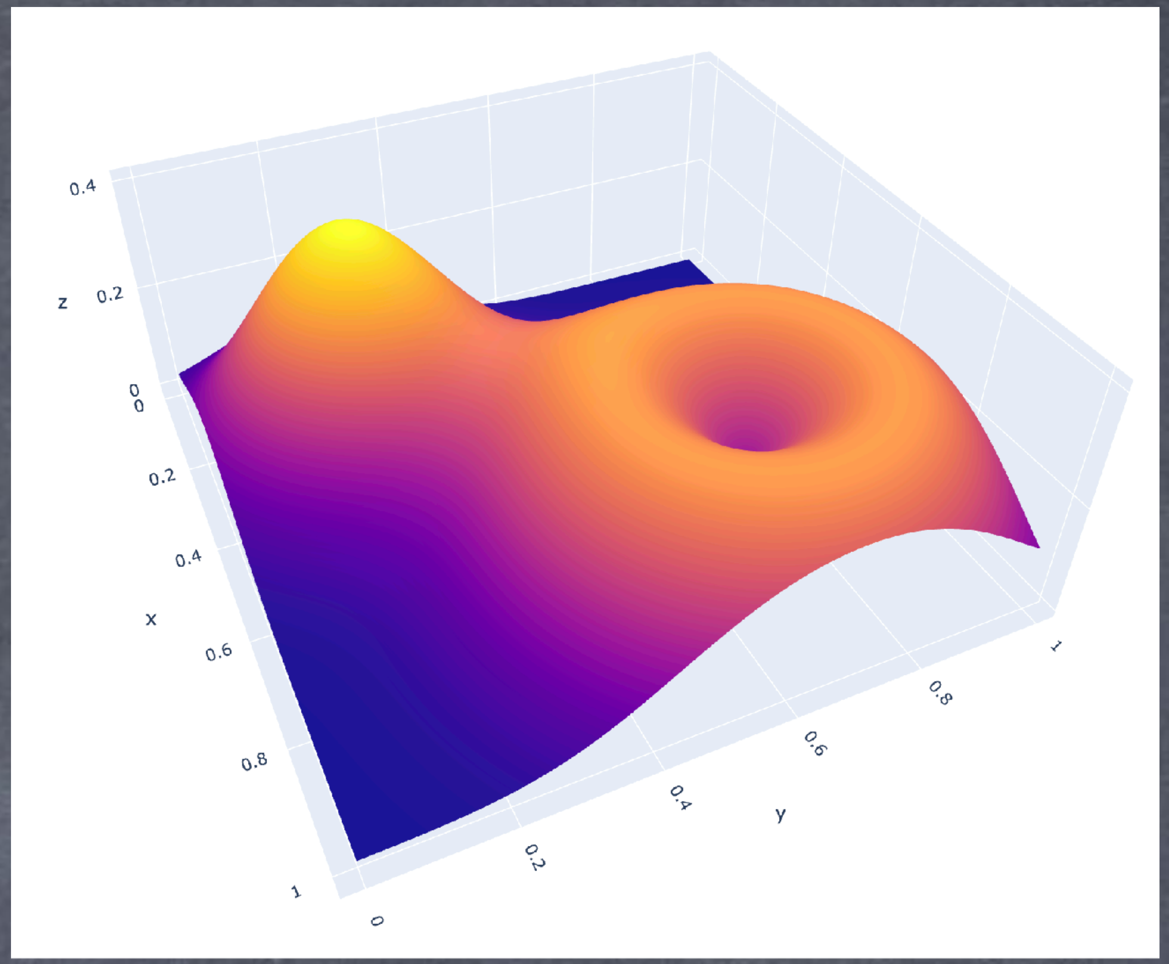

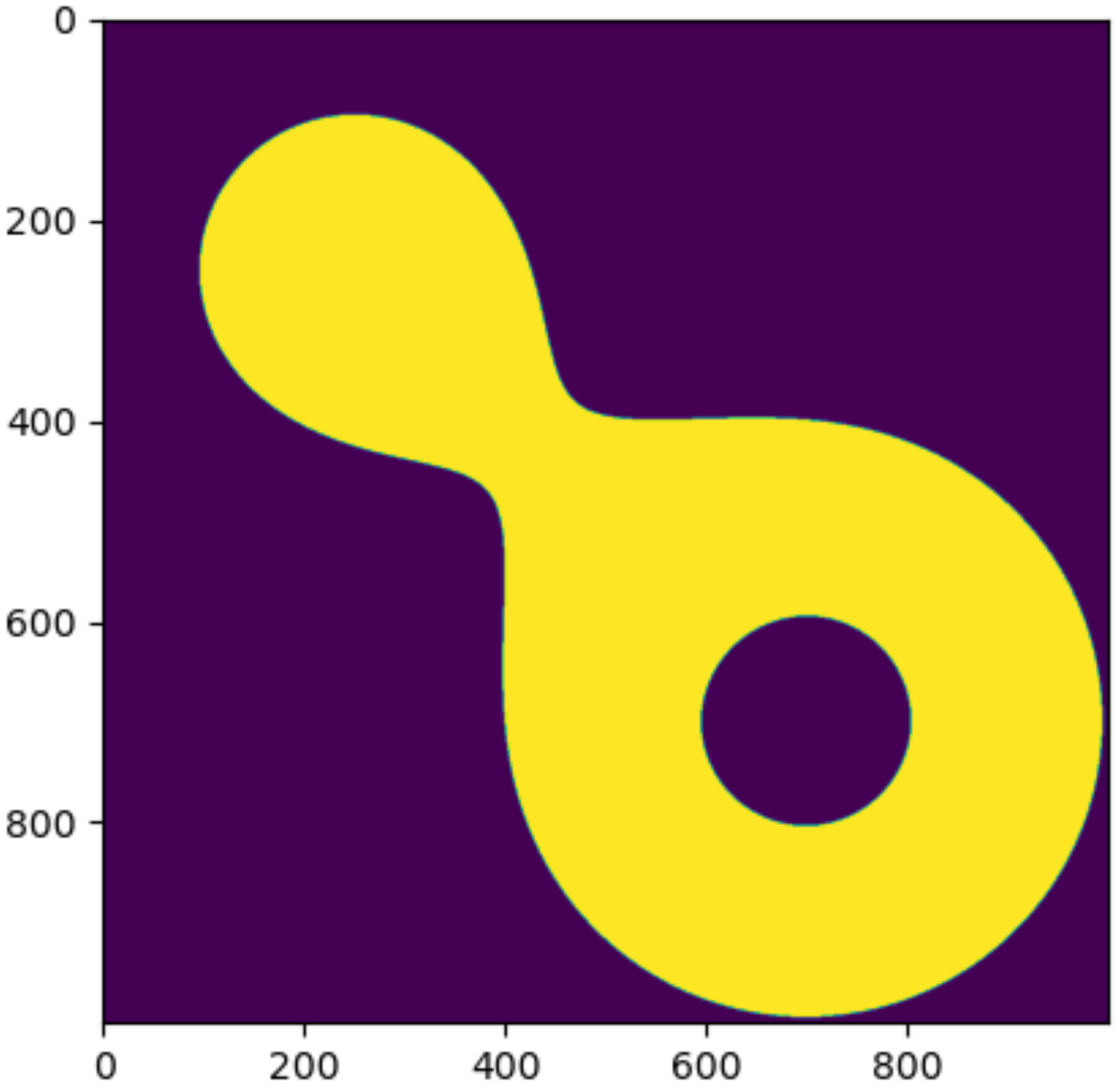
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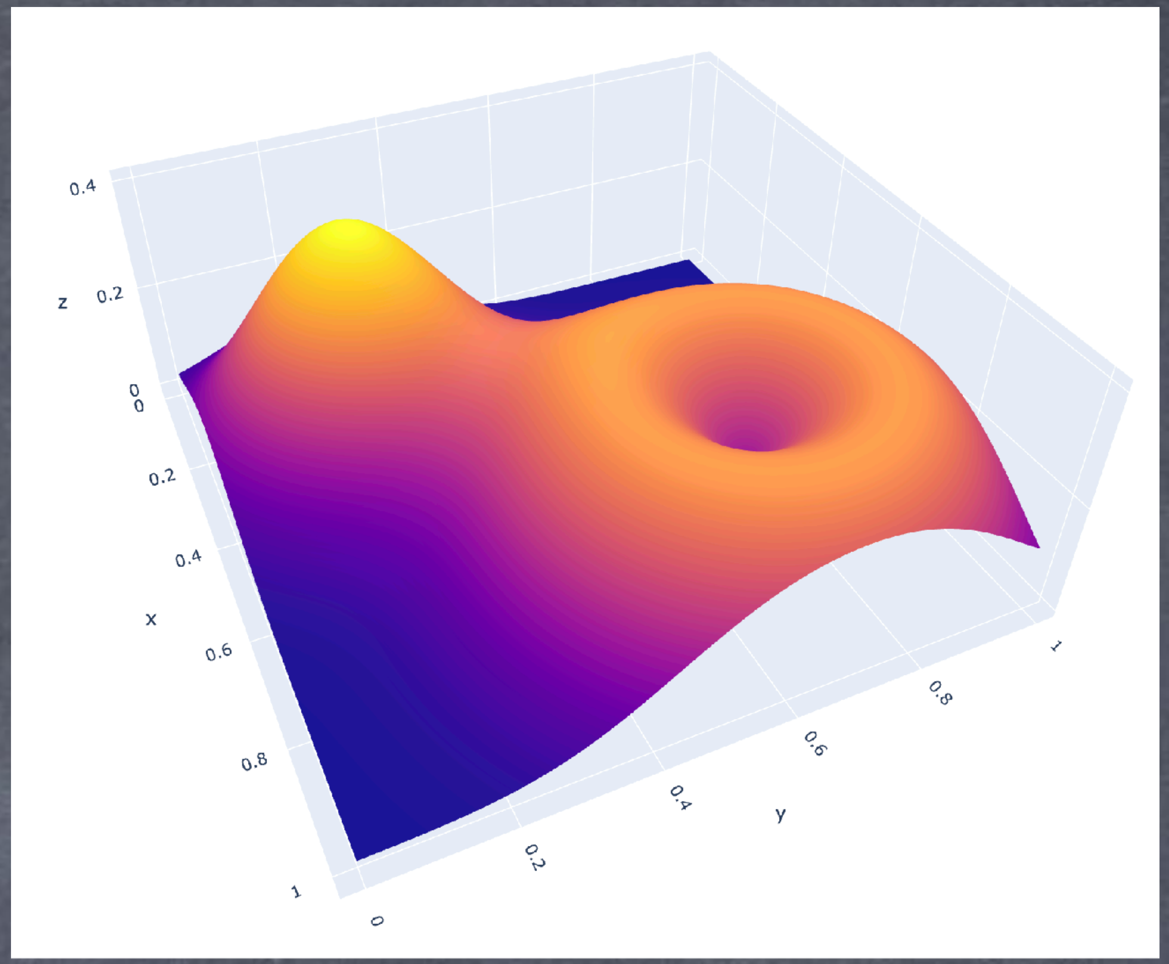

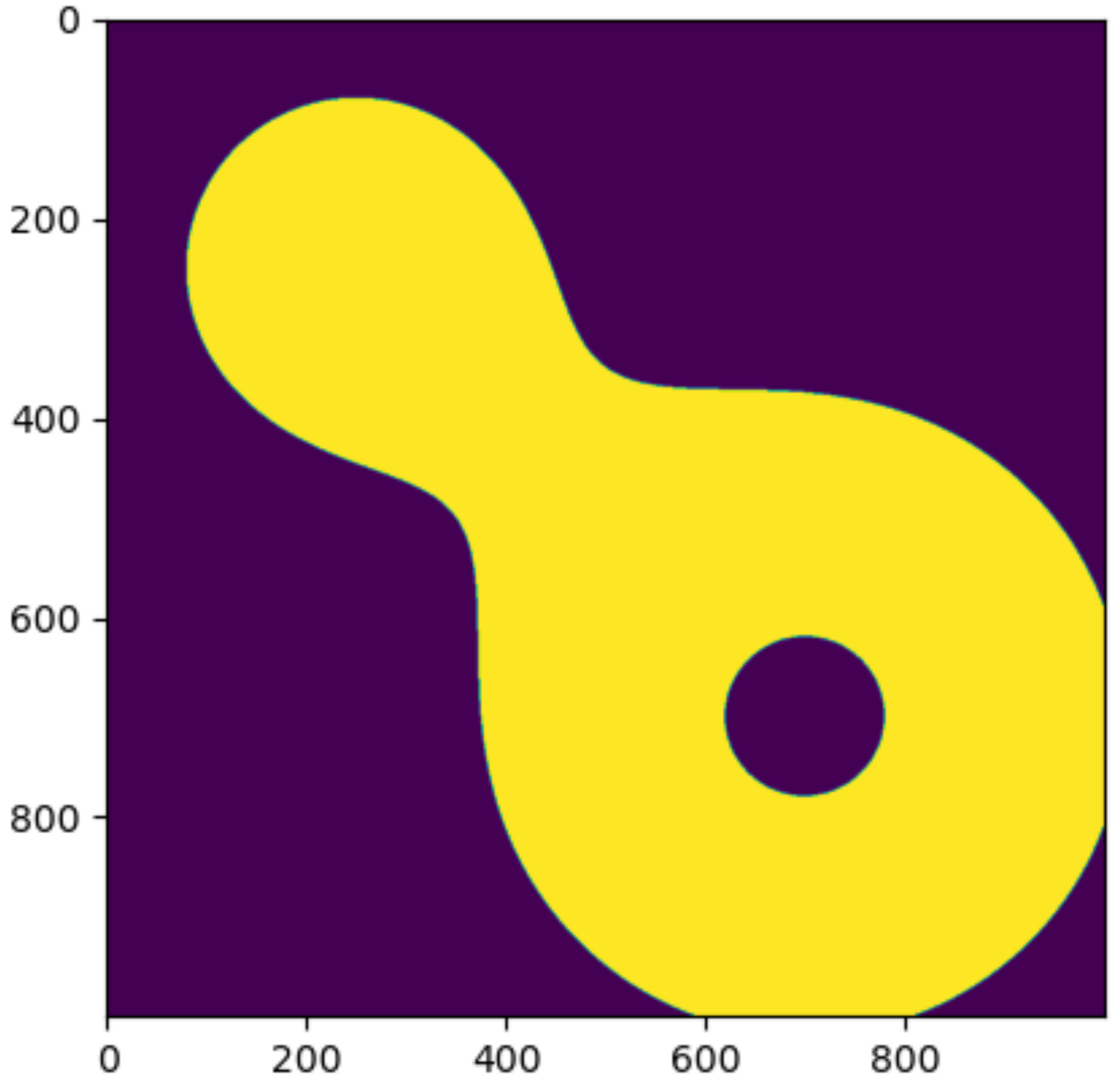

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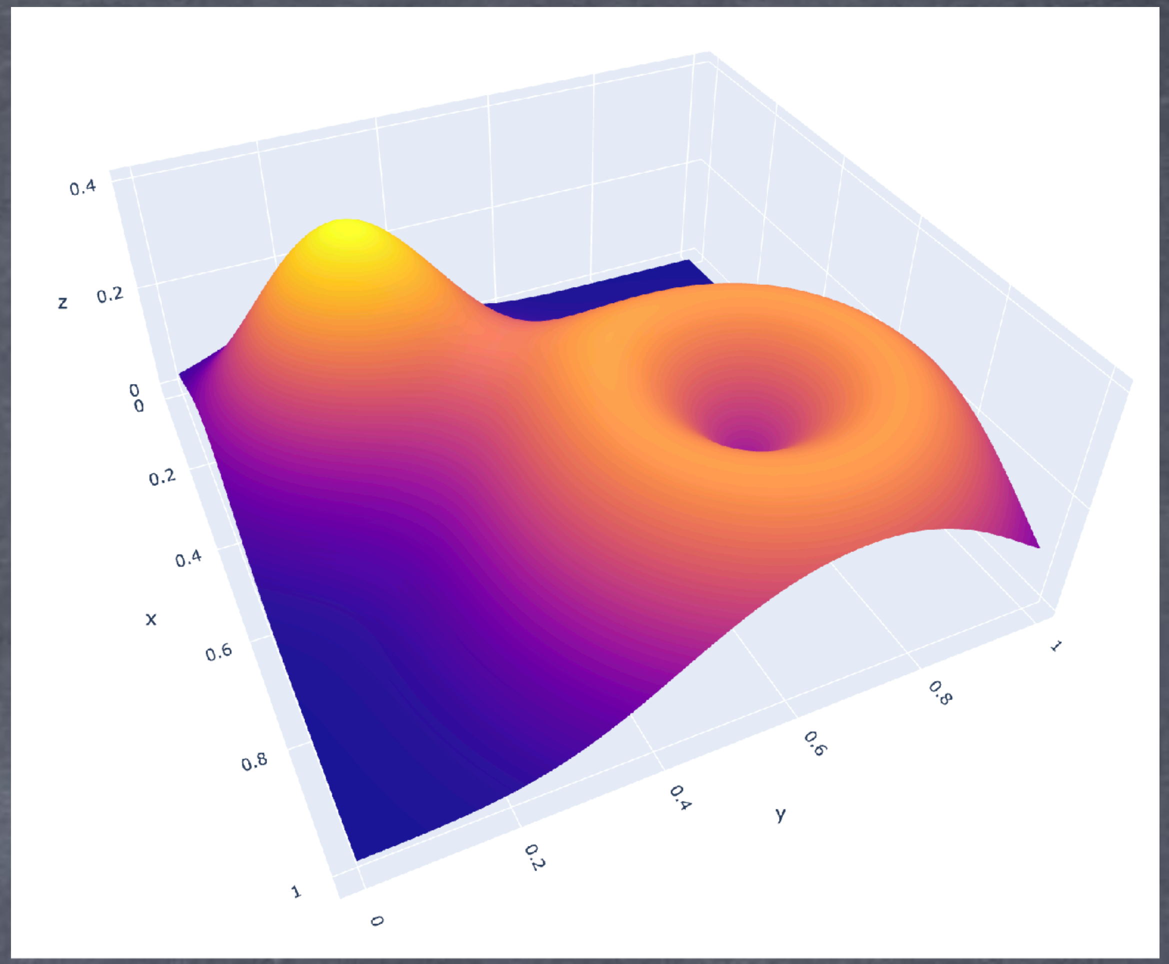

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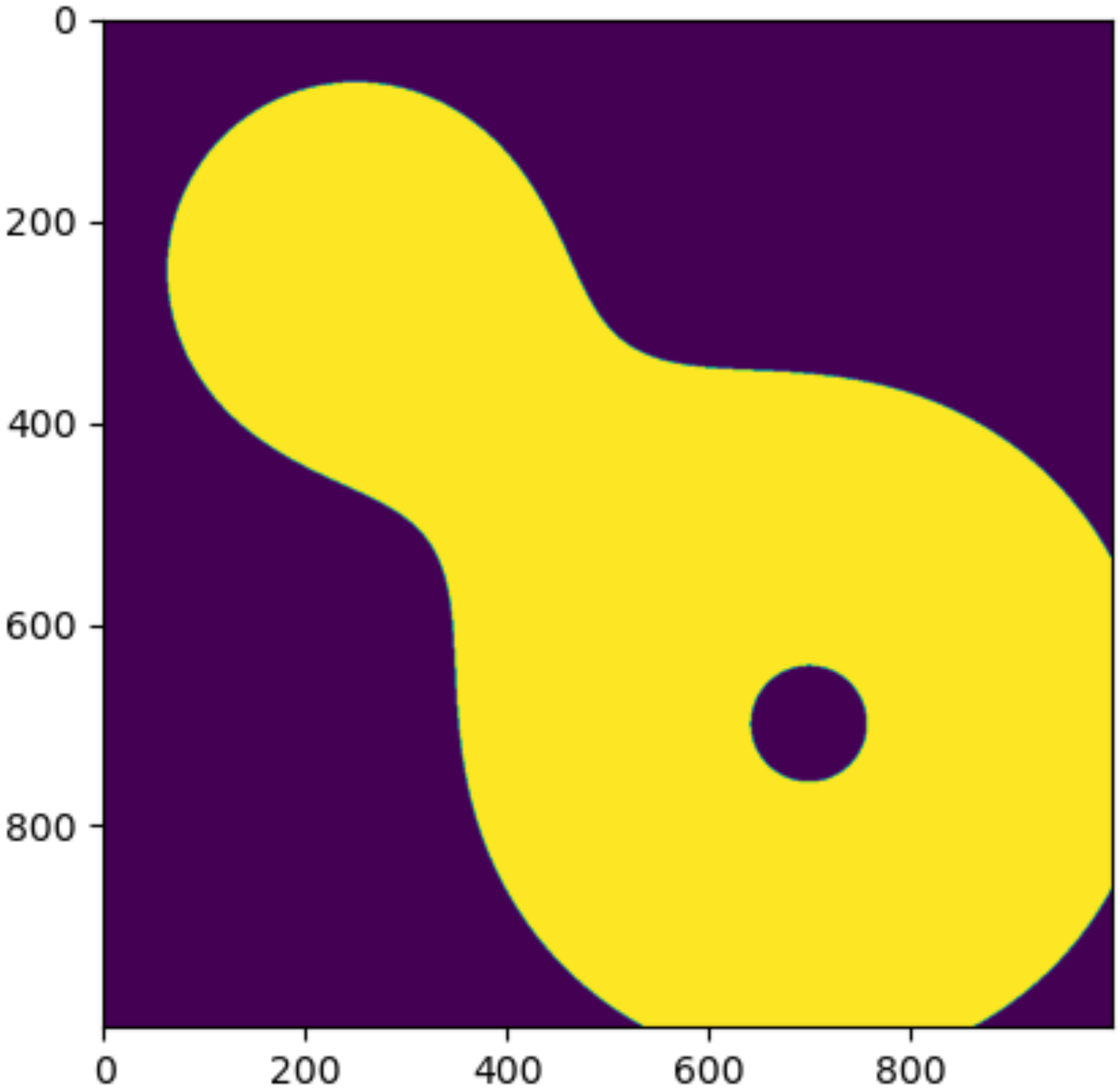
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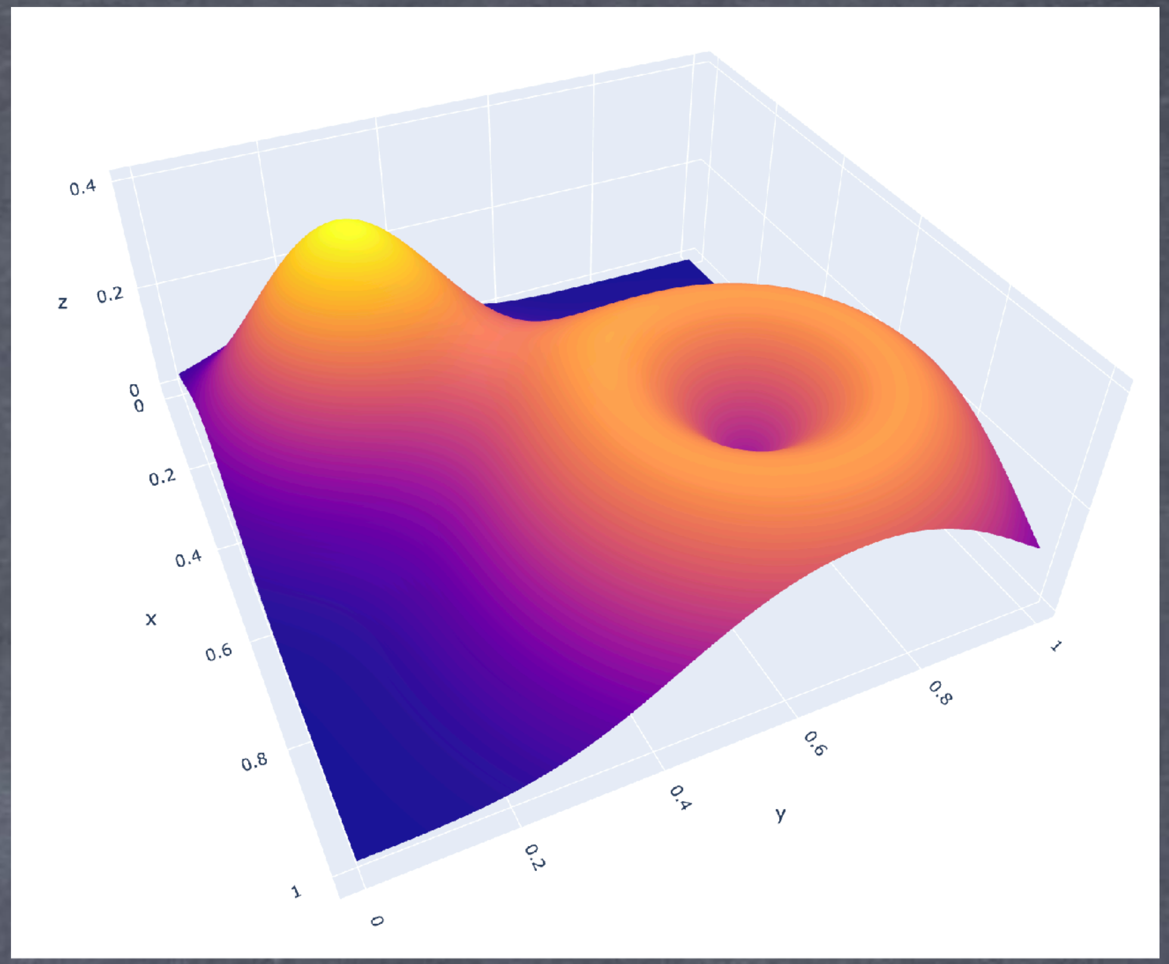

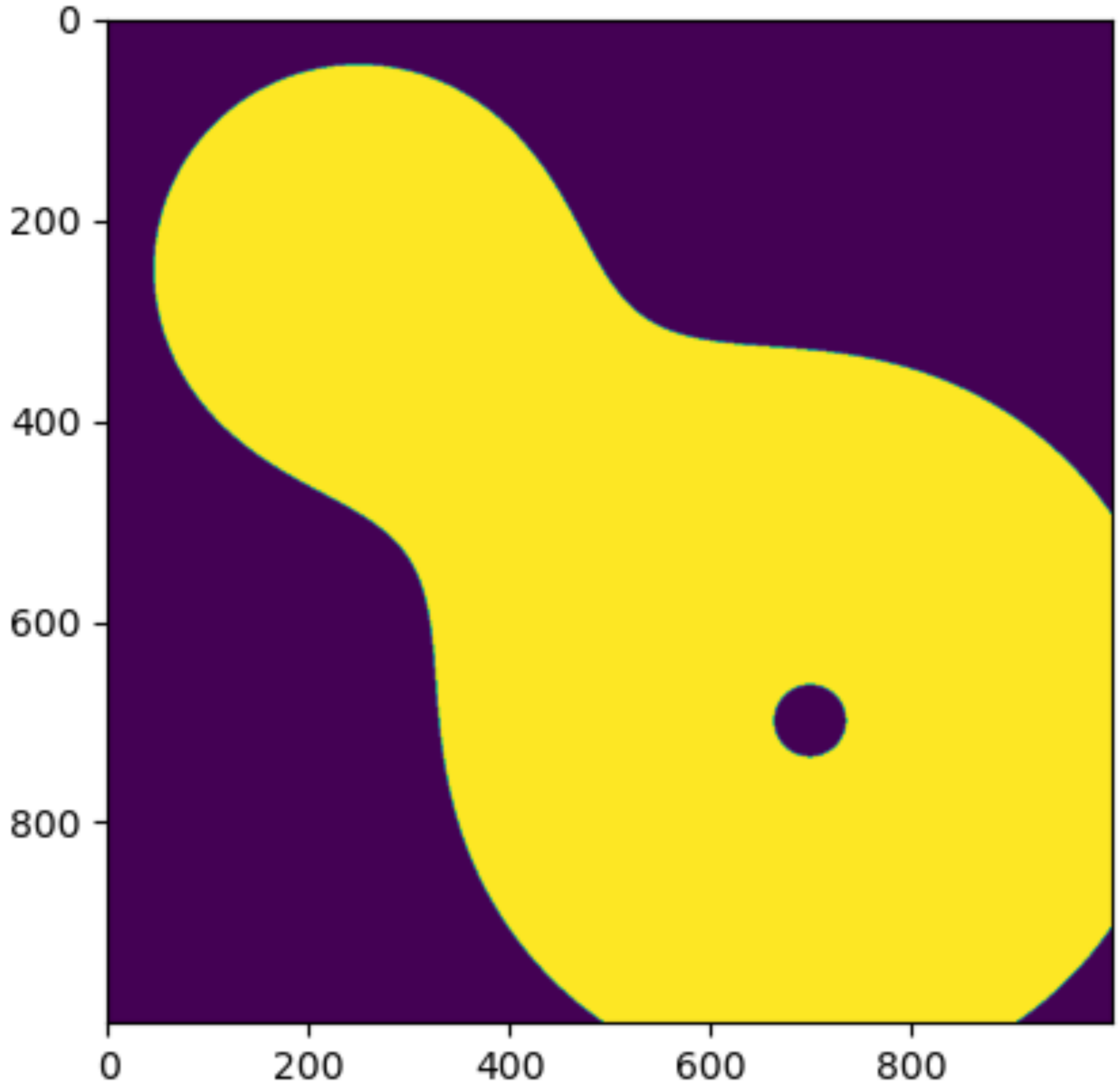
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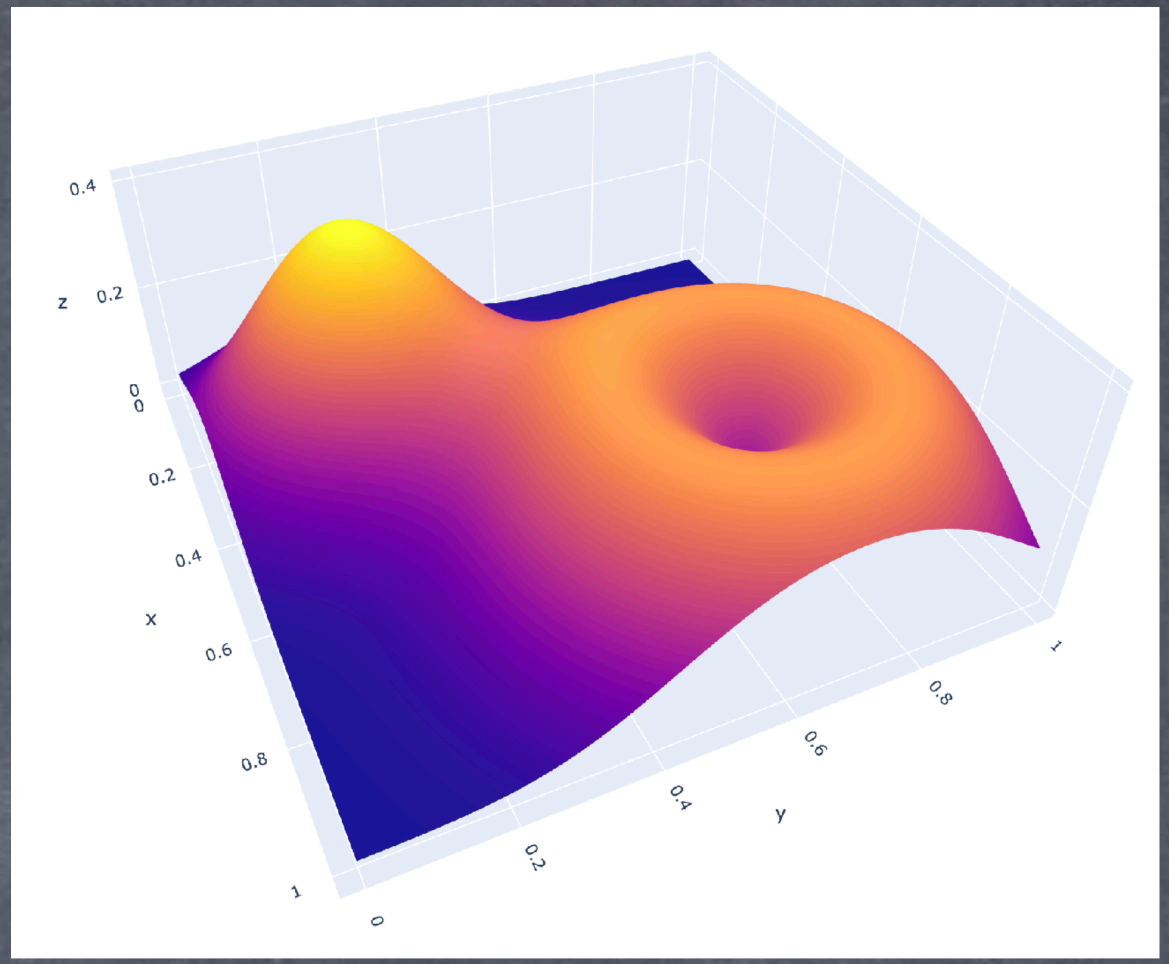

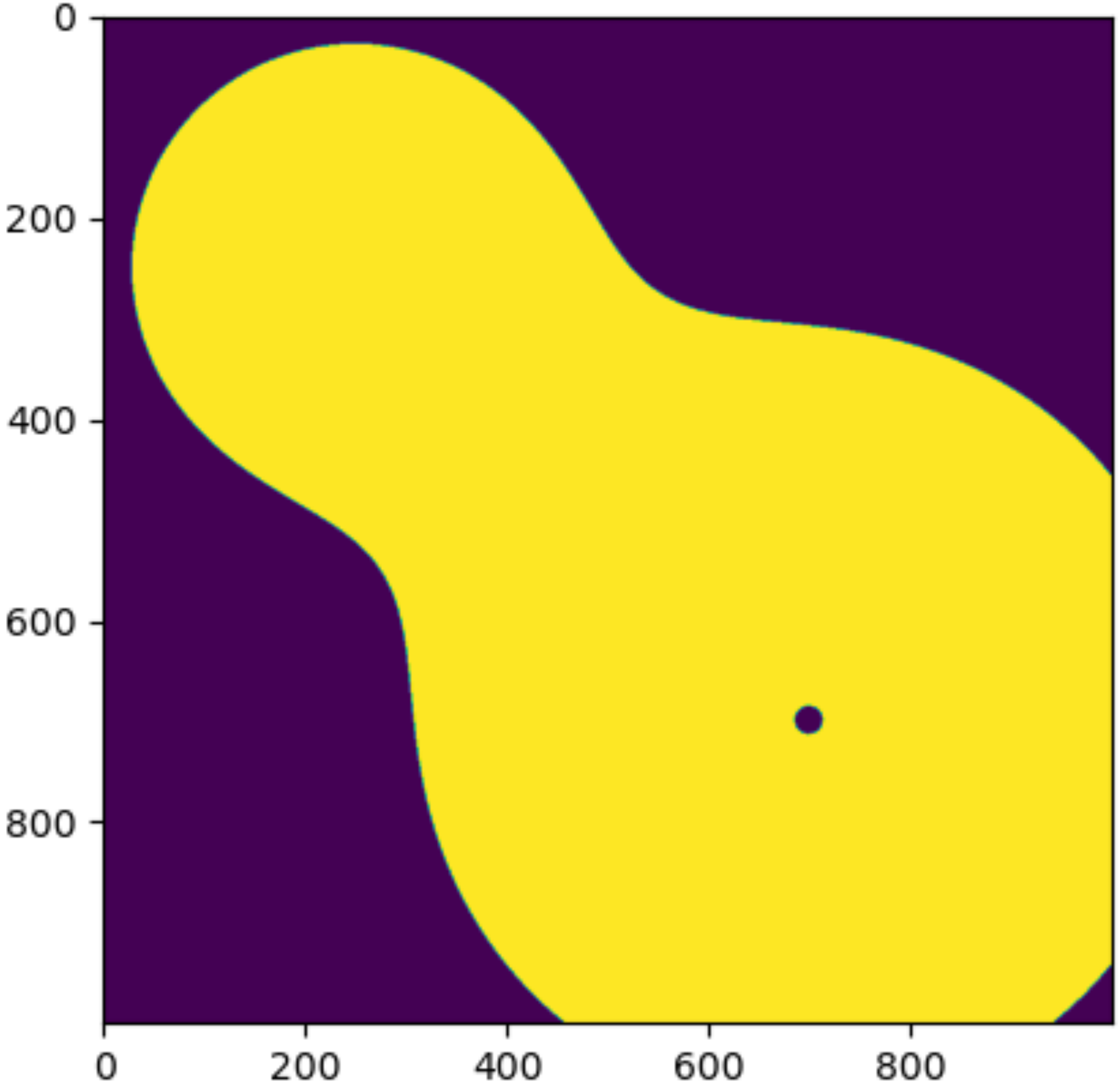
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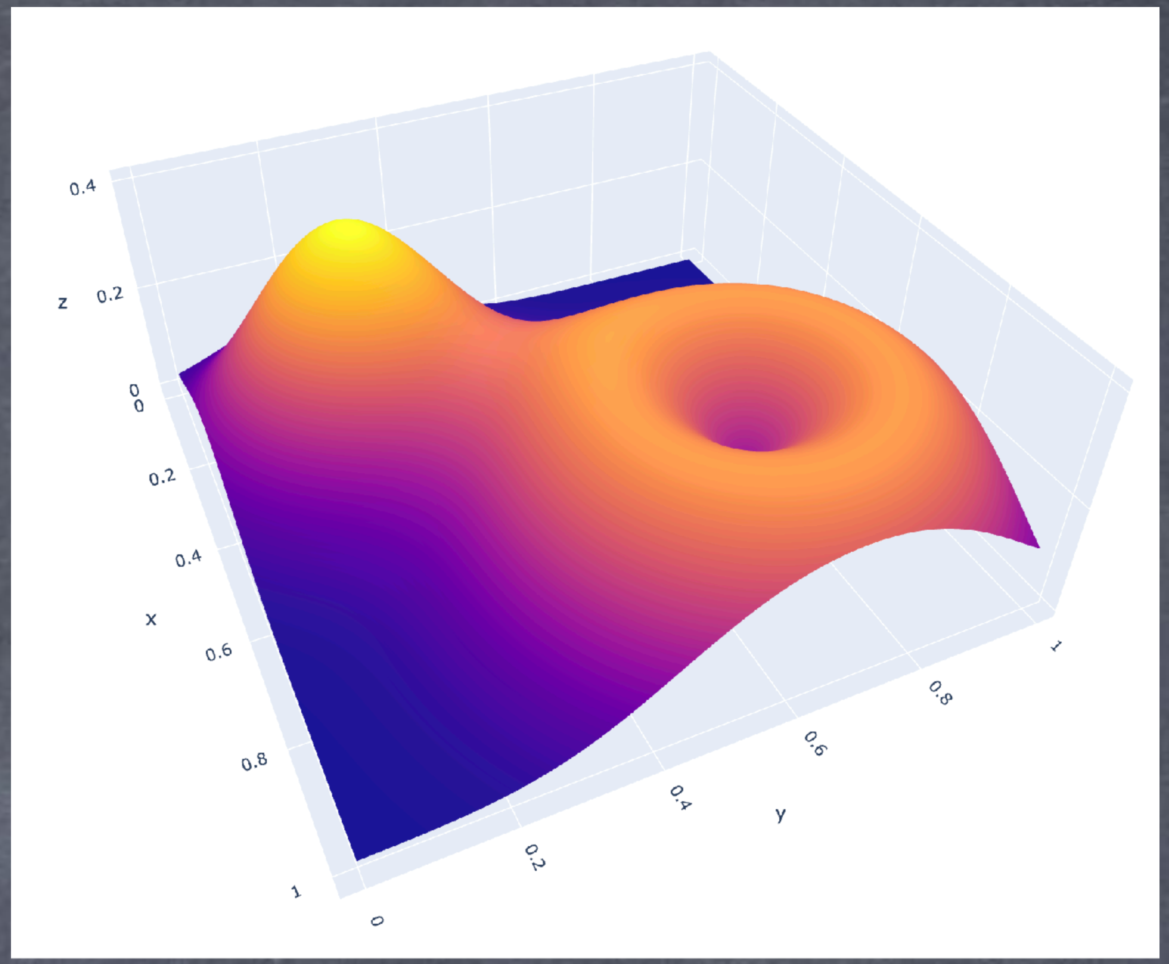

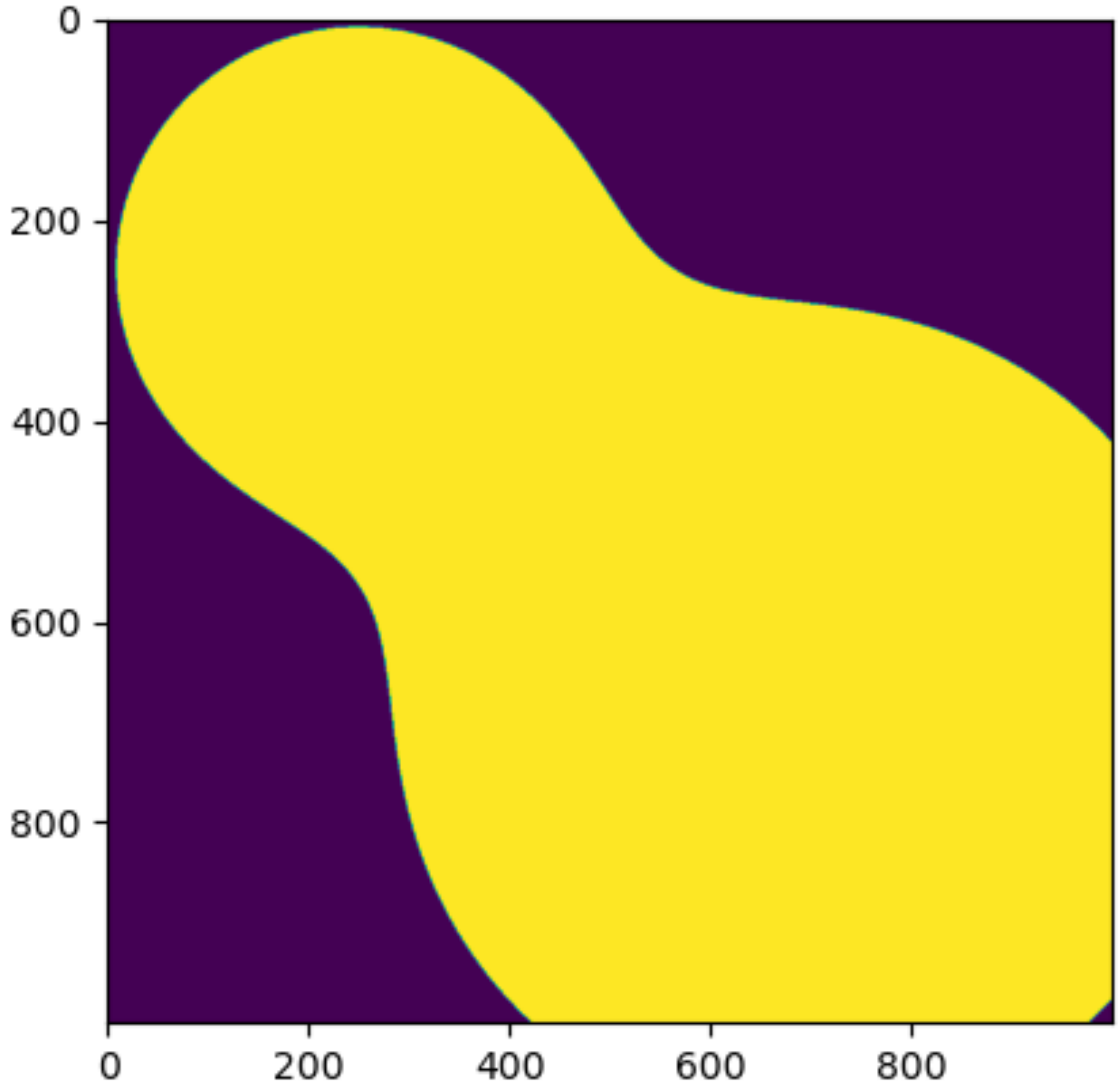

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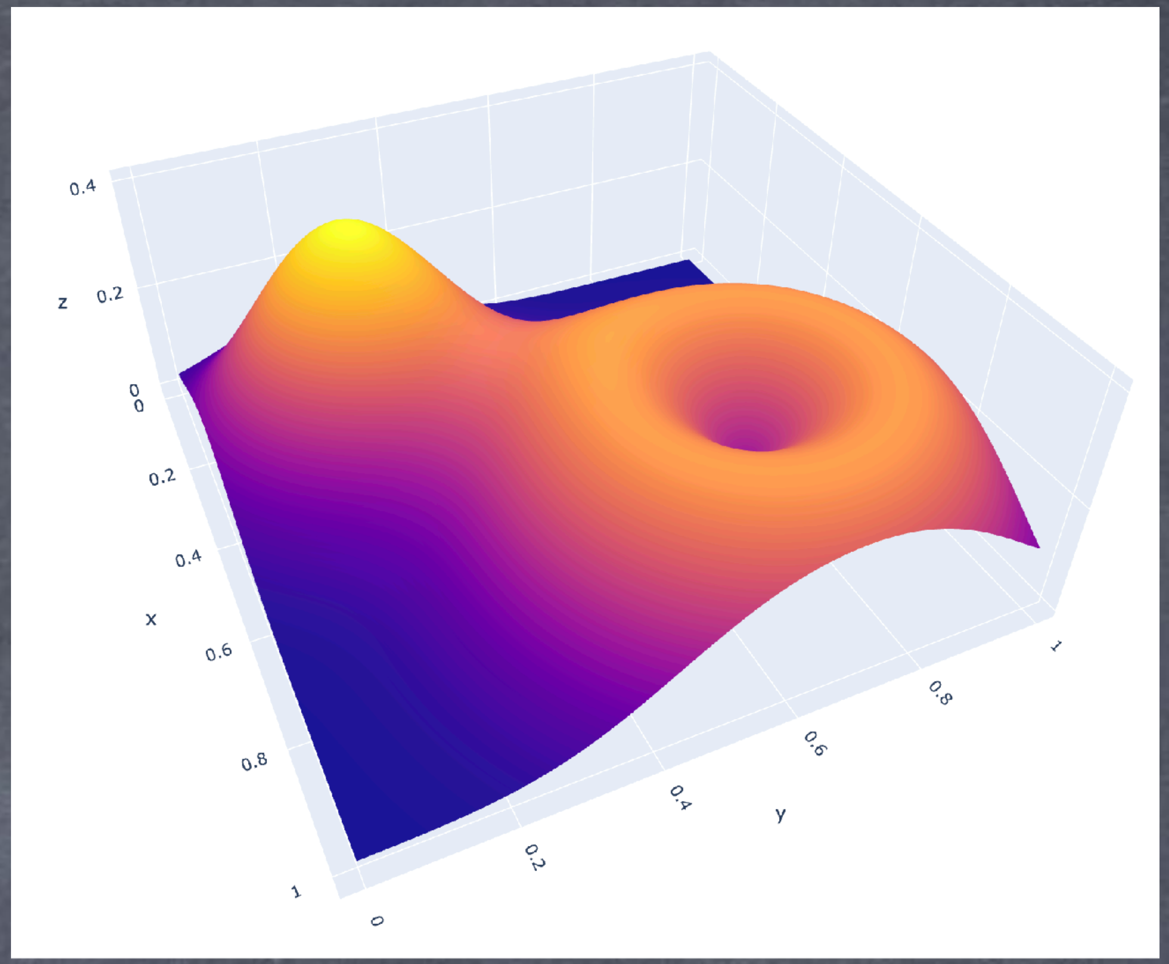

z
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$$H_*(X_1) \to H_*(X_2) \to H_*(X_3) \to H_*(X_4) \to \cdots$$

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One can choose bases
compatible with the linear maps

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Death time

One can choose bases compatible with the linear maps

$$H_*(X_1) \to H_*(X_2) \to H_*(X_3) \to H_*(X_4) \to \cdots$$

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$$H_*(X_1) \to H_*(X_2) \to H_*(X_3) \to H_*(X_4) \to \cdots$$

Birth time

Death time

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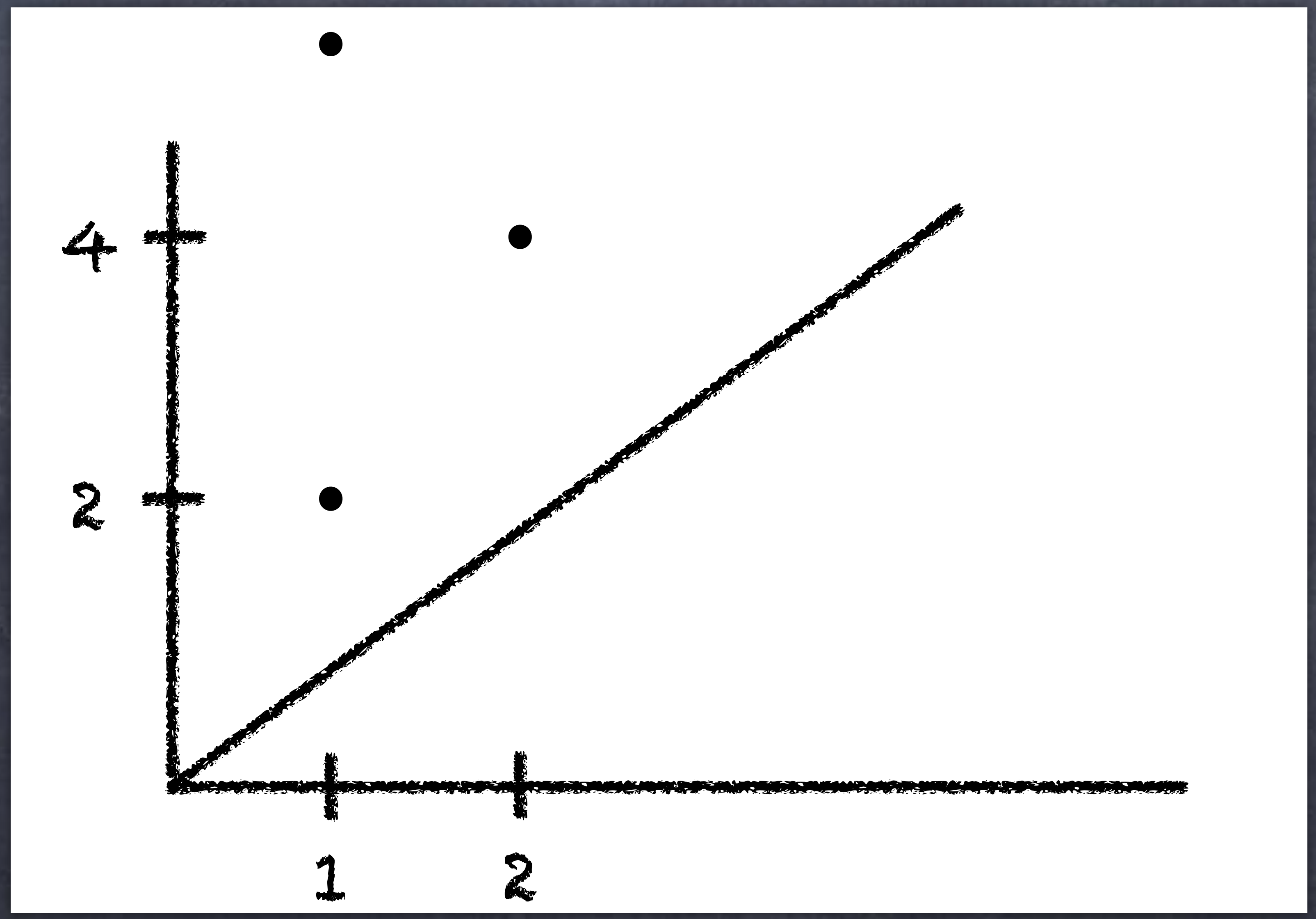

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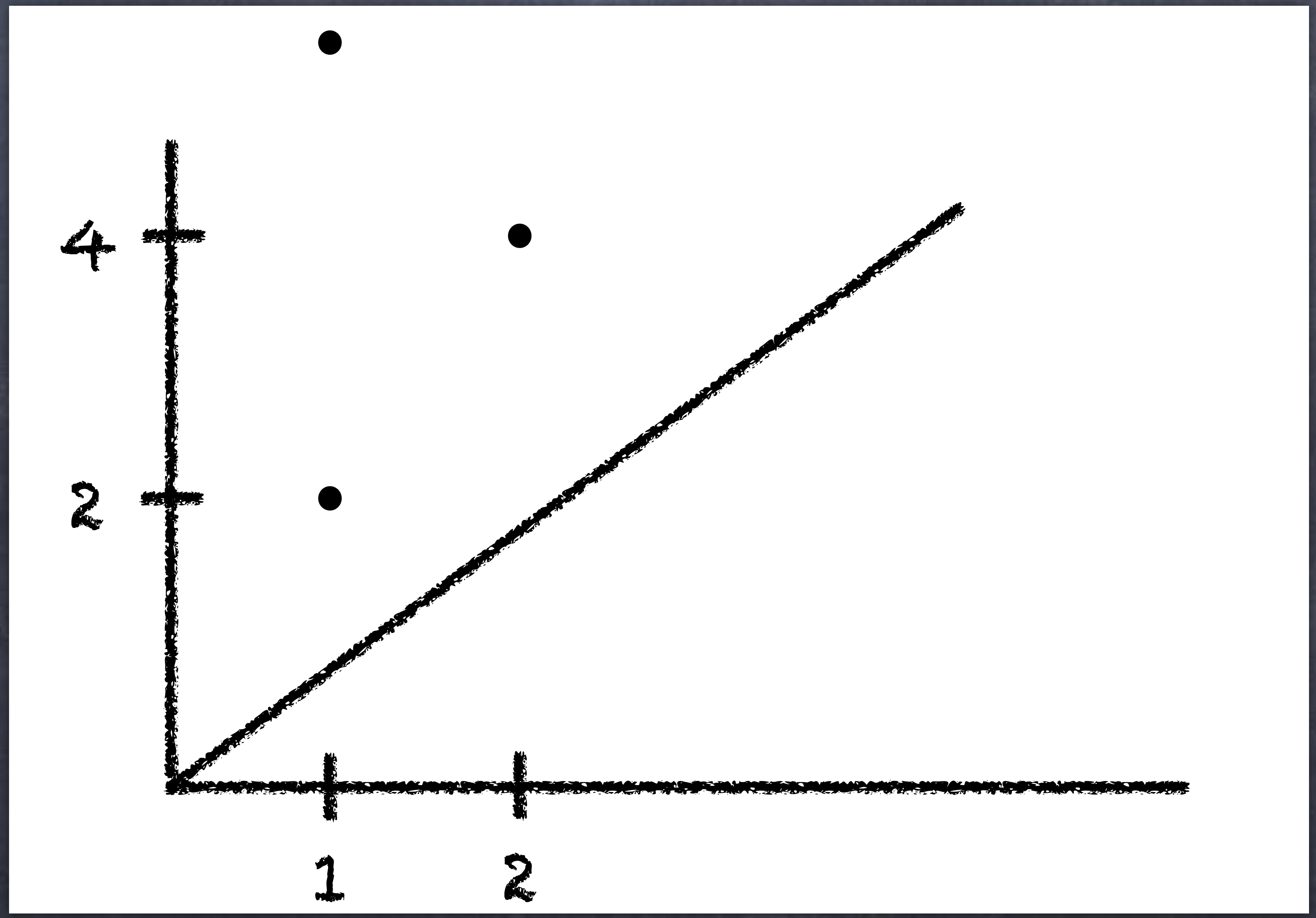
Birth time

Death time

One can choose bases compatible with the linear maps



Persistence diagram

- Vectorise persistence diagrams

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- Feed them into statistical analysis / ML

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- There is good software for these computations:
  GUDHI, Ripser, Giotto-TDA, and more

Thank you

- Nick Sale used these tools to study vortices in SU(2) gauge theory.
- Later today:

  Xavier Crean on monopoles in U(1) gauge theory.

  Biagio Lucini on monopoles in SU(3) gauge theory.

Thank you